THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII

1931

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 1, JANUARY

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

LAGRANGE'S COMPOUND PENDULUM

By H. BATEMAN, California Institute of Technology

1. *The Equations of Motion*

The so called polynomials of Laguerre were introduced into mathematical analysis by J. L. Lagrange[1] in his solution of a dynamical problem in which the oscillations of a vertical chain are represented approximately by those of a set of similar weights equally spaced on a light string which is vertical in the equilibrium position Oy. Let the weights be numbered $0, 1, 2, \cdots$ in order, beginning with the lowest, and let x_n be the small horizontal displacement from the vertical line Oy of the weight W_n associated with the number n. If W is the magnitude of each weight, the tension in the portion $W_{n-1}W_n$ of the string is approximately equal to nW, for the cosine of the angle between the vertical and any portion of the string may be treated as unity when the oscillations are sufficiently small. Let a be the length of string between consecutive weights; then the angle between $W_{n-1}W_n$ and the vertical is approximately

$$(x_{n-1} - x_n)/a;$$

and so the equation of motion of the weight W_n is

$$\frac{W}{g}\frac{d^2x_n}{dt^2} = \frac{nW}{a}(x_{n-1} - x_n) - \frac{(n+1)W}{a}(x_n - x_{n+1}).$$

With a suitable choice of units the equation may be written in the form

$$x_n'' = n(x_{n-1} - x_n) - (n+1)(x_n - x_{n+1}), \tag{1}$$

where primes denote differentiations with respect to a modified time which will be denoted by τ. The actual relation between t and τ is

$$gt^2 = a\tau^2. \tag{2}$$

2. *Lagrange's solution of the equations of motion when the weight W_m is fixed.*

Let us suppose that the initial conditions are

$$x_n = a_n, \quad x_n' = 0, \quad \text{for } t = 0, \quad n = 0, 1, 2 \cdots. \tag{3}$$

Using the abbreviation

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \tag{4}$$

we introduce a set of quantities c_n, defined by the equations

[1] *Miscellanea Taurinensia*, t. III. 1762–1765, *Oeuvres*, I. p. 534. The relation between the polynomials of Lagrange and Laguerre was, I think, first pointed out by Professor E. T. Whittaker after the present author had called his attention to a relation between the polynomials of Abel and Laguerre and his W-function. See Whittaker and Watson, *Modern Analysis*, p. 353.

(5)
$$c_0 = a_0,$$
$$c_n = n!\left[a_0 - \binom{n}{1}a_1 + \binom{n}{2}a_2 - \cdots + (-1)^n a_n\right],$$

and an operator ω with the property that

(6)
$$\omega^n c_q = c_{n+q}, \quad n = 0, 1, 2, \cdots .$$

A solution of the equations (1) and the boundary conditions (3) is then represented symbolically by

(7)
$$x_n = L_n(\omega) \cos (\tau\sqrt{\omega})c_0,$$

where

(8)
$$L_n(z) = 1 - \binom{n}{1}\frac{z}{1!} + \binom{n}{2}\frac{z^2}{2!} - \cdots (-)^n\frac{z^n}{n!}$$

is the polynomial of Lagrange and Laguerre.[2]

The verification is immediate because this polynomial satisfies the difference equation

(9)
$$n[L_{n-1}(z) - L_n(z)] - (n + 1)[L_n(z) - L_{n+1}(z)] = - zL_n(z)$$

and the set of equations (5) gives

(10)
$$a_n = L_n(\omega)c_0.$$

In the particular case when $c_n = u^n c_0$, where u is a constant, we have

(11)
$$x_n = L_n(u) \cos (\tau\sqrt{u})c_0$$

and $x_m = 0$ *for all values of* t if

(12)
$$L_m(u) = 0.$$

This is the result given by Lagrange. It is known that the m roots of the equation (12) are all positive and unequal.[3] We shall denote them by $u_1, u_2, \cdots, u_m$ and shall suppose that these numbers are arranged in order of magnitude, u_1 being the smallest and u_m the largest.

The solution for the initial conditions

(13)
$$x_n = a_n, \quad x_n' = b_n \quad \text{for} \quad \tau = 0, \quad n = 0, 1, 2 \cdots m - 1,$$

and the requirement that $x_m = 0$ for all values of t, is

---

[2] The notation used here is that adopted by E. Hille in the Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 261, 265, 348, and by G. Szegö in Mathematische Zeitschrift, vol. 25 (1926), p. 87.

[3] This was proved by Laguerre, Bulletin de la Société Mathématique de France, vol. 7 (1879). *Oeuvres de Laguerre*, vol. 1, p. 428. The distribution of the roots is discussed by E. R. Neumann in Jahresbericht der Deutschen Mathematiker Vereinigung, Band 30 (1921), p. 15.

$$x_n = \sum_{s=1}^{m}[A_s \cos (v_s\tau) + B_s \sin (v_s\tau)]L_n(u_s), \tag{14}$$

where $u_s = v_s^2$ and

$$\sum_{s=1}^{m} A_s L_n(u_s) = a_n, \tag{15}$$

$$\sum_{s=1}^{m} v_s B_s L_n(u_s) = b_n. \tag{16}$$

These equations can be solved for A_s and $v_s B_s$ if the determinant

$$D \equiv | L_p(u_q) | \qquad (p = 0, 1, \cdots, m-1, q = 1, 2 \cdots m)$$

is different from zero. If D were zero it would be possible with suitable coefficients $C_0, C_1, \cdots C_{m-1}$ to construct a polynomial

$$F_{m-1}(x) = \sum_{r=0}^{m-1} C_r L_r(x)$$

with the m roots, $x = u_1$, $x = u_2, \cdots, x = u_m$. This however, is impossible because the roots are all distinct and the polynomial is only of degree $m-1$.

If the string is supported from a fixed point dividing the line $W_{m-1}W_m$ in the ratio $\alpha:\beta$, there is a solution of a similar type; but the equation for the determination of the quantities u_q is now of type $G(x) = 0$, where

$$G(x) = \alpha L_{m-1}(x) + \beta L_m(x) = 0, \tag{17}$$

α and β being positive constants. The polynomial of Laguerre satisfies the equation

$$xL_m'(x) = mL_m(x) - mL_{m-1}(x), \tag{18}$$

where the prime now denotes a differentiation with respect to x. This equation tells us that if x has a positive value for which $L_m(x) = 0$ the functions $L_m'(x)$ and $L_{m-1}(x)$ have opposite signs, m being positive. Now $L_m'(x)$ has opposite sign at two consecutive roots of $L_m(x)$ and so $L_{m-1}(x)$ has also opposite signs at two consecutive roots of $L_m(x)$. This shows that a root of the equation $G(x) = 0$ lies between two consecutive roots of the equation $L_m(x) = 0$; indeed there cannot be three or more roots in such an interval because this would give the equation $G(x) = 0$ too many roots, since we must allow at least one for each interval. Allowing just one root for each interval there is till one root left. To locate this we observe that if μ is the greatest root of the equation $L_m(x) = 0$, $L_m'(\mu)$ and $L_{m-1}(\mu)$ have opposite signs and so $L_m(x)$ and $L_{m-1}(x)$ have opposite signs for $x > \mu$. Eventually $|L_m(x)|$ is enormously large compared with $|L_{m-1}(x)|$ and so $G(x)$ must vanish for some value of x greater than μ. The roots of the equations $L_m(x) = 0$, $G(x) = 0$ can thus be placed in correspondence in such a way that

each root of the second equation is greater than the corresponding root of the first. The physical meaning of this result is that the natural frequencies of the pendulum are increased by shortening the string. This is simply an illustration of a well known general theorem.

3. *Unlimited string*

When the string is unlimited in length a solution corresponding to the initial conditions (13) may be written down by making use of the fact that the function $L_n(u)$ satisfies the orthogonal relation (discovered by Abel and Murphy)

$$\int_0^\infty e^{-u}L_n(u)L_p(u)du = 0, \quad n \neq p, \qquad = 1, \quad n = p. \tag{19}$$

The appropriate generalisation of (14) is thus

$$x_n = \int_0^\infty [A(u) \cos vt + B(u) \sin vt]L_n(u)du, \tag{20}$$

where $v=\sqrt{u}$ and $A(u)$, $B(u)$ are defined by the infinite series,[4]

$$A(u) = \sum_{p=0}^\infty a_p e^{-u}L_p(u), \tag{21}$$

$$vB(u) = \sum_{p=0}^\infty b_p e^{-u}L_p(u). \tag{22}$$

In particular, if $a_p=0$, $p\neq s$, $a_s=1$, $b_p=0$, the solution is simply

$$x_n = \int_0^\infty e^{-u} \cos v\tau L_n(u)L_s(u)du. \tag{23}$$

The symmetry with respect to n and s should be noted. This is an illustration of Rayleigh's general reciprocal theorem.[5] Similarly, if $b_p=0$, $p\neq s$, $b_s=1$, $a_p=0$, the solution is

$$x_n = \int_0^\infty e^{-u} \sin v\tau L_n(u)L_s(u)\frac{du}{v} \qquad v = \sqrt{u} \tag{24}$$

and again there is symmetry.

---

[4] The convergence of series of this type is discussed by Hille, l.c. See also G. Szegö, Mathematische Zeitschrift, vol. 25 (1926), p. 87.

[5] Proceedings of the London Mathematical Society, (1), vol. 4 (1873), p. 357; Scientific Papers, vol. 1, p. 179.

A solution equivalent to (23) may be obtained directly from (7) and may be written in the form

$$
\begin{aligned}
x_n = c_0 &- \binom{n}{1}\frac{c_1}{1!} + \binom{n}{2}\frac{c_2}{2!} - \cdots \\
&- \frac{\tau^2}{2!}\left[c_1 - \binom{n}{1}\frac{c_2}{1!} + \binom{n}{2}\frac{c_3}{2!} - \cdots\right] \\
&+ \frac{\tau^4}{4!}\left[c_2 - \binom{n}{1}\frac{c_3}{1!} + \binom{n}{2}\frac{c_4}{2!} - \cdots\right] \\
&- \cdots ,
\end{aligned} \tag{25}
$$

the special values of the quantities a_p being introduced into the expression (5) for c_q. A similar expression for the integral in (24) may be obtained by integrating (25) from 0 to τ with respect to τ and replacing a_p' by b_p' in the expression for c_q.

4. *Relation to the Wave-Equation*

Let V be a solution of the equation

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{\partial^2 V}{\partial \tau^2} \tag{26}$$

which arises from the two-dimensional wave equation when there is symmetry about an axis. Denoting this solution by the symbol $V(\rho,\tau)$ and assuming that it is finite and continuous for all positive values of ρ, we may form a definite integral

$$x_n = \frac{1}{2^{2n+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+1}V(\rho, \tau)d\rho \tag{27}$$

which is readily seen to be a solution of the equation

$$x_n'' = n[x_{n-1} - x_n] - (n+1)[x_n - x_{n+1}]. \tag{28}$$

In particular, the wave-function

$$V(\rho, \tau) = J_0(\rho\sqrt{u}) \cos (\tau\sqrt{u}) \tag{29}$$

gives rise to the 'Lagrangian' displacement

$$x_n = e^{-u}L_n(u) \cos (\tau\sqrt{u}). \tag{30}$$

When the string is supported at the weight W_m the condition $x_m = 0$ (for all values of τ) corresponds to the condition

$$\int_0^\infty e^{-\rho^2/4}\rho^{2m+1}V(\rho, \tau)d\rho = 0 \tag{31}$$

imposed on a solution of the wave-equation. This is a particular case of the more general condition

$$\int_0^\infty f(\rho^2)\rho d\rho V(\rho, \tau) = 0, \tag{32}$$

where $f(\rho^2)$ is a specified function. The special condition (31) is particularly interesting because a general type of wave-function V satisfying the requirements may be constructed with the aid of the polynomial of Laguerre and V is found to consist of the sum of m terms each of which is a function of ρ multiplied by a trigonometric function of τ.

The problem may be generalised in another way by considering the equation

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1 - 2m}{\rho}\frac{\partial V}{\partial \rho} = \frac{\partial^2 V}{\partial \tau^2}. \tag{33}$$

The differential equation satisfied by x_n is now

$$x_n'' = (n + m)(x_{n-1} - x_n) - (n + 1)(x_n - x_{n+1}). \tag{34}$$

This may be regarded as the equation of motion of the weight W_n when the weight of the string is taken into consideration. We shall indeed regard W as the weight of W_n and the portion of the string between W_{n-1} and W_n, while the weight of this portion of string alone will be supposed to be mW. A portion of string of weight mW will be supposed also to hang from the weight W_0. The tension of the string is then nW at W_{n-1} and $(n+m)W$ at W_n. The equation of motion is consequently of type (34). A solution of this equation may be expressed in terms of the generalised polynomial of Laguerre which was introduced into mathematical analysis by Sonine.[6] Using the notation of Hille we have[7]

$$\frac{1}{(1 - z)^{m+1}} e^{-xz/(1-z)} = \sum z^n L_n{}^m(x). \tag{35}$$

$$L_n^m(x) = \frac{1}{n!}e^x x^{-m}\frac{d^n}{dx^n}[x^{n+m}e^{-x}], \tag{36}$$

$$\begin{aligned} \int_0^\infty e^{-u}u^m L_n^m(u)L_s^m(u)du &= 0 \quad (n \neq s) \\ &= \frac{\Gamma(n + m + 1)}{\Gamma(n + 1)} \quad (n = s), \end{aligned} \tag{37}$$

---

[6] Mathematische Annalen, vol. 16 (1880), p. 1.

[7] Loc. cit. The second equation is due essentialy to E. Kummer. See Crelle's Journal, vol. 15 (1836), p. 139, Eq. (4); the third equation is given by Sonine, loc. cit.; the fourth equation is ascribed by Szegö to É. Le Roy. Toulouse Annales, (2) vol. 2 (1900), p. 379. But Le Roy considers only the case $m=0$. The general formula is given on p. 103 of the author's *Electrical and Optical Wave Motion*, equation (222) and is also mentioned by Hille (loc. cit.).

(38) $$e^{-u}u^{m/2}L_n^m(u) = \frac{1}{2^{2n+m+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+m+1}J_m(\rho\sqrt{u})d\rho.$$

The solution of (33), which is represented by

(39) $$V = \rho^m J_m(\rho\sqrt{u}) \cos (\tau\sqrt{u}),$$

thus corresponds to the displacement

(40) $$x_n = e^{-u}u^{m/2}L_n^m(u) \cos (\tau\sqrt{u}).$$

For the case of the unlimited string this solution may be generalised with the aid of the orthogonal relation (37) and the method of §3.

When the string is held at the point W_s the equation $x_s = 0$ may be satisfied by choosing u so that

(41) $$L_s^m(u) = 0.$$

Gegenbauer[8] has shown that this equation has s unequal positive roots.A geometrical proof for the special case $m = \frac{1}{2}$ is given by Bôcher.[9]

The foregoing interpretation of equation (34) seems to imply that $m<1$. This restriction can however, be avoided by supposing that.the weights are buoyant instead of heavy. The case $m<0$ can be obtained by supposing the string to be buoyant and the weights heavy.

We may form a rough idea of the amplitude of the oscillation (40) when u is large by using Szegö's inequality (loc. cit.).

(42) $$e^{-u/2}\left| L_n^m(u) \right| \leqq \binom{n+m+1}{n}, \quad u \geqq 0, \quad m \geqq -1,$$

or his alternative inequalities,

(43) $$e^{-u/2}\left| L_n^m(u) \right| \leqq \binom{n+m}{n} \quad \text{for} \quad m \geqq 0, \quad u \geqq 0,$$
$$2 - \binom{n+m}{m} \quad \text{for} \quad -1 \leqq m \leqq 0, \quad u \geqq 0.$$

5. *Connection with the equation of heat conduction.*

If U satisfies the equation

(44) $$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} = \frac{\partial U}{\partial \tau}$$

and is finite and continuous for all positive values of ρ, the definite integral

---

[8] Wiener Berichte (1887), p. 274; Amsterdam Proceedings, vol. 5 (1896–7), p. 185.

[9] Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 469.

(45) $$y_n = \frac{1}{2^{2n+1}n!}\int_0^\infty e^{-\rho^2/4}\rho^{2n+1}U(\rho, \tau)d\rho$$

is generally a solution of the equation

(46) $$y_n' = (n+1)(y_{n+1} - y_n) - n(y_n - y_{n-1}),$$

which is satisfied by $y_n = e^{-u\tau}L_n(u)$, the corresponding function U being given by (38). Furthermore, the fundamental function

(47) $$U = 1/\tau \exp\left[-\rho^2/4\tau\right]$$

gives rise to the simple solution

(48) $$y_n = \tau^n/(1+\tau)^{n+1}.$$

This result may be generalised by taking U to be a solution of the partial differential equation,

(49) $$\frac{\partial^2 U}{\partial \rho^2} + \frac{2m+1}{\rho}\frac{\partial U}{\partial \rho} = \frac{\partial U}{\partial \tau}.$$

The quantity y_n is then found to be a solution of the equation

(50) $$y_n' = (n+1)[y_{n+1} - y_n] - (n-m)[y_n - y_{n-1}],$$

and the fundamental solution

(51) $$U = \frac{1}{\tau^{m+1}}e^{-\rho^2/4\tau}$$

Gives rise to the simple solution

(52) $$y_n = \tau^{n-m}/(1+\tau)^{n+1}.$$

6. *The torsional vibrations of a loaded shaft.*

The differential equation (1) occurs also in the theory of the torsional vibrations of a loaded shaft when the torsional stiffness of the portion of shaft between consecutive loads varies along the shaft in a suitable manner. Indeed, if θ_n is the angular displacement of the load W_n the equation of motion of this load is of type[10]

(53) $$p_n\theta''_n = c_{n,n+1}(\theta_{n+1} - \theta_n) - c_{n-1,n}(\theta_n - \theta_{n-1}),$$

where p_n is an inertia coefficient and $c_{n,\ n+1}$ is the torsional stiffness of the portion W_nW_{n+1} of the shaft. We have only to take $c_{n,\ n+1}$ proportional to $(n+1)$ to obtain an equation reducible to (1).

---

[10] See, for instance, J. Morris. "*The strength of shafts in vibration*" (Crosby Lockwood, London, 1929), p. 118.

A METHOD OF SOLVING A DETERMINATE SYSTEM OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS.[1]

By F. UNDERWOOD, University College, Nottingham, England

1. It is proposed to consider the system of simultaneous equations

$$(r) \qquad F_{r1}x_1 + F_{r2}x_2 + F_{r3}x_3 + \cdots + F_{rn}x_n = 0 \qquad (r = 1, 2, 3, \cdots n),$$

where $x_1, x_2, \cdots x_n$ are functions of an independent variable t, $F_{rs}(r, s=1, 2, 3, \cdots n)$ are polynomials in D with constant coefficients, and D means d/dt.

There are at least three methods by which the full solutions of such a system can be obtained:—

(i) the ordinary method given in text-books on differential equations;[2]

(ii) a method (due to Chrystal)[3] for the reduction of the system to an equivalent diagonal system;

(iii) a method which does not seem to be given in text-books on differential equations, but which is dealt with at some length by Routh.[4]

It is proposed to discuss the third of these, and especially a case of failure which does not appear explicitly in Routh's work.

2. In any of these methods references must be made to the characteristic determinant of the coefficients in the system of equations:

$$\Delta(D) = \begin{vmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1n} \\ F_{21} & F_{22} & F_{23} & \cdots & F_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ F_{n1} & F_{n2} & F_{n3} & \cdots & F_{nn} \end{vmatrix}.$$

A complete theoretical treatment of such systems is given in Chrystal's paper referred to above, but, for most determinate systems of this type, the third method is the most direct and the most expeditious in practice, for it gives the full solution for each dependent variable with the proper number of arbitrary constants, without introducing any superfluous constants. It is, however, subject to certain cases of failure noted below, and, of these, the one which is the most difficult has been discussed by Routh.

The third method depends upon the use of cofactors (or first minors) of the elements in the determinant Δ. Let the given equations be numbered in order (1), (2), $\cdots$, (n). Suppose that we use the $(n-1)$ equations (1), (2), $\cdots$ $(r-1)$, $(r+1)$, $\cdots$, (n) $\{$i.e. all except equation $(r)\}$, to determine the ratios

$$x_1 : x_2 : x_3 \cdots : x_n$$

[1] With special reference to an exceptional case (or case of failure) not noted by Routh.

[2] See, for example, reference 1 or 2 at the end of this article.

[3] See reference 3 or 4.

[4] See reference 5 or 6.

by a purely algebraical process, treating F_{pq}, which is a polynomial in D, as if it were a constant. The result may be written:

$$(r') \qquad \frac{x_1}{G_{r1}} = \frac{x_2}{G_{r2}} = \frac{x_3}{G_{r3}} = \cdots = \frac{x_n}{G_{rn}} = V,$$

where G_{rs} is the cofactor of F_{rs} in the characteristic determinant Δ. Substitution of these values of $x_1, x_2, \cdots x_n$ in the given equation (r) leads to the equation $\Delta(V) = 0$.

If the roots of $\Delta(D) = 0$ are all different, say $a_1, a_2, \cdots, a_k$, then

$$V = A_1 e^{a_1 t} + A_2 e^{a_2 t} + \cdots A_k e^{a_k t}.$$

If the roots of $\Delta(D) = 0$ are not all different, the solution for V requires the usual modifications for repeated roots, but, in any case, it contains the full number (k) of arbitrary constants, where k is the degree of Δ as a polynomial in D. In the cases which usually arise, substitution in equation (r') then gives directly the full solutions for $x_1, x_2, \cdots x_n$ which are equivalent to the given system of equations. It will be noted that no arbitrary constants other than $A_1, A_2, \cdots A_k$ can appear in these solutions and that, with the exception of certain cases considered below, the method indicated obtains these solutions as directly as possible.

3. *Exceptional cases* (*or cases of failure*). If $\Delta(D)$ contains a factor $D-a$, there is in the solution for V a term of the type $A_0 e^{at}$ and, of course, if $\Delta(D)$ contains the factor $(D-a)^\lambda$, where λ is a positive integer, there are terms of the type

$$[A_0 + A_1 t + A_2 t^2 + \cdots + A_{\lambda-1} t^{\lambda-1}] e^{at}.$$

Now it is possible that all the cofactors of the r^{th} row in the characteristic determinant $\Delta(D)$, i.e. $G_{r1}, G_{r2}, \cdots G_{rn}$, may contain the factor $D-a$. In this case the arbitrary constant A_0 will be missing from the solutions for each of the variables $x_1, x_2, \cdots, x_n$. This exception is noted by Routh,[5] who points out that "if all the minors of only one row vanished, we could find the values of $x_1, x_2, x_3, \cdots$, etc., by choosing as our operators the minors of some other row. But this cannot be done if all the minors of all the rows are zero." Routh then gives a full discussion of the case when all the first minors contain $(D-m)^\beta$, all the second minors $(D-m)^\gamma$, etc., while $\Delta(D)$ contains $(D-m)^\alpha$. He shows how the full solutions may be obtained, including what he calls solutions of double type, of triple type, etc.

It is obvious that, whatever the values of $\alpha, \beta, \gamma, \cdots (\alpha > \beta > \gamma > \cdots)$, the above exceptional case is one in which $\Delta(D) = 0$ has repeated roots. There is, however, an exceptional case of a similar character (not mentioned explicitly by Routh and apparently not noticed by him), in which the roots of $\Delta(D) = 0$

[5] See reference 5, p. 164 or reference 6, p. 214.

may be all different, though, as will be explained below, there are at least two simple methods of obtaining the full solutions and so avoiding the apparent difficulty of this case. Thus if all the cofactors of the first row contain a factor $(D-a_1)$ all the cofactors of the second row a factor $(D-a_2)$, and so on, where $a_1, a_2, a_3, \cdots$ are all different, the complete solutions for $x_1, x_2, \cdots, x_n$ cannot be obtained from any single equation of the type (r') and yet $\Delta(D)=0$ may not have any repeated factors. In this case Routh's remark that "we could find the values of x_1, x_2, x_3, etc., by choosing as our operators the minors of some other row" is scarcely justified as it stands, though it is easy to see (as in the example below), that it may be justified if it be extended to include a combination of the results obtained from two different rows, i.e. from two equations (r') and (s'), $(r \neq s)$. A similar result holds if the common factors of the different rows are of the types

$$(D-a_1)^{\lambda_1}, \ (D-a_2)^{\lambda_2},$$

or even combinations of these such as

$$(D-a_1)^{\lambda_1}(D-a_2)^{\mu_1}; \quad (D-a_2)^{\lambda_2}(D-a_3)^{\mu_2}; \quad (D-a_1)^{\lambda_3}(D-a_3)^{\mu_3};$$

etc., provided always that no common factor such as $(D-a_1)$ occurs in all the minors. By taking a combination of the results from an appropriate set of equations of type (r') it must always be possible to obtain the full solutions for $x_1, x_2, \cdots, x_n$.

Example.

$$(1) \qquad D(D+2)x+(D+2)y+(D+2)z=0;$$

$$(2) \qquad x+(D+2)y+z=0;$$

$$(3) \qquad (D+1)x+(D+1)y+D(D+1)z=0.$$

Solving for the ratios $x:y:z$ from (2) and (3),

$$(1') \qquad \frac{x}{(D+1)(D^2+2D-1)}=\frac{y}{-(D+1)(D-1)}=\frac{z}{-(D+1)^2}=V_1.$$

Similarly, from (1) and (3):

$$(2') \qquad \frac{x}{(D+1)(D+2)(D-1)}=\frac{y}{-(D+1)(D+2)(D^2-1)}=\frac{z}{(D+1)(D+2)(D-1)}=V_2.$$

From (1) and (2),

$$(3') \qquad \frac{x}{-(D+2)(D+1)}=\frac{y}{-(D+2)(D-1)}=\frac{z}{(D+2)(D^2+2D-1)}=V_3.$$

On substituting from equation (r') in equation (r) for any one of the cases $r=1, 2, 3$, V_r is found to satisfy the equation $\Delta(V)=0$, where Δ is the characteristic determinant of the given system of equations.

Thus

$$\Delta = D(D+1)(D+2)(D+3)(D-1).$$

Hence

$$V = A + Be^{-t} + Ce^{-2t} + Ee^{-3t} + Fe^{t},$$

and contains the full number (five) of arbitrary constants which must remain in the final solutions for x, y, z in this system of equations. If will be noted that $\Delta(D)$ has no repeated factors, and also that the cofactors (denominators) in equations (1′), (2′), (3′) contain the common factors

$$(D+1),\ (D+1)(D+2)(D-1),\ (D+2),$$

respectively. Thus if we denote by x_r, y_r, z_r the actual solutions obtained from equation (r'), $r=1, 2, 3$,
the solutions x_1, y_1, z_1, will contain no term e^{-t};
" " x_2, y_2, z_2 " " " " e^{-t}, e^{-2t}, or e^{t};
" " x_3, y_3, z_3 " " " " e^{-2t}.

The actual solutions are readily found in the forms:

$$(\alpha)\quad \begin{cases} x_1 = -A + Ce^{-2t} - 4Ee^{-3t} + 4Fe^{t} \\ y_1 = A - 3Ce^{-2t} - 8Ee^{-3t} \\ z_1 = -A - Ce^{-2t} - 4Ee^{-3t} - 4Fe^{t} \end{cases}$$

$$(\beta)\quad \begin{cases} x_2 = -2A - 8Ee^{-3t} \\ y_2 = 2A - 16Ee^{-3t} \\ z_2 = -2A - 8Ee^{-3t} \end{cases}$$

$$(\gamma)\quad \begin{cases} x_3 = -2A - 2Ee^{-3t} - 6Fe^{t} \\ y_3 = 2A + 2Be^{-t} - 4Ee^{-3t} \\ z_3 = -2A - 2Be^{-t} - 2Ee^{-3t} + 6Fe^{t}. \end{cases}$$

No single set of solutions (α), (β), or (γ) can provide the full (i.e. the most general) solutions of the given system of equations, but these full solutions may be obtained by taking a combination of (α) and (γ). It will be noted that these full solutions cannot be obtained from the combination (α) and (β), since the denominators in (1′) and (2′) all contain the factor $(D+1)$, and this causes terms in e^{-t} to be absent from sets (α) and (β). For similar reasons a combination of (β) and (γ) does not give any term in e^{-2t}.

Taking the combination (α) and (γ) and putting the arbitrary constants in as simple forms as possible, the most general solutions of the system of equations may be given as:

$$\begin{aligned} x &= A + Ce^{-2t} + Ee^{-3t} + Fe^{t}. \\ y &= -A + Be^{-t} - 3Ce^{-2t} + 2Ee^{-3t}. \\ z &= A - Be^{-t} - Ce^{-2t} + Ee^{-3t} - Fe^{t}. \end{aligned}$$

4. *An alternative method of dealing with the exceptional case* noted above may be illustrated by means of the preceding example. This method may be described as a combination of the first and third methods of paragraph 1. More explicitly, it uses the first (or ordinary) method with its excessive number of arbitrary constants to supply just that part of the solution which must be absent when, in the third method, the cofactors of any one row have a common factor. But it is important to notice that the first method as a whole is not required, and it is used only for that portion of the solution (usually small) where it is known that the third method must be deficient.

Thus in the preceding example x_1, y_1, z_1 may be found as before by using equation (1′) and, since all the denominators in (1′) contain the factor $(D+1)$, these solutions cannot contain any term in e^{-t}.
Hence put

$$x = x_1 + He^{-t}; \quad y = y_1 + Ke^{-t}; \quad z = z_1 + Le^{-t}.$$

Now x_1, y_1, z_1, satisfy the given equations, so that He^{-t}, Ke^{-t}, Le^{-t} taken separately must also satisfy these equations.
Substituting in (1) and (2),

$$-H + K + L = 0; \qquad H + K + L = 0.$$

[(3) gives only $0=0$].
Hence

$$H = 0; \qquad L = -K.$$

The solutions now obtained can easily be put in the same forms as those given above.

Perhaps it may be added that this alternative method is successful also in dealing with the exceptional case noted by Routh and so avoids his solutions of double type, triple type, etc. Thus in the example[6] constructed by Routh to illustrate his methods, each cofactor on the first row contains a factor $(D-1)^3$. Hence after finding x_1, y_1, z_1 as before we can put

$$x = x_1 + (A_1 + B_1t + C_1t^2)e^t,$$

etc., and proceed to the construction of the full solutions of the system.

References

1. A. R. Forsyth, *Treatise on Differential Equations*, 5th edition (1921), pp. 342–350.
2. Ch. J. de la Vallée Poussin, *Cours d'Analyse*, vol. 2, 4th edition (1922), pp. 251–261.
3. G. Chrystal, Transactions of the Royal Society of Edinburgh, vol. 38 part I (1894–95), pp. 163–178.
4. E. L. Ince, *Ordinary Differential Equations* (1924), pp. 144–155.
5. E. J. Routh, Proceedings of the London Mathematical Society. vol. 14 (1883), pp. 144–170.
6. E. J. Routh, *Rigid Dynamics*, part 2, 6th edition (1905), pp. 210–219.

[6] See reference 6, pp. 217–218.

ON THE NUMERICAL SOLUTION OF A BOUNDARY VALUE PROBLEM

By W. E. MILNE, University of Oregon

The approximate solution of linear boundary value problems has been attacked by various methods, such as the method of Ritz,[1] depending in principle on the calculus of variations, the method of least squares,[2] the method of least mth powers,[3] and the method of constant coefficients.[4] The object of this note is to show how the method of numerical integration may be applied to the determination of the characteristic values of λ belonging to the homogeneous differential system

$$(d^2u/dx^2) + G(x, \lambda)u = 0, \tag{1}$$

$$fu(a) + gu'(a) = 0, \; f^2 + g^2 = 1, \tag{2}$$

$$hu(b) + ku'(b) = 0, \; h^2 + k^2 = 1. \tag{3}$$

The function $G(x, \lambda)$ is assumed to be real and continuous and to possess a positive partial derivative with respect to λ for all real values of λ and for x in the interval $a \leqq x \leqq b$. It is further assumed that

$$\lim_{\lambda=-\infty} G(x, \lambda) = -\infty, \quad \lim_{\lambda=\infty} G(x, \lambda) = +\infty.$$

Then there exists[5] an infinite sequence of values of λ, $\lambda_0, \lambda_1, \lambda_2, \cdots$ for which (1), (2), and (3) have non-vanishing solutions. In the case in which $G(x, \lambda)$ is of the form $\lambda - q(x)$ the approximate value of λ_n can readily be determined by means of asymptotic formulas when n is large, but the first values, λ_0, λ_1, etc., (which are usually of the greatest physical importance) cannot be so easily determined. On the other hand the method here given is well adapted to the calculation of the smaller characteristic numbers. The particular case in which the interval is infinite has been treated in an earlier paper,[6] where an illustrative problem is solved by this method.

1. Let $u_1(x)$ and $u_2(x)$ be two solutions of (1) satisfying the conditions,

$$\begin{aligned} u_1(a) &= 1, & u_2(a) &= 0, \\ u_1'(a) &= 0, & u_2'(a) &= c > 0. \end{aligned} \tag{4}$$

---

[1] Ritz, Crelle's Journal, vol. 135 (1909), pp. 1–61.

[2] See, e.g., Kryloff, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 346–350. Kryloff and Bogoliouboff, Bulletin of the Academy of Sciences, U.S.S.R. 1929, p. 471. Also Picone, Rendiconti del Circolo Matematico di Palermo, vol. 52 (1928), pp. 225–253.

[3] W. H. McEwen's *Thesis* (Unpublished).

[4] See for example Bogoliouboff and Kryloff, Bollettino dell' Unione Matematica Italiana, April, 1928.

[5] Cf. Birkhoff, Transactions of the American Mathematical Society, vol. 10, (1909), p. 264.

[6] Milne, Physical Review, vol. 35 (1930), pp. 863–867.

The constant c is arbitrary and may be chosen so as to make the calculation as simple as possible. We define $w(x)$ by the equation

$$w(x) = [u_1^2(x) + u_2^2(x)]^{1/2}$$

and easily show[7] that $w(x)$ satisfies the equation

$$(d^2w/dx^2) + G(x, \lambda)w = c^2w^{-3}, \tag{5}$$

with the initial values $w(a)=1$, $w'(a)=0$. The general solution of (1) can now be expressed by the formula

$$u(x) = Cw(x) \sin [\phi(x) - \theta_1], \tag{6}$$

in which

$$\phi(x) = c\int_a^x w^{-2}dx$$

and C and θ_1 are arbitrary constants. There is no loss in generality in setting $C=1$. To determine θ_1, we substitute $u(x)$ from (6) into (2) and obtain

$$\theta_1 = \tan^{-1}(cg/f).$$

In order to satisfy condition (3) we let

$$y(x) = hu(x) + ku'(x). \tag{7}$$

After substitution from (6) into (7), and after some transformations, we obtain

$$y(x) = U(x) \sin [\psi(x) - \theta_2], \tag{8}$$

in which

$$U^2(x) = [hw(x) + kw'(x)]^2 + k^2c^2w^{-2}(x),$$

$$\psi(x) = c\int_a^x [h^2 + k^2G(x, \eta)]U^{-2}(x)dx, \tag{9}$$

and

$$\theta_2 = \tan^{-1}(ck/h) - \tan^{-1}(cg/f).$$

It is clear from (7) and (8) that the necessary and sufficient condition that λ be a characteristic number is that

$$\psi(b) = \theta_2 + n\pi \quad (n \text{ an integer}). \tag{10}$$

The quantity $\psi(b)$ is a function of λ only, and we now establish the important fact that if f, g, h, k and c are independent of λ, the function $\psi(b)$ is a continuously increasing function for all values of λ. To prove this we first note that $\psi(b)$ is independent of f and g. Now choose any value of λ, $\lambda=l$. Corresponding

[7] Milne, loc. cit.

to $\lambda = l$ there exists a solution $u(x)$ of (1) which satisfies (3). Then f and g may be determined so that this solution $u(x)$ of (1) also satisfies (2). Next let $x_0 < b$ be a value of x which varies with λ so that

$$hu(x_0) + ku'(x_0) = 0 \tag{11}$$

and such that, as x_0 approaches b, λ approaches l. Then by differentiation and substitution we obtain from (11),

$$dx_0/d\lambda = -[u'(\partial u/\partial\lambda) - u(\partial u'/\partial\lambda)][u^2 + u'^2]^{-1}[h^2 + k^2G(x_0, \lambda)]^{-1}. \tag{12}$$

We have also the equation [obtained from (1) by differentiating with respect to λ, and integrating with respect to x, after eliminating $G(x, \lambda)$ $(u\partial u/\partial x)$]

$$u'(\partial u/\partial\lambda) - u(\partial u'/\partial\lambda) = \int_a^x (\partial G/\partial\lambda)u^2 dx, \tag{13}$$

and finally, since $\psi(x_0) = \theta_1 + n\pi$, we have

$$(\partial\psi/\partial\lambda) + (\partial\psi/\partial x_0)(dx_0/d\lambda) = 0. \tag{14}$$

From (12), (13) and (14), together with (9), it follows that

$$\partial\psi/\partial\lambda = cU^{-1}[u^2 + u'^2]^{-1}\int_a^x [\partial G/\partial\lambda]u^2 dx,$$

which is always positive. This holds in particular when $x_0 = b$, $\lambda = l$, and therefore establishes the fact that $\psi(b)$ is an increasing function of λ.

2. We now show how the foregoing results enable us to determine characteristic numbers by numerical integration. We select k values of λ, namely $\lambda^{(1)}, \lambda^{(2)}$ $\lambda^{(3)}, \cdots, \lambda^{(k)}$ (usually equally spaced for convenience) chosen so that the desired characteristic number lies somewhere between $\lambda^{(1)}$ and $\lambda^{(k)}$. For each of these values of λ we solve equation (5) by numerical integration. Then using in turn each of the solutions of (5) now obtained we substitute in the formula

$$N = (1/\pi)\left\{c\int_a^b [h^2 + k^2G(x, \lambda)][(hw + kw')^2 + k^2c^2w^{-2}]^{-1}dx - \theta_2\right\} \tag{15}$$

and determine N for each by numerical quadrature, obtaining k values $N_1 < N_2 < \cdots < N_k$. If a characteristic value of λ lies between $\lambda^{(1)}$ and $\lambda^{(k)}$, there will be an integer $N = n$ between N_1 and N_k. The corresponding value of λ can be determined by inverse interpolation and will be one of the desired characteristic numbers. It is clear that θ_2 can be determined so that the characteristic number corresponding to $N = n$ will be precisely λ_n.

3. For certain special types of boundary conditions the integral (13) assumes particularly simple forms; for example in the case of the conditions

$$fu(a) + gu'(a) = 0, \quad u(b) = 0,$$

we have

$$N = (1/\pi)\left\{ c\int_a^b w^{-2}dx - \theta_2 \right\},$$

and similarly of course for the conditions

$$u(a) = 0, \quad fu(b) + gu'(b) = 0.$$

Likewise for the conditions

$$fu(a) + gu'(a) = 0, \; u'(b) = 0$$

we have

$$N = (1/\pi)\left\{ c\int_a^b G(x, \lambda)\left[w'^2 + c^2w^{-2}\right]^{-1}dx - \theta_2 \right\}.$$

4. It is evident from the definition of $w(x)$ that the value of N in (13) can be expressed in terms of the two solutions of (1), $u_1(x)$, $u_2(x)$, just as well as in terms of $w(x)$. Therefore one might obtain $u_1(x)$ and $u_2(x)$ by the process of numerical integration applied to equation (1), and then proceed as before. However, several of trial computations indicate that to obtain $w(x)$ from (5) rather than $u_1(x)$ and $u_2(x)$ from (1) results in a notable saving of time and labor.

A DISCUSSION OF A DIFFERENTIAL EQUATION

By CHARLES E. WILDER, Dartmouth College

Frequently the discussion of the families of line elements determined by a differential equation gives more information about the solution of a problem than does the formal solution of the differential equation. The following problem, proposed to Professor B. H. Brown by some doubtless well-meaning freshman, is an excellent example.

The Problem

An automobile moves along a straight road with a constant speed (v) while a man in a field beside the road walks with a constant speed (u) along such a path as to always keep a tree between him and the automobile. Determine his path.

The Discussion

First let us note that there is an obvious particular solution, namely, a straight line parallel to the road and on the opposite side of the tree at a distance from the tree equal to ua/v, where a is the distance from the tree to the road. The proof of this is a mere matter of similar triangles.

We shall set up the problem in both polar and rectangular coordinates. In Fig. 1.

O is the tree,

Q is the automobile at the instant (t),

P is the man at the same instant,

and the remainder of the notation is as shown in the figure. The time t is measured from the time of crossing the Y axis. From the figure we conclude that

$$\tan\theta = -vta^{-1} \text{ or } xy^{-1} = -vta^{-1},$$

and from this we obtain by differentiation:

(1) $$\sec^2\theta d\theta = -va^{-1}dt \quad \text{or} \quad (ydx - xdy)y^{-2} = -va^{-1}dt.$$

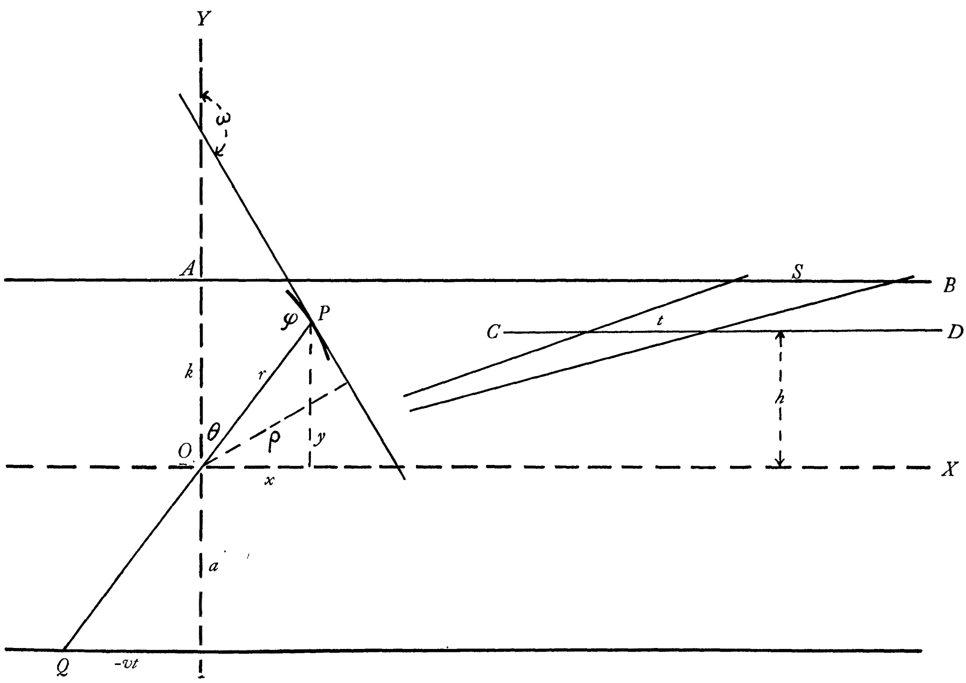


FIG. 1

If s is the length of arc along the path, $ds = udt$, and the substitution of dt from (1) gives

$$ds = -uav^{-1}\sec^2\theta d\theta \text{ or } ds = -uav^{-1}(ydx - xdy)y^{-2}.$$

Finally the appropriate formula for ds is used and the constant $-ua/v$ is replaced by k. This yields as the differential equation of the path

(2) $$(dr/d\theta)^2 + r^2 = k^2\sec^4\theta \text{ or } (y^4 - k^2x^2)(dy/dx)^2 + 2k^2xy(dy/dx) + y^2(y^2 - k^2) = 0.$$

The particular solution already noted may be written

$$r = k\sec\theta \quad \text{or} \quad y = k.$$

We should note that the differential equation does not require the tree to be between the man and the automobile.

The discussion will follow essentially the same method that is used in curve plotting. First we note that changing x into $-x$, or y into $-y$ does not change the equation. Hence if we can determine the solutions or parts of solutions in the first quadrant, the solutions in the remainder of the plane can be obtained by reflections in the axes. We do not want, however, any solutions below the axis of x, since these would not have the tree between the man and the automobile. Next we note that since the derivative is given in either system of coordinates by an expression containing a radical, there will be points of the plane through which there are no solutions and through such points as there are solutions there will be just two in general. If in the differential equation we set the radical equal to zero we obtain the equation of the curve, which we shall call the bounding curve, which bounds that part of the plane in which solutions lie. This curve is

$$r^2 = k^2 \sec^4 \theta \text{ or } y^4 = k^2(x^2 + y^2).$$

It passes through the point $(0, k)$ and has points of inflection at

$$x = \pm \tfrac{1}{2}k\sqrt{3}, \quad y = \tfrac{1}{2}k\sqrt{6}.$$

If we use the ordinary methods of polar coordinates, the angle ϕ shown in Fig. 1 is given by

$$\cot \phi = r^{-1}dr/d\theta,$$

and from the differential equation we have

$$\cot^2 \phi = (k^2 \sec^4 \theta - r^2)r^{-2}, \tag{4}$$

which, being interpreted, means that the two solutions through any point make equal angles with the radius vector drawn to that point. In particular at points of the bounding curve both solutions are perpendicular to the radius vector while at the origin (the tree) there are two solutions for each direction of approach.

Since the two line elements at any point make equal angles with the radius vector their lines must both be tangent to the same circle about the origin. The radius ρ of this circle is the perpendicular distance from the origin to the line determined by the line element, and again referring to Fig. 1 we have

$$\rho = r \sin \phi;$$

and using the above expression for $\cot \phi$ we get

$$\rho^2 = r^4k^{-2} \cos^4 \theta = y^4k^{-2}.$$

Thus we have the following construction for line elements. If P with coordinates (x, y) is any point, construct the circle with center at the origin and radius

$y^2/|k|$; then the line elements through P have the directions of the tangents to this circle from P. In particular the line elements along any horizontal are all tangent to the same circle.

We next inquire, where may solutions have horizontal or vertical tangents? This of course could be answered directly from the differential equation, but it is now easier to make use of the formula

$$\rho^2 = y^4k^{-2}.$$

For if a line element is horizontal, $\rho^2 = y^2$; and we have

$$y^4 = k^2y^2.$$

This gives us the axis of x which we can rule out at once, and the particular solution. We then conclude that no other solution except one through $(0, k)$ can have a horizontal tangent, for at no other point of the particular solution can the other solution as given by our construction be horizontal. If a line element is vertical, $\rho^2 = x^2$, and we have

$$y^4 = k^2x^2,$$

which represents two parabolas through the origin whose axes are along the axis of x. At all points of these parabolas one of the two line elements is vertical.

After horizontal and vertical tangents come points of inflection. These we shall obtain from the polar coordinate condition which is

$$r^2 - r^2(d^2r/d\theta^2) + 2(dr/d\theta)^2 = 0.$$

When the derivative and second derivative are found from the differential equation and substituted in this, after some reduction we obtain

$$r^2 = k^2 \sec^2 \theta,$$

which gives us only the particular solution. This time we can conclude, after due consideration of our method of construction for such line elements as lie near the particular solution, that no other solution can have a point of inflection.

We might note now that solutions can cross each other at right angles only when $\cot \phi = \pm 1$, which by using (4) gives

$$2r^2 = k^2 \sec^4 \theta \quad \text{or} \quad 2y^4 = k^2(x^2 + y^2),$$

a curve similar to the bounding curve. At each point of this curve the two line elements are perpendicular to each other.

If we desire to go still farther we can find the curvature of a solution at any point. To do this it is best to turn to the definition of curvature. Curvature is

$$C = d\omega/ds,$$

where, from Fig. 1, $\omega = \phi + \theta$. Hence

$$C = (d\phi/ds) + (d\theta/ds) = [1 + (d\phi/d\theta)](d\theta/ds).$$

Now from (4)

$$r^2 \csc^2 \phi = k^2 \sec^4 \theta,$$

and from this we derive by differentiation

$$(d\phi/d\theta) = 1 - 2 \tan \theta \tan \phi,$$

which is merely another form of our original differential equation. We already have, in the deduction of the differential equation,

$$(ds/d\theta) = k \sec^2 \theta,$$

and so we can write the curvature

$$C = 2(1 - \tan \theta \tan \phi) \cos^2 \theta / k.$$

In particular when $\theta = 0$, $C = 2/k$, so all solutions cross the axis of y with the same curvature which is that of the parabolas, $y^4 = k^2x^2$ at the origin.

From the construction of line elements it is clear that there are solutions which remain below the particular solution as x increases. As we go out along such a solution the slope is always decreasing, as is easily seen from the method of construction of line elements. Hence the solution is asymptotic to the particular solution or else to a horizontal line below it. We can disprove the latter possibility by an argument suggested by Professor B. H. Brown.

In Fig. 1, AB is the particular solution and CD any other horizontal below it. Draw two radii from O and let the segments cut off by these on AB and CD be respectively s and t. Let h be the height of the line CD above the axis of x. Then

$$ts^{-1} = hk^{-1} < 1.$$

Now assume a solution asymptotic to CD. Then as we go out along the solution in the direction of increasing x the length of the segment cut on the solution by two near-by radii approaches the length cut by them from CD. That is, the ratio of the length cut from the solution to the length cut from AB approaches a number less than unity. But from the statement of our problem the lengths of the segments cut from any two solutions by two radii must always be equal. Hence there is no solution asymptotic to CD.

Similarly we can show that there is a family of solutions asymptotic to the particular solution from above.

While we are discussing asymptotic properties let us prove that our parabolas $y^4 = k^2x^2$ are asymptotic to the bounding curve. Let (x_0, y_0) be a point on the bounding curve and (x_0, y_1) be on the parabola. We have

$$y_0^4 - k^2y_0^2 = k^2x_0^2$$

and

$$\begin{array}{rcl} \underline{y_1^4 \qquad\qquad} & \underline{=} & \underline{k^2x_0^2\,;} \\ y_0^4 - y_1^4 & = & k^2y_0^2\,; \end{array}$$

and from this

$$y_0 - y_1 = k^2 y_0^2 (y_0^2 + y_1^2)^{-1}(y_0 + y_1)^{-1},$$

which shows that $y_0 - y_1$ approaches zero as we go out along the curve.

We are now ready to collect our various bits of information and draw our conclusions. In Fig. 2 the auxiliary curves are in full lines and solutions in dotted lines. At points of the bounding curve (a) line elements are constructed by means of the fact that they are perpendicular to the radius vector drawn to the point. Line elements at points of the particular solution (e) are on the tangents from the points to the circle about the origin of radius k. Any other

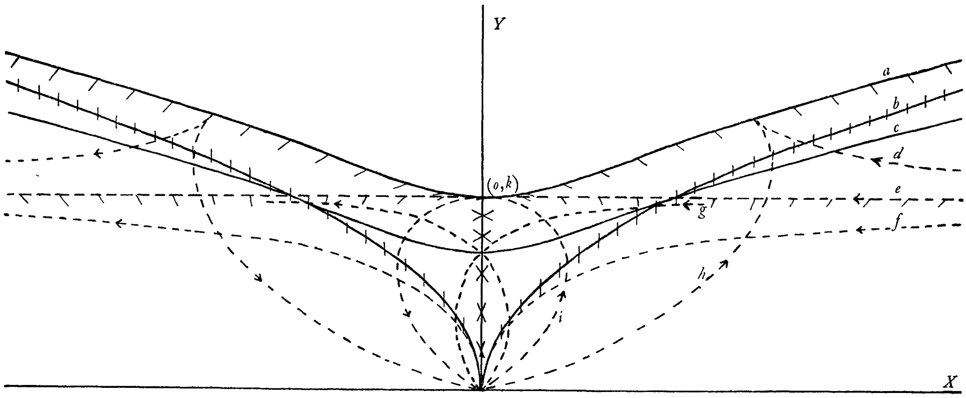


FIG. 2

a. boundary of region in which solutions lie.
b. parabola along which one family of line elements are vertical.
c. curve along which solutions cut at right angles.
d. sample solution asymptotic to particular solution from above.
e. particular solution.
f. discriminating solution.
g. sample solution asymptotic to particular solution from below.
h. solution of finite length.
i. discriminating solution.

If the automobile moves from left to right then the man follows solutions above in the directions shown by the arrow heads.

line elements desired may be drawn by means of our construction method. The fact that solutions do not have horizontal tangents or points of inflection helps us in joining up line elements into solutions.

Through the point (0, k) passes the particular solution and another solution (i) which also has there the slope zero. This solution (i) may be approximately constructed by means of successive line elements or by numerical integration. If forms a closed curve as shown, entering the origin with slopes we shall call $\pm\lambda$. Through the origin are two solutions which are tangent to the axis of y. These also are constructed by line elements. In the figure, (f) is the one in the first quadrant.

Starting from the origin with any given slope are two solutions one of which curves up and the other down. If the given slope is less than λ that solution that curves down remains below (f) but is roughly similar to it and, like it, is asymptotic to the particular solution. On the other hand the solution that curves up has its slope constantly increasing till it crosses the parabola (b) vertically. It then proceeds to the bounding curve with a negative slope. An example of this type of solution is (h). If we trace the other solution (d) which strikes the bounding curve at the same point we find it is asymptotic to the particular solution from above. When we choose a slope at the origin greater than λ the solution that curves downward has the same character as before, but the other solution remains in the first quadrant within the solution (i), crosses the parabola vertically and then crosses the axis of y and finally goes off to the left asymptotic to the particular solution from below. These curves and their reflections in the y axis are the only solutions that cross this axis. The solution (g) and its reflection is the particular case in which the two line elements at the point in which they cross the axis of y are orthogonal. It is not difficult to see that all the line elements defined by the differential equation are used up by the various types of solutions we have considered so that our discussion is complete.

To connect the solutions of the differential equation thus found with our original problem we assume that the automobile moves along the road, which is of course, somewhere below the axis of x and parallel to it, from left to right. By means of lines through O (the tree) we can then determine the direction in which the man must move along the various solutions. These directions are shown in the figure by arrow heads. We now demand that the man make no abrupt changes in direction. If then the man starts initially from some point on the particular solution he can follow it indefinitely. If he starts in the first quadrant below (e) and outside of (i) he may go to the bounding curve where he must give up the game, or else to the tree where his only way out is to make an abrupt change in direction. If he starts between the particular solution and the bounding curve both solutions lead him to the bounding curve. If he starts inside of (i) he may go to the origin or he may continue indefinitely along one of the solutions on the left asymptotic to the particular solution from below. From any point in the second quadrant and below the bounding curve one of his two possible paths is a solution asymptotic to the particular solution and so may be followed indefinitely.

And now we turn back to the differential equation itself to see just what can be done by way of a proper solution. Contrary to all precedent known to us the first actual solution came by means of a contact transformation. Briefly, the parabola $y^2=kx$ seemed intrinsic in the problem and vague memories of some text suggested polar reciprocation in it. Thus we have the contact transformation

$$X = -(px-y)p^{-1}, \quad Y = \tfrac{1}{2}kp^{-1}, \quad P = \tfrac{1}{2}ky^{-1},$$

in which $p=dy/dx$. The use of this transformation on the differential equation yields

$$k^2 - 16P^4X^2 + 4Y^2 = 0$$

in which the variables are separable and the usual method leads to the solution. Incidentally the simpler transformation

$$X = p^{-1}, \quad Y = yp^{-1} - x, \quad P = y,$$

would have worked just as well.

Next we arrived at a solution from the differential equation in form (5):

$$d\phi/d\theta = 1 - 2 \tan\theta \tan\phi.$$

The substitution,

$$z = \tan\phi, \quad \tau = \tan\theta,$$

gives

$$(dz/d\tau) = (1 + z^2)(1 + \tau^2)^{-1}(1 - 2\tau z),$$

which is of the form

$$dz/d\tau = P_0 + P_1 z + P_2 z^2 + P_3 z^3,$$

in which the P's are functions of τ. This form is preserved if we make the further substitution $z=au+b$, in which a and b are functions of τ. We then determine a and b so that the new P_0 and P_1 shall both be identically zero. This gives us

$$a = (1 + \tau^2)\tau^{-4}, \quad b = \tau^{-1},$$

b being a particular solution which we fortunately know. Our new equation is

$$du/d\tau = - 5\tau^{-4}u^2 - 2(1 + \tau^2)\tau^{-7}u^3,$$

and the substitution $u=v^{-1}$ reduces this to what may be called a normal form:

$$v(dv/d\tau) = 5v\tau^{-4} + 2(1 + \tau^2)\tau^{-7}.$$

Equations of this type can sometimes be solved. In this case the further substitution $s=\tau^2 v+\tau^{-1}$ reduces it to

$$d\tau/ds = \tfrac{1}{2}s(s^2 + 1)^{-1}\tau - \tfrac{1}{2}(s^2 + 1)^{-1}$$

which is linear in τ.

Lastly, of course, we found what seems to be the easiest method. Solve the original differential equation for x, thus:

$$x = [ky \pm y^2(1 + p^2)^{1/2}]k^{-1}p^{-1},$$

in which $p=dy/dx$. The sign merely determines the sense of s (the measure-

ment of arc) and we shall take it negative. Differentiate with respect to x and simplify to

$$0 = 2p^2(1 + p^2) - k(dp/dx)(1 + p^2)^{1/2} + y(dp/dx).$$

Since $(dp/dx) = p(dp/dy)$, this becomes

$$2p(1 + p^2)(dy/dp) = k(1 + p^2)^{1/2} - y,$$

or

$$(dy/dp) + \tfrac{1}{2}p^{-1}(1 + p^2)^{-1}y - \tfrac{1}{2}kp^{-1}(1 + p^2)^{-1/2} = 0,$$

and again we have an ordinary linear differential equation.

If the reader will take the trouble to finish up any or all of the methods of solution suggested he will see why the formal solution is a useless form of answer to our given problem. We refrain from writing them out. They would take too much space.

THE PROBABILITY FUNCTION

By N. R. WILSON, University of Manitoba

The determination of the even probability function is a basic mathematical problem for students in statistics and in engineering and other exact sciences, but the mathematics involved places it quite beyond the average of these groups. Though receiving attention since the time of Gauss and Laplace, methods of derivation leave much to be desired in the matter of rigor, and have been severely criticized by an eminent economist.[1] Grouping together Gauss's first proof in his *Theoria Motus* and others in the elementary treatment of statistics making no mathematical pretensions, derivations fall into four classes. The commonest follows the lines of Gauss's second proof, based on the assumption that the "most probable value" is the average.[2] Other assumptions are that the function can be expanded in an infinite Maclaurin series and is always positive, with minor assumptions eight in number.[3] The first assumption has always been questioned. Also while it follows from the definition of probability that the function cannot be negative, experimental evidence certainly does not justify discarding zero values. This is the method especially attacked by Keynes (loc. cit., p. 208); the real criticism, however, being what is read into the misleading term "most probable value." The other two methods refer more particularly to the law of error in measuring lengths. Of these Hagen's[4] is based on the as-

---

[1] J. L. Keynes, *Probability*, pp. 205–214.

[2] Gauss, *Theoria Comb. Obs.*; Merriman, *Least Squares*, p. 22; Leland, *Least Squares*, p. 211; Coolidge, *Probability*, p. 113.

[3] Coolidge, l.c., p. 118.

[4] Hagen, *Grundzüge der Wahrscheinlichkeitsrechnung*, p. 31; Merriman, *Least Squares*, p. 17.

sumption that these errors are the resultant of infinitesimal errors obeying the binomial law. The other, due to Lord Kelvin, appears to have passed unnoticed.[5] This is based on the assumption that, if carried to two dimensions, points located by the measured lengths will lie symmetrically about the true position. The assumption is justified by experience with falling stones; evidence hardly bearing very directly on the measurement of lengths.

Probability is a mathematical concept—subjective, that is to say—and the derivation in question should, so far as possible, be made to depend upon other mathematical concepts rather than upon objective considerations. The method last cited above suggests a method of so doing which materially simplifies the discussion. The additive law for the probability of the alternative occurrence of independent constituent events, and the multiplicative law for the concurrence of such events, are usually regarded as theorems, not as assumptions. Whether such or not, they are taken as true here, and the solution, essentially contained in equation (2) below, rests on them.

A variable t is said to be determined by the even law of probability, $\phi(t^2)$, when, in the usual phrasing of applied mathematics, the probability that t lies between t and $t+dt$ is $\phi(t^2)dt$. The function ϕ is then subject to the condition following from the definition of probability, (i) of being not negative and (ii) of satisfying the relation

$$\int_{-\infty}^{\infty} \phi(t^2)dt = 1. \tag{1}$$

With reference to (i), the discussion is simplified by assuming as is usually done that ϕ is positive. We make this assumption in the body of the text, indicating in brackets how it may be avoided. We add the analytic assumptions that ϕ and its first derivative are continuous.

Consider a continuum of points $P(x, y)$ located in a plane with reference to a pair of rectangular axes, QX and QY; the variables x and y being determined by the laws of probability $\phi(x^2)$ and $\phi(y^2)$, respectively. Consider now the laws of probability for the determination of the coordinates (x', y') of the same set of points with reference to a set of axes, QX' and QY', bisecting the angles between the former pair. From the complete symmetry of the set with reference to the new axes, it follows that the law is even and the same for both axes, say $f(x'^2)$ and $f(y'^2)$.

[Analytically, integrating for the probability, fdx', that a point lies within a strip of width dx', parallel to QY', we have

$$f = \int_{-\infty}^{\infty} \phi(u^2)\phi(v^2)dy',$$

where $u=(x'+y')/\sqrt{2}$, $v=(x'-y')/\sqrt{2}$. By inspection, f is unchanged on

[5] Thomson & Tait, *Natural Philosophy*, Part I, p. 449.

reversing the sign of x, proving its evenness; and becomes the same function of y on interchanging the axes. The existence of this integral follows from equation (1) since neither $\phi(u^2)$ nor $\phi(v^2)$ is negative. It may be added that this is the only place where condition (i) does not appear unnecessary, being elsewhere deducible from equation (1).]

By the multiplicative law, the probability that a point lies in the usual infinitesimal neighborhood of $P(x, y)$ is $\phi(x^2)\,\phi(y^2)dx\,dy$, and in that of $P(x', y')$ is $f(x'^2)\,f(y'^2)dx'\,dy'$. Equating the probabilities per unit area that it lies in the neighborhood of $P(x, y; x', y')$, we have

$$\phi(x^2)\phi(y^2) = f(x'^2)f(y'^2).$$

Putting $y=0$ in equation (2), so that $x'=-y'=x/\sqrt{2}$ we have

$$a\phi(x^2) = \{f(x^2/2)\}^2, \tag{3}$$

where $a=\phi(0)$; whence $A=f(0)$.

[To show that $a=\phi(0)$ is positive, we cannot have $f(0)=0$. For if so, putting $x'=0$ in (2), so that $y=-x$, we have $\phi(x^2)=0$ for all values of x, contradicting equation (1). Since from (2) $\phi(0)=\pm f(0)$, we have $\phi(0)\neq 0$. From (i), ϕ cannot be negative. Hence $\phi(0)$ is positive. For a similar reason the relation $\phi(0)=\pm f(0)$, leads to $f(0)=a$.]

Reversing the above process—i.e. starting with the continuum $P(x', y')$, where x' and y' are determined by the laws of probability $f(x'^2)$ and $f(y'^2)$, and examining the laws along OX and OY—we obtain a relation of the form

$$af(x'^2) = \{g(x''^2/2)\}^2,$$

where $g(x''^2)$ gives the law along OX, x'' denoting temporarily the coordinate along this axis. Since $x''=(x'+y')/\sqrt{2}=x$, the arguments in the deduced law $g(x''^2)$ along OX and the original law $\phi(x^2)$ are the same. Hence g and ϕ are identical, or

$$af(x'^2) = \{\phi(x'^2/2)\}^2. \tag{4}$$

Eliminating f between (3) and (4),

$$a^3\phi(x^2) = \{\phi(x^2/4)\}^4.$$

Replacing x by $x/2^n$, where $n=1, 2, 3, \cdots$, and eliminating intermediate values of the arguments, we have, on writing m for 4^n,

$$a^{m-1}\phi(x^2) = \{\phi(x^2/m)\}^m. \tag{5}$$

Writing $x^2=u$, $\log\phi(u)=\psi(u)$, and differentiating,

$$\psi'(u) = \psi'(u/m).$$

[To legitimize these steps when zero values of ϕ are not excluded by hypothesis, suppose that $\phi(x^2)=0$ for some value of x. Then by (5) $\phi(x^2/m)=0$ for

all positive integral values of n, m being 4^n. Since ϕ is continuous, $\phi(0)=\lim \phi(x^2/m)$ as $n\to\infty$ and therefore $m\to\infty$; or $\phi(0)=0$, contradicting the second paragraph (which is in brackets.]

Since $\phi'(u)$ is continuous and $\phi(u)$ positive, $\psi'(u)$ is continuous. Hence, on taking the limit of (6) as $n\to\infty$ and therefore $m\to\infty$, we have $\psi'(u)=\psi'(0)$, or

$$\phi'(u)/\phi(u) = \psi'(0). \tag{7}$$

Since $\phi(x^2)$ is positive for all values of x^2, the existence of the integral in equation (1) requires that ϕ decrease for some value of x^2, and therefore that $\phi'(u)$ should be negative or zero for the same value of u. Rejecting the zero value as it leads to a constant value of ϕ, we may write $-h^2$ for $\psi'(0)$ in (7). Integrating this, $\phi(x^2)$ is of the form $ke^{-h^2x^2}$. Substituting this value of ϕ in the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x^2)\phi(y^2)dxdy = 1,$$

deducible on inspection from equation (1), and expressing the area integral in terms of polar elements of area, we have

$$2\pi k^2\int_0^{\infty} e^{-h^2r^2}rdr = 1, \text{ where } r^2 = x^2 + y^2;$$

whence $k=h/\sqrt{\pi}$; or $\phi(x^2)=he^{-h^2x^2}/\sqrt{\pi}$, the form desired.

If we add the usual assumption that $\phi(x^2)$—and therefore $\log\phi(x^2)$—is a power series in x^2, the solution may be simplified, $\log\phi(x^2)$ being derived from the equation immediately preceding equation (5) by undetermined coefficients.

ON THE INVARIANCE OF THE DIVERGENCE OF A VECTOR FUNCTION

By REED LAWLOR, Los Angeles, California

Let $F(r)=F_1(x, y, z)i+F_2(x, y, z)j+F_3(x, y, z)k$ be a vector function of the vector $r=xi+yj+zk$, where i, j, and k are the unit vectors for a right-hand, orthogonal, Cartesian system of axes and where ∇F_1, ∇F_2, and ∇F_3 exist at each point of a domain D of three space. It is well-known that $\nabla\cdot F$, the divergence of F, has the form $(\partial F_1/\partial x)+(\partial F_2/\partial y)+(\partial F_3/\partial z)$ and that its value does not depend upon the choice of axes. The present note shows that $\nabla\cdot F$ has the above form when any three non-coplanar unit vectors i, j, and k are used. This seems of importance when one realizes that it is often desirable to use an oblique system of axes (for example, when a crystal is studied and the coordinate axes are chosen to coincide with those of the crystal) and it is very convenient to know that the divergence of a vector function has the same analytic form in

this system as in an orthogonal system. The curl of F does not have this property, neither does the gradient of a scalar function have the property.

Let a, b, and c be any three unit vectors such that $[abc]=a\cdot(b\times c)\neq 0$ and let i, j, and k be the unit vectors of a right-hand orthogonal system. Let $r=xi+yj+zk=ua+vb+wc$ be the vector from a common origin of both of the above systems to any point of D. Let

$$F(r)=F_1(x,y,z)i+F_2(x,y,z)j+F_3(x,y,z)k=f_1(u,v,w)a+f_2(u,v,w)b+f_3(u,v,w)c$$

be a vector function such that ∇F_1, ∇F_2, ∇F_3 exist at each point of D.

Theorem: $\nabla\cdot F=(\partial F_1/\partial x)+(\partial F_2/\partial y)+(\partial F_3/\partial z)=(\partial f_1/\partial u)+(\partial f_2/\partial v)+(\partial f_3/\partial w)$ at each point of D.

Proof:

$$\begin{aligned}(\partial F_1/\partial x) &= \nabla F_1\cdot i = i\cdot\nabla(F\cdot i) = i\cdot\nabla[f_1(a\cdot i)+f_2(b\cdot i)+f_3(c\cdot i)] \\ &= \nabla f_1\cdot(a\cdot i)i+\nabla f_2\cdot(b\cdot i)i+\nabla f_3\cdot(c\cdot i)i.\end{aligned}\tag{1}$$

Similarly[1]

$$(\partial F_2/\partial y) = \nabla f_1\cdot(a\cdot j)j+\nabla f_2\cdot(b\cdot j)j+\nabla f_3\cdot(c\cdot j)j,\tag{2}$$

$$(\partial F_3/\partial z) = \nabla f_1\cdot(a\cdot k)k+\nabla f_2\cdot(b\cdot k)k+\nabla f_3\cdot(c\cdot k)k.\tag{3}$$

Combining (1), (2), and (3), we obtain

$$\begin{aligned}\nabla\cdot F &= \nabla f_1\cdot[(a\cdot i)i+(a\cdot j)j+(a\cdot k)k]+\nabla f_2\cdot[(b\cdot i)i+(b\cdot j)j+(b\cdot k)k] \\ &\quad+\nabla f_3\cdot[(c\cdot i)i+(c\cdot j)j+(c\cdot k)k] \\ &= a\cdot\nabla f_1+b\cdot\nabla f_2+c\cdot\nabla f_3 = (\partial f_1/\partial u)+(\partial f_2/\partial v)+(\partial f_3/\partial w),\end{aligned}$$

since $a\cdot\nabla f_1=(\partial f_1/\partial u)$, $b\cdot\nabla f_2=(\partial f_2/\partial v)$, $c\cdot\nabla f_3=(\partial f_3/\partial w)$. This completes the proof of the theorem and shows that $\nabla\cdot F=(\partial F_1/\partial x)+(\partial F_2/\partial y)+(\partial F_3/\partial z)$ for any choice of axes of x, y, and z so long as they are non-coplanar, where F_1, F_2, and F_3 are the respective components of F parallel to these axes.

---

[1] A proof of the existence of ∇f_1 is as follows:

$$f_1=[Fbc]/[abc]=F_1[ibc]/[abc]+F_2[jbc]/[abc]+F_3[kbc]/[abc].$$

Hence ∇f_1 exists and

$$\nabla f_1=\nabla F_1[ibc]/[abc]+\nabla F_2[jbc]/[abc]+\nabla F_3[kbc]/[abc].$$

Similar proofs show the existence of ∇f_2 and ∇f_3.

NEW BOUNDS FOR THE ROOTS OF AN ALGEBRAIC EQUATION[1]

By E. C. WESTERFIELD, University of Colorado

§1.

From Descartes' rule of signs we know that the real equation

$$x^n = \sum_1^n p_r x^{n-r}, \quad p_r \geqq 0, \quad p_n > 0, \tag{1}$$

has exactly one positive root[2] and, since the left member of this equation is of higher degree than the right member, we have

LEMMA 1: *Any value of x satisfying the inequality*

$$x^n \leqq \sum_1^n p_r x^{n-r} \tag{2}$$

will form a lower bound for the positive root of equation (1).

LEMMA 2: *Any value of x satisfying the inequality*

$$x^n \geqq \sum_1^n p_r x^{n-r} \tag{3}$$

will form an upper bound for the positive root of (1).

Since every root of the equation

$$z^n + \sum_1^n b_r z^{n-r} = 0, \qquad b_r \text{ complex}, \tag{4}$$

must satisfy inequality (2) when (2) and (4) are related thru the equalities

$$x = |z|, \quad p_r = |b_r|, \qquad r = 1, \cdots, n; \tag{5}$$

it follows from Lemma 1 that the modulus of any root of (4) will form a lower bound for the positive root of equation (1). Stated differently we have[3]

LEMMA 3: *When equations* (5) *are satisfied the sole positive root of* (1) *will form an upper bound for the modulus of any root of* (4).

We now employ a simple identity to obtain a few special results. We write

$$(h+k)^n \equiv h(h+k)^{n-1} + k(h+k)^{n-1} \tag{6}$$

$$\equiv h(h+k)^{n-1} + hk(h+k)^{n-2} + k^2(h+k)^{n-2} \tag{7}$$

$$\equiv k(h+k)^{n-1} + hk(h+k)^{n-2} + h^2(h+k)^{n-2}. \tag{8}$$

---

[1] Abstract from a thesis presented for the A. M. degree at the University of Colorado, June, 1930.

[2] G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis I, (1925), p. 87; Oscar Perron, Algebra II, Theorie der algebraischen Gleichungen, (1927), p. 20.

[3] See reference 2.

Continuing this process, we may choose either of the terms in (7) or (8) which contain the explicit expression $(h+k)^{n-2}$ and break it up into two terms of degree $(n-3)$ in $(h+k)$. Eventually the process will end with two terms of degree unity. Since at each step after the first, we must choose between two terms, it is evident that 2^{n-1} such expansions are possible. Half of these expansions may be obtained from the other half, however, by the interchange of h and k; and, since h and k enter symmetrically in the left member of (6), it follows that only 2^{n-2} of these expansions are distinct. These expansions will be of the general type

$$(9) \qquad (h+k)^n \equiv \sum_1^n P_r(h+k)^{n-r}, \qquad P_r \equiv P_r(h, k),$$

where the functions $P_r(h, k)$ are special polynomials determined in the manner indicated by (6), (7), and (8). Since such an expansion for $(h+k)^n$ is identical with the left member of (1) when $x=h+k$, it follows that when h and k are positive quantities so chosen that the coefficients of the right member of (9) are not less than the corresponding coefficients of the right member of (1),— then the value $x=h+k$ will satisfy (3) and, applying Lemma 2, will form an upper bound for the positive root of (1). Applying Lemma 3, this gives us

Theorem: *Every root of* (4) *must satisfy the inequality*

$$(10) \qquad |z| \leqq h+k,$$

where k and h are two arbitrary positive quantities so chosen that they satisfy each of the inequalities

$$(11) \qquad |b_r| \leqq P_r(h, k), \qquad r = 1, \cdots, n,$$

the n functions $P_r(h, k)$ being determined from one of the 2^{n-1} expansions of type (9).

One expansion of type (9) gives us

$$P_r = hk^{r-1}, \qquad r = 1, \cdots, n-1,$$
$$P_n = hk^{n-1} + k^n.$$

We see that $h=|b_1|$ and $k=\mathrm{Max}\,|b_{r+1}:b_1|^{1/r}$, $r=1, \cdots, n-1$, satisfy (11); and applying the above theorem we find that every root of (4) must satisfy the inequality[4]

$$(a) \qquad |z| \leqq |b_1| + \mathrm{Max}\,|b_{r+1}: b_1|^{1/r}, \qquad r = 1, \cdots, n-1.$$

Taking account of both terms in the expression for P_n, one obtains the more exact but much more cumbersome expression

$$(b) \qquad |z| \leqq |b_1| + \mathrm{Max}\left\{ |b_{r+1}: b_1|^{1/r},\ [|b_n|: 2]^{1/n},\ |b_n: 2b_1|^{1/(n-1)} \right\}, \qquad r = 1, \cdots, n-2.$$

[4] The fractional exponent, $1/r$, is used throughout to designate the positive rth root.

If, on the other hand, one takes $k=1$ and $h=\text{Max}\,(|b_r|,\ |b_n|-1)$, $r=1,\cdots,n-1$, (11) is again satisfied and one obtains

$$(c)\qquad |z| \leqq \text{Max}\,(1+|b_r|,\ |b_n|), \qquad r=1,\cdots,$$

These three simple expressions for the bounds are also obtained as corollaries to some theorems due to Kojima[5].

Another of these expansions gives us

$$\begin{aligned} P_1 &= h, \\ P_r &= k^2h^{r-2}, \qquad\qquad r=2,\cdots,n-1, \\ P_n &= k^2h^{n-2}+kh^{n-1}. \end{aligned}$$

Applying the theorem as before, we may obtain the bound

$$(d)\qquad |z| \leqq |b_1| + \text{Max}\ |b_r : b_1^{r-2}|^{1/2}, \qquad r=2,\cdots,n,$$

while, as before, a more exact but much more cumbersome expression may be obtained by taking into consideration the second term in the expression for P_n.

Still another expansion of type (9) gives us

$$\begin{aligned} P_r &= hk^{r-1}, \qquad\qquad r=1,\cdots,m-1, \\ P_m &= k^m, \\ P_r &= h^2k^{r-2}, \qquad\qquad r=m+1,\cdots,n-1, \\ P_n &= h^2k^{n-2}+hk^{n-1}, \end{aligned}$$

where m may be any integer between 0 and n exclusive. When $m=n$ we have $P_m=P_n=k^n+hk^{n-1}>k^n$. For simplicity of notation we will now introduce the notation $q_r^r \equiv |b_r|$, q_r being positive. If, now, we take m to be the index of the greatest of the terms q_r and take k equal to this greatest term and h equal to the second greatest term, we see that (11) is satisfied. This gives us the simple bound

$$(e)\qquad |z| \leqq \text{Max}\,(q_r+q_s), \qquad r,s=1,\cdots,n,\ r\neq s.$$

Maintaining m as the index of the greatest of the terms q_r, we may again take $k=q_m$ and obtain the closer bound

$$(f)\qquad |z| \leqq q_m + \text{Max}\,(|b_r| : q_m^{r-1},\ [|b_s| : q_m^{s-2}]^{1/2}),$$
$$r=1,\cdots,m-1,\quad s=m+1,\cdots,n.$$

These last two expressions may be compared with a simple bound

$$(g)\qquad |z| < 2\,\text{Max}\,(q_r), \qquad r=1,\cdots,n,$$

---

[5] Tôhoku Mathematical Journal, vol. 5 (1914), p. 58.

obtained as corollary to a general theorem due to Fujiwara,[6] and also with a related bound obtained by Carmichael[7]

(h) $$|z| \leqq \sum_{1}^{n} q_r.$$

Table 1 gives some numerical upper bounds obtained by applying the eight formulae above to the three following polynomial equations:

(A) $$z^8 + 246z^7 - 305z^6 - 321z^5 + 355z^4 - 400z^3 + 420z^2 + 446z + 450 = 0,$$

(B) $$z^8 - 2z^7 + 38z^6 + 32z^5 - 160{,}000z^4 + 35{,}000z^3 - 4{,}200z^2 - 200z + 120 = 0,$$

(C) $$z^8 + z^7 + 10z^6 - 10z^5 + 700z^4 + 300z^3 + 100z^2 + 20{,}000z - 100{,}000{,}000 = 0.$$

TABLE 1

	(A)	(B)	(C)
(a)	247.3	45.1	14.9
(b)	247.3	45.1	13.6
(c)	447.0	160,001.0	100,000,000.0
(d)	263.5	202.0	25,001.0
(e)	263.5	28.1	15.2
(f)	263.5	22.1	11.0
(g)	492.0	40.0	20.0
(h)	285.3	47.4	30.9

§2.

Remembering that in equation (4) we have $b_0 = 1$, we may write a bound due to Carmichael and Mason[8] in the convenient form

(i) $$|z| < \left[\sum_{0}^{n} | b_r^2 | \right]^{1/2}.$$

If we multiply (4) thru by $z - a$ (where a is an arbitrary complex quantity), we see that any bound for the roots of the resulting equation will also bound the roots of (4). Since, as with (4), the coefficient of the highest power term is unity, we have the resulting bound

[6] Tôhoku Mathematical Journal, vol. 10 (1916), pp. 167–171.

[7] Bulletin of the American Mathematical Society, vol. 24 (1917–18), pp. 286–296.

[8] Bulletin of the American Mathematical Society, vol. 21 (1914), pp. 14–22; also, M. Kuniyeda, *Note on the roots of algebraic equations*, Tôhoku Mathematical journal, vol. 9 (1916), pp. 167–173; vol. 10 (1916), pp. 187–188; M. Fujiwara, *Ueber die Wurzeln der algebraischen Gleichungen*, Tôhoku Mathematical Journal, vol. 8 (1915), pp. 78–85; S. B. Kelleher, *Des limites des zéros d'un polynome*, Journal de Mathématiques, (7), vol. 2 (1916), pp. 169–171; and reference 2.

$$(12)\qquad |z| < \left[\sum_{0}^{n+1} |b_r - ab_{r-1}|^2\right]^{1/2}, \qquad b_{n+1} = b_{-1} = 0.$$

Taking $a=1$ gives us a special result due to Williams[9]

$$(\mathrm{j})\qquad |z| < \left[\sum_{0}^{n+1} |b_r - b_{r-1}|^2\right]^{1/2}, \qquad b_{n+1} = b_{-1} = 0.$$

We now seek to determine the arbitrary quantity a so as to make the bound determined through (12) a minimum. To this end we write

$$\begin{aligned}\sum_{0}^{n+1} |b_r - ab_{r-1}|^2 &= \sum_{0}^{n+1}(b_r - ab_{r-1})(\bar{b}_r - \overline{ab}_{r-1})\\ &= (1+|a|^2)\sum_{0}^{n}|b_r^2| - 2\sum_{1}^{n}R(b_r\overline{ab}_{r-1})\\ &= (1+|a|^2)A - 2R\left(\bar{a}\sum_{1}^{n} b_r\bar{b}_{r-1}\right), \qquad A \equiv \sum_{0}^{n}|b_r^2|,\\ &= (1+\rho^2)A - 2\rho B\cos(\theta-\phi), \qquad \rho, B \geqq 0,\end{aligned}$$

where $\bar{a}$ denotes the conjugate of a, $R(\cdots)$ denotes the real part of the expression in the brackets, $\rho e^{i\phi} \equiv a$, and $Be^{i\theta} \equiv \sum_{1}^{n} b_r\bar{b}_{r-1}$. Considering this as a function of ρ and examining it for a minimum as ρ varies, we find that $\rho = (B:A)\cos(\theta-\phi)$ provides this minimum; and setting this value into the original expression we have

$$\sum_{0}^{n+1}|b_r - ab_{r-1}|^2 = A - (B^2 : A)\cos^2(\theta - \phi).$$

Examining this expression as ϕ varies, we see that $\phi=\theta$ gives us a minimum. The resulting minimum bound is, therefore,

$$(\mathrm{k})\qquad |z| < \left[\sum_{0}^{n}|b_r^2| - \left|\sum_{1}^{n} b_r\bar{b}_{r-1}\right|^2 : \sum_{0}^{n}|b_r^2|\right]^{1/2},$$

and the value of a providing this minimum is given by

$$a = \sum_{1}^{n} b_r\bar{b}_{r-1} : \sum_{0}^{n}|b_r^2|.$$

Table 2 gives the numerical upper bounds obtained by applying (i), (j), and (k) to the following simple polynomial equations:

$$(\mathrm{D})\qquad z^5 + z^4 + z^3 + z^2 + z + 1 = 0,$$

$$(\mathrm{E})\qquad z^5 - z^4 + z^3 - z^2 + z - 1 = 0,$$

$$(\mathrm{F})\qquad z^5 - 4z^4 + 3z^3 - z + 4 = 0.$$

[9] Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 394–6.

TABLE 2

	(D)	(E)	(F)
(i)	2.45−	2.45−	6.56−
(j)	1.41+	4.69+	11.22+
(k)	1.35+	1.35+	5.81−

For a comprehensive bibliography of literature on the location of roots of polynomials the reader is referred to an article by E. B. VanVleck in the Bulletin of the American Mathematical Society vol. 35 (1929), pp. 643–683.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

An Introduction to Mathematics, with Applications to Science and Agriculture. By I. L. Miller. New York, F. S. Crofts & Co., 1930. xiv+298 pages.

Contributions to the History of Determinants, 1900–1920. By Sir Thomas Muir. London, Blackie & Son, 1930. xxiv+408 pages. 30s net.

Theory of Functionals, and of Integral and Integro-Differential Equations. By Vito Volterra. London, Blackie & Son, 1930. xiv+226 pages. 25s net.

Advanced Algebra. By P. H. Graham and F. W. John. New York, Prentice-Hall, 1930. x+258 pages. $1.85.

Von Zahlen und Figuren. By Hans Rademacher and Otto Toeplitz. Berlin, Julius Springer, 1930. vi+164 pages.

Bericht über Neuere Untersuchungen und Probleme aus der Theorie der Algebraischen Zahlkörper. By H. Haase. Teil I: Klassenkorpertheorie; Beweise zu Teil I. 134 pages. 7.40 marks. Teil II: Reciprozitatsgesetz. iv+204 pages. 11.80 marks. Leipzig, B. G. Teubner, 1930.

Mathematics of Finance, preceded by Elementary Commercial Algebra. By B. E. Crenshaw, Z. M. Pirenian, and T. M. Simpson. New York, Prentice-Hall, 1930. xiv+384 pages, with tables and answers. $3.75.

Higher Arithmetic, designed for the use of high schools and colleges and for self-instruction. By Frederick A. Smith, C. E. Copyrighted, 1929, by the author. iii+138 pages, with portrait. $2.00.

Analytic Geometry. By A. M. Harding and G. W. Mullins. Revised Edition. New York, The Macmillan Company, 1930. x+316 pages. $2.50.

Introduction to Higher Geometry. By William C. Graustein. New York, The Macmillan Company, 1930. xvi+486 pages. $4.50.

Algebraic Charts. By Edgar Dehn. New York, The Nomographic Press, 1930. 14 sheets in a folder. $1.00.

Addition-Subtraction Logarithms, to Five Decimal Places. By L. M. Berkeley. New York, White Book and Supply Company, 1930. xii+136 pages. $3.25.

Plane and Spherical Trigonometry with Tables. By Leonard M. Passano. Revised Edition. New York, The Macmillan Company, 1930. $2.10.

Geometry Workbook. By H. B. Kingsbury and R. R. Wallace. Chicago, Bruce Publishing Company, 1930. 78 practice tests and 10 review tests. Paper, 76 cents.

The Logic of Discovery. By Robert D. Carmichael. Chicago, the Open Court Publishing Company, 1930. $2.00.

Debunking Science. By E. T. Bell. University of Washington Chapbooks, No. 44. Paper, 40 pages.

REVIEWS

Differential Geometry of Three Dimensions, Volume II. By C. E. Weatherburn Cambridge University Press, 1930. xii+240 pages. $4.25.

The second volume of this work on metric differential geometry continues the discussion of the subject along lines which are a natural extension of those followed in the first volume, which appeared in 1927. It seems appropriate, therefore, to make a few comments on certain characteristics which the two volumes have in common, before considering the second volume specifically.

The first thing that strikes the reader on turning the pages of the two volumes is that consistent use of vector analysis is made throughout. The classical notation of Gibbs is employed, symbols for vectors being printed in Clarendon type. Certain economies are thus effected in the way of simplifying and condensing the presentation of the subject. Those who have been in the habit of lecturing to graduate students on the subject of metric differential geometry, using the conventional methods, might do well to consider the advisability of trying out a presentation by vector methods. To anyone who ventures on this undertaking these two volumes before us will be very useful; they should be in the hands of the students as well as on the lecturer's desk.

Another commendable characteristic of the entire work is that the author has not allowed himself to forget that he is, before all, writing a treatise on geometry. The geometry is the thing that holds the center of the stage, and the analytical machinery is relegated to a subordinate place in the background where it ought to be in a book on geometry. The author does *not* make the mistake of becoming so immersed in the intricacies of his machinery, or so occupied with juggling his tools, that he loses sight of his main undertaking. Moreover, it is worthy of remark in this connection that the author's geometric insight and intuition are as clear and penetrating as his geometric interest is dominant.

The treatise lacks the profundity of such monumental works as those of Bianchi and Darboux. The more elementary parts of the subject are fairly adequately treated, but the more advanced portions are touched upon rather lightly. There is an abundance of good exercises. So the volumes are, we should

say, well suited for beginners, and would quite likely be very satisfactory as texts in the classroom, but would perhaps be only of secondary value on the reference shelf alongside Bianchi and Darboux.

There are certain features of the work which will perhaps not meet with unqualified approval. One of these is the entire omission of the Christoffel symbols. Another concerns a change in terminology. For reasons which seem compelling to the author, and which he explains in the preface to the first volume and more fully in a note at the end of the same volume, he insists on replacing the old terminology *mean curvature* and *total curvature* at a point of a surface by the expressions *first curvature* and *second curvature* respectively. We dissent from the author's opinion regarding this change in terminology, and believe, with a recent reviewer of the first volume in the Bulletin of the American Mathematical Society, that the total or Gaussian curvature should not "be subjected to the ignominy of the name 'second curvature.' "

Let us glance quickly at the contents of the first volume. We find treated here the most fundamental topics in the metric theory of curves, surfaces, and rectilinear congruences. The sequence in which the topics are arranged does not diverge very far from that which has become more or less conventional and is somehow reminiscent of Eisenhart's Differential Geometry, which has been the first avenue of approach to the subject for so many American students.

As we turn to the second volume and start through it, the reviewer gets an increasingly strong impression that we have here a collection of "selected topics" in metric differential geometry instead of a closely organized and coordinated whole. "Most of the chapters are based on recent papers by the author," as he himself asserts. There is, however, much other material included. There are thirteen chapters. After a preliminary chapter on differential invariants for a surface, two chapters are devoted to families of curves on a surface, and particularly, oblique trajectories. A fourth chapter is taken up with ruled surfaces and Weingarten surfaces. In Chapter V the author lays the foundations for a theory of curvilinear coordinates in ordinary metric space. He discusses in this connection the three-parametric gradient, divergence, and rotation as well as other differential invariants of functions in space which in recent times have entered largely into the theory of mathematical physics and here play also an important part in differential geometry. After a chapter on families of surfaces, and particularly Lamé families, the author discusses in Chapter VII dyadics, which are substantially tensors of the second order, and in the next chapter families of curves and functions of direction on a surface. Chapter IX is devoted to Levi-Civita's parallelism, especial attention being given to parallel displacement of a vector along a given curve on a surface; the results are applied to Tchebychef systems of curves on a surface. Next come three chapters on representation of surfaces, small deformations of curves and surfaces, and applicability of surfaces. The final chapter presents two methods of studying curvilinear congruences.

The book is well written. The author evidently understands the funda-

mental principles of good mathematical exposition. The typography is excellent. And it would not be surprising if some of the new portions of the material should prove to be stimulating to other writers and should thus lead to further new contributions to the literature of differential geometry.

ERNEST P. LANE

The Adjustment of Errors in Practical Science. By R. W. M. Gibbs. Oxford University Press, 1929. 112 pages.

This brief book is intended to serve as a practical guide to workers dealing with experimental and statistical data in other than the pure physical sciences. Among other topics it touches lightly on probability, least squares, give some detail on the normal curve of errors, and deals at some length with correlation theory. Numerous examples are worked out so that the methods of the text can be applied directly, although the treatment is non-mathematical, all rigorous proofs being reserved for the appendix of some twenty-seven pages Many graphs adequately illustrate the ideas which are expressed with clarity and simplicity. Beyond its expressed purpose this text may well interest the non-specialist in further study in the theory of errors and of statistics.

S. B. LITTAUER

Agricultural Mathematics. By L. C. Plant. McGraw-Hill Book Co. New York, 1930. x+200 pages. $2.50.

This book is written as a text for freshmen in agriculture and presupposes only the usual preparation in high school mathematics. It contains a large number of problems that are typical of those that the agriculturist will meet in his future study, each chapter has a group to which it is necessary to apply the principles developed in it, for example, balancing rations from a given list of feeds, or mixing a certain fertilizer from given materials furnish problems in the solution of simultaneous linear equations, while the capacity of a silo or the size of a drainage tile needed under given conditions furnish applications for quadratic equations. A whole chapter is given to the applications of probability to problems in heredity.

One of the longest chapters is "Statistics," in which enough of the fundamentals of the subject to enable the student to understand the statistical bulletins of the department of agriculture are supplied. Sufficient trigonometry to enable the student to solve triangles is contained in the last chapter.

The book is well written and has very few typographical errors for a first edition. However a few questions arose in the mind of the reviewer. Can the author justify the use of the word "transformations" as it occurs on page 9 where each of the steps in the solution of the equation, $ax+b=cx+d$, is spoken of as a transformation? On p. 45 the author defines $a^{-m}=1/a^m$, $a^{1/m}=\sqrt[m]{a}$, etc., without showing any reason for doing so. Is it not better to give the student the so-called proofs found in any college algebra or at least tell him why we

use these definitions? The explanation given of how to look up the logarithm of a number and more especially that for the converse problem seems to be inadequate for the average student.

In chapter III the author gives a brief but quite satisfactory discussion of the significant figures in a number obtained by measurement and of the computations with such approximate numbers. This feature might be added, with profit to the student, to other texts in freshman mathematics. The symbolic treatment of probability found on pp. 94–95 is quite good.

As a whole this text impresses the reviewer as being well adapted to use by the group of students for whom it was written. It should do much to popularize the study of mathematics with agricultural students, for the choice of problems in it furnishes the answer to the question, "Why should I study mathematics?", so often asked by this group.

HERMAN W. SMITH

A REVIEW OF A REVIEW

In the February, 1929 number of this Monthly there appeared a review by James Byrnie Shaw of Professor Keyser's book "The Pastures of Wonder" on which I believe comments should be made. It is my belief that this review fails to represent the book fairly, and also that it contains many statements that are at least open to question. For this reason I am asking the Monthly to give space for this rejoinder. In what follows excerpts from the review are in quotes.

"He has departed widely from his view, once expressed, that science is a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eagle sporting with the mountain winds." This figure of speech, containing as it does, and properly so, such general and vague words as "sublimated," "play," "austere," "lofty," and "analogue" should not, I believe, be regarded as the statement of a "view" as to how the word "science" should be delimited nor could Dr. Keyser have supposed that anyone might be fatuous enough to take it for such a statement. I, for one, do not find the slightest discrepancy between that figure of speech and the book reviewed.

"The book is a natural outcome of the notions of 'logistic' development by Russell and others. The antidote to this *disease* is . . .". (A "disease," is it?) I should say that the book—at least much of it—is a natural outcome of the work of all who have occupied themselves with the foundations of mathematics from Euclid down to the present. Is it fitting to characterize all this work as a *disease*?

"The premises may be false, they may be mere empty symbols [how gloriously confused this is], and the conclusion may be nowhere applicable, but if the logic is correct . . . we have a mathematical statement. It is the old and *absurd* statement of Russell, that in mathematics we do not know what we are talking about [we start with undefined symbols] nor if our conclusions are true [we start with unproved propositions]." The comments in brackets are mine. I confess that in my simplicity I have interpreted this "absurd" epigram so that

The proof is invariably deductive. "Verification" by testing the result in special cases is not part of the proof.

It may be that ultimately propositions will be characterized by the type of proof on which they depend—according to their basis for validity—though this is closely connected with their form, the more completely the proposition is stated the more close is this connection. Whether or not Dr. Keyser's classification will be adopted just as it now stands, it seems fairly evident that some such classification is in the air. The role of deductive reasoning in thought is, I believe, due for a new type of recognition and attention. Whether we agree with Dr. Keyser on all points or not, he deserves our thanks and appreciation for producing "The Pastures of Wonder."

N. J. Lennes

Missoula, Montana, June 23, 1930.

MATHEMATICS CLUBS

All reports of club activities should be sent to Professor F. M. Weida, George Washington University, Washington, D. C.

CLUB TOPICS

MECHANICS—A DRAMATIC SKIT

By Tomlinson Fort, Lehigh University

Dramatis Personae

1. Newtonian Mechanics, alias Brigham Young.
2. Physics
3. Chemistry
4. Astronomy
5. Engineering
6. Wave Mechanics
7. Matrix Theory
8. Relativity
9. Geometry

(The scene opens with an empty room. *Newtonian Mechanics* enters. He is a large man with rough dress, beard and deep masculine voice. Telephone rings. He goes to the phone.)

Newt. Mechs.: Yes, Yes. This is Young. Yes Brigham Young—B-R-I-G-H-A-M Y-O-U-N-G, Brigham Young. What is that? No, not always. That's right. That's what they used to call me. Newtonian Mechanics, from my father Isaac Newton. (Irritably) Why did I change my name? Well I got religion, that's why. If it were'nt for that I couldn't manage this family of mine. How are they? (Angrily) Go to Paradox! (Hangs up, moves over to another chair. Sits down.) Well I've had a day of it. By the Ghost of Galileo, I'm tired. (Relaxes). (Sits up and rings bell.)

(Enter the wives: (1) Physics who has recently had her face lifted. She is very frivolous. Her dress is covered with waves and lightning flashes. (2) Chemistry. She has a many-colored dress decorated with crucibles etc., light peroxide hair. She is sarcastic and cynical. (3) Astronomy. She is dignified and of lofty demeanor. Her dress is covered with stars. (4) Engineering. She is elderly. Her dress is that of an ordinary scrub woman decorated with a few faded levers or other machines. Her hair is slicked back and her sleeves rolled up. She is homely and practical. Each has her name on her breast.)

Physics: (With giggles and to no one in particular) You know I think I look much younger since my face-lifting operation. I. . . . .

Chemistry: (Interrupting) Really you do try to attract attention. Don't you?

(*Newtonian Mechanics* looks at them with resigned air. He takes out roll book and pencil.)

Newt. Mechs.: The wives will answer to roll call: Physics.

Physics: Present, b—u—t—.

Newt. Mechs.: (Roughly) Chemistry.

Chemistry: Here.

Newt. Mechs.: Astronomy.

Astronomy: (Resignedly) Why this eternal roll call. Have I not always been present, always inspired your best work? Do you not trust your old sweetheart? Come back to the enthusiasm of your youth. Do something for my perihelions.

Newt. Mechs.: Shut up! Engineering.

Engineering: You can count on me. If I were not here who would cook the dinner?

New. Mechs.: Well I'm ready for dinner now. Galileo knows I work hard enough for all of you.

Physics: (To herself). He has such beautiful hair.

Newt. Mechs.: You talk too much.

Physics: I must tell you of my electrons.

Newt. Mechs.: (Exasperated) For Newton's sake!

Chemistry: I don't feel so very well either."

Newt. Mechs.: Both of you take a dose of "law of motion oil" and go to bed. (Aside) That always fixes them.

(*Newt. Mechs.* gets up) I'll be back in a few minutes. No foolishness now. (Goes out).

Astronomy: (Sweetly) I can't help worrying about my perihelions. (*Physics* giggles. *Astronomy* assumes dignified look.)

Engineering: (Disgustedly) Smokestacks! (She begins to straighten things.)

Physics: (To *Chemistry*) But my dear you should see my new doctor. He has the most wonderful hair, all blonde and as if it had been marcelled. He thought that I was a debutante. He really did.

Chemistry: (Aside) Great Priestley, I wonder how I ever got into this family. Priestley knows old *Newtonian Mechanics* never did anything for me. (To *Physics*) Darling I should like to see this young man of yours.

(Door bell rings)

Physics: (Excitedly) I told him to call.

Engineering: Come in.

(Enter two up to the minute young men; both wear glasses and carry medical cases.)

Physics: How do you do? (To others) Allow me to present *Dr. Wave Mechanics.*

Chemistry and Astronomy: So glad to know you.

Engineering: (Slowly) Good evening.

Wave Mechs.: How do you do? Allow me to present *Dr. Matrix Theory.*

Matrix Theory: Pleased to meet you, I'm sure.

(*Physics* powders her nose; *Chemistry* uses bright red lipstick.)

Physics: My electrons, Doctor, don't you think an examination will be necessary?

Engineering: (Disgustedly) Smokestacks! (Goes to work at something.)

Wave Mechs.: (Paying no attention to *Engineering*) My beautiful young lady: I'm afraid so, and such a building-up process as you'll get. Come to our sanatorium.

Physics: (Aside to *Astronomy*) Notice his hair. Did you ever see such eyes?

Astronomy: My dear; remember our husband.

Physics: (To *Wave Mechanics*) Do you know our husband, *Newtonian Mechanics*?

Wave Mechanics: He is all wrong.

Matrix Theory: No, simply incompetent, but a trusting old soul.

Physics: He is old-fashioned but good-hearted. He married me when I was just a child. You know he is much older.

Matrix Theory: You should see our chief surgeon, *Mathematics*! There's a man for you! You should watch him operate. And our head nurse, *Mrs. Quantum.* It's really she that gives us our work to do.

Engineering: (Disgustedly) Smokestacks! I'll call *Newtonian Mechanics.* He will throw you upstarts out.

Physics: Please!

Chemistry: Possibly they can do *Physics* good. I'm all upset myself. Old *Newtonian Mechanics* never does anything for me. I sympathize with *Physics.*

Wave Mechs.: *Physics*, come with me. I'll look to your electrons myself, even your free electrons. You may have whatever you want. *Physics*, I love you.

Physics: I must pack my things.

Wave Mechs.: You need nothing. *Mrs. Quantum* will supply all.

Physics: I must have *Laws of Motion.*

Wave Mechs.: Leave them.

Physics: Let me take *Conservation of Matter.*

Wave Mechs.: No.

Physics: Boo hoo! Boo hoo! Please let me take *Conservation of Energy.* It is such a beautiful garment and can be worn on so many occasions. It is most economical.

Wave Mechs.: The most pernicious of them all. No! Come.

Physics: Please!

Astronomy: But she will be coming back soon. She is the wife of *Newtonian Mechanics.*

Wave Mechs.: No.

Physics: Boo hoo. Just a little *Conservation of Energy.*

Wave Mechs.: No.

Matrix Theory: Define energy and he will listen to you.

Physics: Boo hoo.

Chemistry: Define *Energy*! Who ever heard of such a thing?

Engineering: That's easy. My baby, *Work*, can do more than that. (All laugh.)

All: Isn't she old-fashioned? Ha Ha, Ha Ha.

(*Matrix Theory* and *Wave Mechanics* go out followed by *Physics.*)

(Enter *Newtonian Mechanics.*)

Newt. Mechs.: Where's *Physics*?

All: (in chorus) Gone.

Newt. Mechs.: Where?

Astronomy: With a handsome stranger.

Engineering: (with disgust) Smokestacks! Deserted her husband in the Lord for an upstart who called himself *Wave Mechanics* and another one who said he was *Matrix Theory.* Scandalous, I call it.

Astronomy: She loves him.

Chemistry: Yes and they promised to fix her electrons; Doctors, you know, and their head nurse is to look after all her little quanta.

Engineering: (with disgust) Smokestacks! If she ever has any.

Chemistry: She'll get description anyhow, and description is more than I get in this family.

Newt. Mechs.: The ungrateful woman! But I have the rest of you.

(Nervously) Answer to roll call.

(All answer, Here, but *Physics.*)

Newt. Mechs.: Let's have dinner. (The bell rings twice, *Astronomy* shivers, the door opens and a handsome stranger is there.

Newt. Mechs.: Who are you?

Relativity: A distant cousin of yours, *Relativity* by name, to help you with your work.

Newt. Mechs.: Welcome stranger. Wives, make room for *Relativity.* Here, Cousin. We must help *Engineering* with the dinner. Bring in the chairs.

Engineering: Smokestacks!

(*Newtonian Mechanics* and *Relativity* seize a chair together. *Relativity* drops it.)

Relativity: Oh my Universe. (Grabs his back) Sir, this is not the way. I stand here, you there, I on Mars, you on distant Neptune. I take a clock, you a vibrating atom.

Astronomy: Wonderful!

Newt. Mechs.: Nonsense!

Engineering: Smokestacks!

Relativity: We must have a plan for any chair, anywhere, without a care.

Newt. Mechs. and Engineering: Get to work or get out.

Relativity: Well cousin, you scorn my generality. No romance, no poetry, all triviality. You cannot even manage a perihelion. Adieu!

Engineering: Smokestacks! Get out. (Raises broom).

Astronomy: Hold, I must go with you.

All: What?

Chemistry: You his favorite wife, you his boasted sweetheart, Ha, Ha, Ha.

Astronomy: I must go. My perihelions.

Newt. Mechs.: (stutters, waves arms) S . . . . S . . . .

Relativity: Come. (Exit *Relativity* and *Astronomy*)

Chemistry: Well *Newt.* old boy, what do you think of that? Two gone. This family of saints is in a bad way. Ha, Ha, Ha.

Newt. Mechs.: By the ghost of Galileo! By my father *Newton!*

(*Engineering* begins to sweep violently)

Chemistry: Ha, Ha, Ha I'm tired of it. I'm going too. What have you ever done for me? Ha, Ha, Ha, you didn't know that your servant, Calculus, was my lover. Ha, Ha, Ha, Thought to hold us all, Ha, Ha, Ha. I have a new lover too, another servant of yours, *Differential Equations.* I'm going to *Differential Equations.* Ha, Ha. To *Differential Equations*, To *Differential Equations*! (Goes out)

(*Engineering* and *Newtonian Mechanics* stand looking at each other. *Mechanics* is unable to speak. He waves his arms and stutters.)

Engineering: It's all right, *Newt.*, I'm glad to get rid of them. Now you'll come back to the Baptist Church and stick by me your one and only lawful wife, *Engineering.* Be strictly mathematical and we shall live long and prosper.

(Loud knocking at door)

Newt. Mechs.: Oh . . . . (Shivers)

(Knocking repeated)

Engineering: Come in. (Knocking repeated. She goes to door and opens it.)

(Powerful stranger presents himself.)

Engineering: Who are you?

Geometry: I am called *Geometry*, I am the sheriff. See my badge. (He displays a badge marked with a large circle)

Newt. Mechs.: Oh . . . . (Covers his face with his hands) Oh . . . . .

Engineering: What do you want? A certain geometry, *Elementary* by name, used to serve us well.

Geometry: That was my infant self. I have come for *Mechanics.* (boastfully) See how big I am.

Newt. Mechs.: Woe! Woe! Woe!

Geometry: (to *Newtonian Mechanics*) I shall put you in my pocket. I shall consume you alive. (Declaims) Yes, Yes, *Geometry* will swallow up *Mechanics.* Live *Geometry*, down with *Mechanics.*

Newt. Mechs.: (Weeps out loud) Oh . . . . .

Geometry: Ha, you sniveling creature. Get in my pocket. In with you, old working man. You infinitesimal. In with you, I say.

Engineering: Now, *Master Geometry*, you are a big fine fellow there is no doubt about that. How you have grown! But away with you; put on your elementary aspects; leave *Mechanics* with me. The world cannot get on without me and I need this old classical fellow. (Aside) Cheer up

Newt., it's all right. (To Geometry again) I'll take him to *Physics* lots of times. I'll promise, We old folks need a girl like that. Maybe she'll come back week ends. These girls will run after handsome strangers. We'll behave ourselves. *Mathematics* will regulate our every move. Be a generous fellow, *Geometry*. Leave us alone now.

Newt. Mechs.: Oh leave us alone, leave us alone, Oh, Oh, Oh . . . . .

Geometry: Well, *Madam Engineering*, if you want the old fellow you may keep him. Possibly all of us can use him some. I'm quite sure that *Astronomy* and *Physics* can. As for me; I'll keep coming back and shall polish him up from time to time. He'll get on well enough but after a while you'll find that I have him completely rewritten in geometric language, and that (declaims) verily *Geometry* has swallowed up *Mechanics*. If you expect to use him then you, yourself, must know my higher dimensions. You too must come to *Geometry*, *Madam Engineering*, and I'll not wear my elementary aspects. No indeed there'll be no limit to the dimensions I'll wear. Adieu *Madam Engineering*.

Geometry: (To *Newtonian Mechanics*) Au revoir, old fellow. Good luck, but I'll be back. I'll have you in my pocket, do you understand, but for the present au revoir, auf wiedersehen. (Exit) (*Engineering* and *Newtonian Mechanics* fall into each others arms weeping.)

CLUB ACTIVITIES

Albion College Mathematics Club, Albion, Michigan.

Officers: President, Allene Day; Vice-President, Grace Ulbright; Secretary-Treasurer, Alberta Wocholz.

The programs for the year 1929–1930 were as follows:

October: Roll call—geometric figures in everyday life; A report of "The Cultural Value of Mathematics," by Aubrey J. Kempner.

November: Roll call—famous mathematicians and their work. Two reports," Mathematics in the junior high schools" and "The value of the history of mathematics in teaching."

December: A lecture by Dr. L. A. Hopkins, secretary of the College of Engineering and Architecture of the University of Michigan. Open to the student body.

January: Roll call—geometric theorems. A report, "Graded algebraic abilities in teaching"; and a contest conducted on an "Ancient duodecimal system."

February: A special report and discussion on "The plotting of the cubic."

March: Roll call—algebraic and geometric formulas. Two reports, "Mathematics in England and Germany" and "Mathematics in everyday life."

April: Roll call—geometric figures in nature. A report, "Watching the meteors."

May: Professor Evarts of Kalamazoo College gave a talk on "Uncomputative mathematics."

(Report by Alberta Wocholz)

Napierian Club of De Pauw University, Greencastle, Indiana.

Officers for the year 1929–1930 were: President, Charles Stunkel ('30); Vice-President, Josephine Newkirk ('30); Secretary, Margery Joslin ('30); Treasurer, Roy Holwager, ('30).

The following programs were given at the regular monthly meetings:

October 3, 1929. Election of new members.

October 10. "History of the Napierian Club and the life of John Napier" by Josephine Newkirk.

November 14. "Scales of notation" by Richard Jay; Book review of "Flatland" by Margery Joslin.

December 12. "Vector analysis" by Professor R. W. Babcock.

January 9, 1930. "Problems of a high-school teacher" by Christine Dietrick.

February 13. "History of algebra" by Mary McCord; "Probability in gambling" by Howard Fetters.

March 13. "The system of homogeneous coordinates" by Professor W. C. Arnold.

April 10. "Discussion of the new planet" by Roy Holwager; "Discussion of the coming eclipse of the moon and that of the sun" by Horace Barnett; Observations from McKim Observatory.

May 8. "Mathematics in music" by Professor H. E. H. Greenleaf; Election of new officers.

(Report by Margery Joslin)

Irrational Club of the University of Wyoming, Laramie, Wyo.

The officers for the year 1929–1930 were: President, Robert Hill; Vice-President, Imogene Montgomery; Secretary-Treasurer, Rowene Danielson; Faculty Advisor, Greta Neubauer.

The programs of the club for the year 1929–1930 were as follows:

October 9, 1929. Election of officers. Fall party.

November 7. "Development of the nine point circle" by Miss Montgomery.

November 21. "Mathematical cross-word puzzle" by club officers; "Graphical solutions for the complex roots of simultaneous quadratic equations" by Professor C. F. Barr.

December 5. "Life history of Réné Descartes" by Mr. Call; "Relation of mathematics and architecture" by Mr. Hitchcock.

January 16, 1930. "Logarithms" by Miss Neubauer; "Mechanical demonstrations of logs and cosines" by Robert Hill.

January 30. "Mathematics in music" by Mrs. Shaw; "The Irrational Club Song" by Mr. Barr.

February 21. "The Simpson line" by Miss Achenbach; "The problem of Apollonius" by Mrs. Shaw; "The locus of the point of similitude of two similar triangles, similarly placed, during rotation of equal amounts about dissimilar points" by Miss Montgomery and Miss Wortheim; A mechanical device showing the locus of points of similitude was demonstrated by Mr. Mize.

March 7. "History of the slide rule" by Miss Neubauer; "The slide rule" by Professor Goodrich.

March 21. "Determining pi by tossing matches" by Mr. Warner.

April 10. "Astronomy and mathematics" by the astronomy class.

May 8. "History of pi" by Rowene Danielson; "History of the equals sign" by Ada Burke; "History of e and of logarithms" by Grace Reid.

May 26. Annual beefsteak fry held in the mountains of the Laramie Range.

(Report by Rowene Danielson)

Mathematics Club of the College of the City of Detroit, Detroit, Michigan.

Any student of mathematics, or for that matter any student interested in mathematics at the College of the City of Detroit, may become a member of the club by simply coming to the meetings. The organization is rather loose, hence the president's job is a responsible one. Although no time is specified for meetings they have generally been held at least once a month, at the discretion of the president. The feature of each meeting is a paper, relating to some branch of mathematics, presented by a student. The widely varying nature of these papers is made apparent in the following paragraphs.

The first meeting of the year 1929–30 was held in October, and at that time Clarence Wylie was elected president. Immediately following the election, Mr. Wylie presented a paper on "Three dimensional plane geometry," in which he treated the imaginary element in plane analytics as a third dimension using i as a revolving operator.

Early in December, Gordon Wilcox presented a paper on "Conditions for exact division," in which he treated the relations between the digits of a number and the remainder or lack of remainder when divided. Here Mr. Wilcox introduced an interesting application of certain series.

At the April meeting the writer presented a paper on "Determinants," in which the algebra and elementary properties of this useful device were dealt with. Later in April Clarence Wylie discussed "Elementary analytic projective geometry"; he took up such topics as the principle of duality, homogeneous coordinates, and transformations involving the line at infinity.

Early in May, Miss Jean Persons, a graduate of the college teaching in the Detroit Public Schools, spoke on "Geometrical constructions by paper folding." Later in the same month Kenneth E. Stecker presented a paper on the "Approximate division of a circumference." This was the final meeting of the year.

(Report by Glen Brookes)

The Undergraduate Mathematics Club of the University of Iowa, Iowa City, Iowa.

Because of the untimely death of the president of the club, Mr. Wilbur B. Elliff, the first meeting of the club was not held until the second semester. At that time the following officers were elected for the remainder of 1929–30 and also for 1930–31: Professor L. E. Ward, Faculty Adviser; Mr. Carl H. Fischer, President; Miss Thelma Coate, Secretary-Treasurer.

The program for the semester was as follows:

February 20, 1930. "Methods of extracting square roots" by Professor L. E. Ward.
March 14. "A method of approximating to the nth root of a number" by Mr. Milton S. Weinberger.
March 27. ""Meteoric phenomena" by Mr. Lloyd O. Ritland.
April 11. "Confocal conics" by Miss Gerturde Rickey

(Report by Thelma Coate)

The Mathematics Club of the University of Georgia.

This informal club has no regularly appointed officers, no membership requirements, and no dues. By common consent it meets twice a month, and is attended by the faculties of the University of Georgia, State Teachers College, and Lucy Cobb Institute, together with the graduate students in mathematics. Those attending take turns as speakers. The programs for the year 1929–1930 were historical in character and D. E. Smith's *Source Book in Mathematics* served as inspiration and outline.

(Report by David F. Barrow)

The Mathematics Club of the Cooper Union Institute of Technology, New York, N.Y.

The officers for the year 1929–30 were: President, Leo Rubinowitz ('30); Vice-President, G. D. Champlin ('30); Secretary, David Rabinovich ('31); Treasurer, W. T. Eddy ('32); Assistant Secretary. J. J. Murphy ('31). The club has a paid membership of 170, the yearly dues being 25 cents.

The programs for the year were as follows:

Nov. 12, 1929. "Probability and its applications" by Mr. E. C. Molina, research engineer, American Telephone & Telegraph Co.
Dec. 3. "Application of the catenary to railway electrification" by G. D. Champlin ('30).
Dec. 17. "Elements of the theory of integers" by Fitzgerald Bramwell ('33.)
Jan. 14, 1930. "Bernoulli numbers" by Leo Rubinowitz ('30.)
Jan. 28. "The bearings of mathematics" by Professor C. J. Keyser, Columbia University.
Feb. 18. "Fundamental ideas of projective geometry" by Joseph Stitelman ('32).
Mar. 11. "The slide rule and its uses" by Professor W. E. Breckenridge, Stuyvesant High School.
Mar. 25. "Euler's problem" by Robert Wolins, '31; "The Russian peasant multiplication system" by L. A. Kenworthy ('31).
April 15. "Curious properties of numbers" by Professor H. W. Reddick. Election of officers.

(Report by C. H. Lehmann)

Bryn Mawr Mathematics Club of Bryn Mawr, Pennsylvania.

The program of the Bryn Mawr Mathematics Club for the year 1929–1930 was as follows:

October 16, 1929. "The definition of trigonometric functions by means other than geometric" by Dr. D. V. Widder.

November 5. "The motion of an electrically charged particle in a magnetic field, as developed by Dr. Richard Gans on the principles of vector analysis" by Ruth Unangst ('30), and "The Life of Evariste Galois" by Elizabeth Baker ('30).

Announcement was made at this meeting of a $10 prize to be awarded for the best undergraduate club report of the year.

November 26. "Regular polygons" by Dr. Anna Pell Wheeler, Bryn Mawr College.

December 18. "The history and computation of π" by Pauline Huger ('32), and "Homogeneous point coordinates" by Anne Lea Nicholson ('30).

January 7, 1930. "Groups—the theory being built up from facts observed of the group four" by Dr. Thomas, University of Pennsylvania.

February 11. "Isolation of the real roots of an equaton of degree n—Sturm's functions and Bowidan's" by Miss Lucile Anderson, Bryn Mawr College ('30).

March 18. "Inversions" by Constance Cole ('30), and "Fundamental propositions of algebra" by Agnes Hannay.

April 15. "Historical problems of the calculus of variations especially the brachistochrone of the Bernoulli brothers" by Gretchen Mueller ('32), and "Applications of the calculus of variations," by Mary Peters ('30).

May 13. "Postulates of a miniature geometry—indicating what a sizable geometry could be inferred without a continuity assumption" by Dr. Marguerite Lehr, Bryn Mawr College. Dr. Widder awarded the prize, referred to above (Nov. 5), to Miss Agnes Hannay for her report on "Fundamental propositions of algebra."

The officers for the year 1929–1930 were: Miss Agnes Hannay, President; Miss Janet Wise, Vice-President and Treasurer; Miss Ruth Unangst, Secretary.

The officers elected for 1930–1931 are: Miss Gretchen Mueller, President; Miss Ruth Unangst, Vice-President and Treasurer; Miss Pauline Huger, Secretary.

The club is composed of graduates, and undergraduates who have had or who are taking the second year mathematics course. Meetings were held from about 4:00 to 6:00 o'clock, the reports and discussions being followed by tea, food, and choice puzzles.

(Report by Ruth Unangst)

The Mathematics Club of the University of Alberta, Edmonton, Alberta, Canada.

The programs for the year 1929–1930 were as follows:

October 22, 1929. "Present day tendencies in secondary mathematics" by A. J. Cook and E. W. Sheldon.

November 6. "Units" by E. S. Keeping.

November 19. "Infinity" by Louise Miller.

December 3. "Congruences" by A. J. Cook.

January 14, 1930. "Superposition" by E. D. M. Williams.

January 28. "The vogue of statistics" by E. W. Sheldon.

Feburary 11. "Mechanisms" by J. S. Beggs.

Feburary 25. "Difficulties in secondary mathematics" by D. L. Shortliffe.

March 7. "Relativity and geometry" by I. F. Morrison.

(Report by E. W. Sheldon)

The Mathematics Club of Brown University, Providence, R.I.

The following programs were given in 1929–1930:

October 29, 1929. "The Martian counts on his fingers" by Lulu Amelia Vorleck (graduate); "Cocoanuts, sailors, and a monkey" by John Bernard Chaffee ('31.)

December 3. "Moving in four dimension" by John Otis Prouty ('31); "Nomograms" by David Moskovitz (graduate).

January 14, 1930. "Measuring the distances of the stars" by Clincton Harvey Currier, associate professor of mathematics, Brown University.
February 18. "Mathematics of the pyramid builders" by Mary Taylor ('30); "The golden section and Fibonacci series" by Harold Irving Brown ('30).
March 18. "Newton and the calculus" by Donald Leigh Fowler, Jr. ('31); "Leibniz and the calculus" by Enis Eva DeMagistris ('31).
April 22. "A mathematical cinderella" by Heinrich Wilhelm Brinkmann, assistant professor of mathematics, Harvard University.
May. Picnic.
Committee on Program: Professor Bennett; Mr. Thurston; Lulu Amelia Vorleck (graduate); Mary Taylor ('30); John Otis Prouty ('31); Donald Leigh Fowler, Jr. ('31).
Committee on Arrangements: Mr. Krall; Ruth Barden Eddy ('32); Mary Elizabeth Brooks ('31); James Benjamin Brown ('31); Delbert Swan Wicks, Jr. ('32).

PROBLEMS AND SOLUTIONS

EDITED *by* B. F. FINKEL, OTTO DUNKEL, *and* H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3469. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

3470. *Proposed by F. L. Wren, George Peabody College for Teachers.*

If the hypotenuse of a right triangle be divided into n equal parts and the vertex of the right angle be joined to these points of equal division, then, if d_i be the length of the lines so drawn, we have

$$\sum_{i=1}^{n-1} d_i^2 = \frac{(n-1)(2n-1)}{6n}h^2,$$

where h is the length of the hypotenuse.

3471. *Proposed by W. R. Ransom, Tufts College.*

In assigning dormitory rooms, a college gives preference to pairs of students in this order:

$$AA,\ AB,\ AC,\ BB,\ BC,\ AD,\ CC,\ BD,\ CD,\ DD,$$

in which AA means two seniors, AB means senior and junior, etc. Determine numerical values to assign to A, B, C, D so that the set of numbers $A+A$, $A+B$, $A+C$, $B+B$, etc., corresponding to the order indicated above, will be in descending magnitude. Find the general solution, and also the solution in least integers.

3472. *Proposed by Morgan Ward, California Institute.*

Let S_n denote the sum of the nth powers of the roots of $F(x)=x^3-Px^2+Qx-1$, where P, Q are integers; p, a prime of the form $3n+2$ chosen so that $F(x)$ is irreducible modulo p, and μ, the least value of n such that $S_n\equiv S_0$, $S_{n+1}\equiv S_1$, $S_{n+2}\equiv S_2$, mod p. Prove that if n is not divisible by μ, $S_n\equiv 0$ mod p, when and only when $S_{3n}\equiv 3$ mod p.

3473. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the incircle is tangent to CA at D and to CB at E. Through a variable point P on DE, AP and BP are drawn meeting CB at X and CA at Y.

Find the envelope of the line XY.

3474. *Proposed by R. E. Gaines, University of Richmond, Va.*

From a point P on the ellipse $b^2x^2+a^2y^2=a^2b^2$ two chords are drawn, one through $(ae, 0)$ and the other through $(-a, 0)$; for what value of e and what position of P will these two chords trisect the area of the ellipse?

SOLUTIONS

3417. [1930, 157]. *Proposed by C. O. Williamson, The College of Wooster, Wooster, Ohio.*

Construct a square so that each side shall pass through a given point.

I. *Solution by E. M. Berry, Lynchburg College*

Let A, B, C, D, be the given points. Draw a line BL through B perpendicular to AC and so that BL is equal to AC. Let d be the line joining the points D and L. Draw b parallel to d through B, and draw a and c perpendicular to d through the points A and C respectively. The lines a, b, c, d, form the required square, since the distance between a and c is the same as the distance between b and d. This follows from the fact that the angle between a and AC is the same as the angle between b and BL and that the segments AC and BL are equal.

In general there are six solutions, for there are two solutions with A and C on opposite sides of the square and there are three ways of picking the pairs of points for opposite sides.

II. *Solution by R. Goormaghtigh, Bruges, Belgium*

The following is a solution of the generalized problem: *Construct a rectangle ABCD similar to a given rectangle MNPQ, being given four circles tangent to the*

sides of ABCD and placed, with respect to the corresponding sides, in given regions of the plane, compared with the center of the rectangle.

Let a, b, c, d be the centers and r_a, r_b, r_c, r_d the radii of the given circles, respectively tangent at α, β, γ, δ to AB, BC, CD, DA; call m and n two numbers proportional to the sides MN and NP of $MNPQ$, MN and NP being homologous to AB and BC.

We first consider the case when the four circles are, with respect to the sides of $ABCD$, in the same regions of the plane as the center of the rectangle and when the rectangles $ABCD$ and $MNPQ$ are directly similar. If b', d' be the projections of b, d on AB and a', c' those of a, c on DA, the projections of $\delta d b \beta$ and $\gamma c a \alpha$, respectively, on AB and DA are

$$AB = r_d + d'b' + r_b, \quad DA = r_c + c'a' + r_a.$$

But these projections are in the same ratio as m and n. Therefore, if, on the perpendicular from d on ac, we take dw equal to $m \cdot ca$, so that the right angle (ca, dw) has the same sign as $MNPQ$, and if, on db, we take dv equal to $n \cdot db$, the projection of wv on AB is

$$n \cdot d'b' - m \cdot c'a' = \rho = m(r_a + r_c) - n(r_b + r_d).$$

Hence the following construction:

If, on the perpendicular from d on ac, we take dw equal to $m \cdot ca$, so that the angle (ca, dw) has the same sign as $MNPQ$ and then on db a length dv equal to $n \cdot db$, the directions of DA corresponding to the problem are those of the tangents drawn from v to a circle having w as center and ρ as radius.

If dw' is taken equal to $m \cdot ca$ but so that the angle (ca, dw') has the same sign as $QPNM$, the construction gives rectangles $ABCD$ inversely similar to $MNPQ$.

If, further, some of the given circles are, with respect to the corresponding sides, in regions opposite to those containing the center of the rectangle, the signs of the corresponding radii will be changed in the foregoing construction.

In the special case of problem 3417, $r_a = r_b = r_c = r_d = 0$, $m = n = 1$, and the construction becomes the following:

On the perpendicular from d on ac take $w'd = dw = ac$; then the side BC of the required square passes through w or w' (two solutions for this selection of pairs of points).

Also solved by S. F. Bibb, J. D. Deutsch, P. S. Dwyer, V. F. Ivanoff, J. H. Neelley, William Orange, A. Pelletier, W. A. Rees, J. Rosenbaum, Hazel E. Schoonmaker, E. F. Sweet, and E. C. Westerfield.

3418. [1930, 157]. *Proposed by David H. Dodge, Los Angeles, California.*

Prove that

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2+n+p}{n^2+n-p}.$$

where

$$\sum n^6 = (6n^7 + 21n^6 + 21n^5 - 7n^3 + n)/42;$$

$$\sum n^4 = (6n^5 + 15n^4 + 10n^3 - n)/30;$$

and

$$\sum n^2 = (2n^3 + 3n^2 + n)/6.$$

Solution by Beatrice Aitchison, The Johns Hopkins University

Set

$$5 \sum n^4 + \sum n^2 = f_1(n) \text{ and } 7 \sum n^6 + 5 \sum n^4 = f_2(n);$$

then the problem requires the proof of the identity

$$\frac{f_2(n) + pf_1(n)}{f_2(n) - pf_1(n)} = \frac{n^2 + n + p}{n^2 + n - p}.$$

It is easily found that $f_2(n) = (n^2+n)f_1(n)$, and this suffices to prove the equality above.

Also solved by S. A. Corey, Mabel S. Graham, A. Pelletier, Irwin Roman, and Margaret M. Young.

3419 [1930, 196]. *Proposed by the late F. P. Matz.*

Determine the law of force in order that the orbit be a Cassinian oval.

Solution by Norman Anning, University of Michigan

Assume that the center of attraction is at the center of figure. It is proved in treatises on dynamics that P, the force per unit mass, is given by the formula

$$P = h^2p^{-3}dp/dr, \tag{1}$$

where h is a constant and where p and r are connected by the pedal equation of the path. The Cassinian oval defined by the relation $PF_1 \cdot PF_2 = a^2$, where F_1 is the point $(-c, 0)$ and F_2 is $(+c, 0)$, has the equation

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = a^4. \tag{2}$$

From this, by standard methods which need not be reproduced here, we find the "p-r" equation

$$2a^2pr = r^4 + a^4 - c^4. \tag{3}$$

Now, with equation (1) and with p expressed as a function of r, the rest is shop-work and leads to the result

$$P = 4a^4h^2r(3r^4 - a^4 + c^4)(r^4 + a^4 - c^4)^{-3}.$$

References: For (1), Norris & Legge, *Mechanics via the Calculus*, p. 274; for (2) and (3), Hilton, *Plane Algebraic Curves*, p. 324. For finding p-r equations, Edwards, *Differential Calculus*, p. 161.

Also solved by William Hoover.

3421 [1930, 196]. *Proposed by Otto Dunkel, Washington University.*

A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

I. *Solution by W. A. Bristol and W. R. Church, University of Pennsylvania*
Mr. Bristol's Contribution

Let K_n represent the number of ways a polygon of n sides can be divided in accordance with the conditions of the problem. Consider the vertices of any polygon as numbered consecutively from 1 to n. Draw diagonal $n2$ cutting off triangle $n12$, leaving a polygon of $n-1$ sides. This polygon of $n-1$ sides can be divided into triangles in accordance with the rules of the problem in K_{n-1} ways. Next, draw diagonal 13 and again the polygon of $n-1$ sides left can be divided into triangles in K_{n-1} ways, all different from the first set of K_{n-1} ways. Proceed to cut off successive vertices by diagonals joining each time the two vertices adjacent to the vertex to be cut off. The number of new ways of dividing the original polygon, for the ith diagonal, $i>2$, will be given by subtracting from K_{n-1} the sum of the first $i-2$ terms in the series (see Table 1 below) for the polygon of $n-1$ sides.

A triangle will be considered as being divided in one way. The above shows, as is also evident, that a quadrilateral can be divided in $1+1=2$ ways; a pentagon in $2+2+(2-1)=5$ ways. It is easy to see that the number of new ways added when $i=n-2$ will always be 1 and that there can be no new ways when $i=n-1$. Hence K_n is given by a series of $n-2$ terms which decreases after the second term according to the law stated in the above paragraph.

Thus we have

TABLE 1

n		K_n
3	1	1
4	1 + 1	2
5	2 + 2 + 1	5
6	5 + 5 + 3 + 1	14
7	14 + 14 + 9 + 4 + 1	42
8	42 + 42 + 28 + 14 + 5 + 1	132
. . .	. . . .	. . .

To illustrate further a point in the argument it will be noted that in obtaining the 5th term in the series for an octagon, the term is given by subtracting from K_7 twice K_6 and also K_6-K_5. But K_6-K_5 is the third term in the series for K_7.

Mr. Church's Contribution

Delete the first term in each series and renumber the rows according to the number of the row from top down. Next, reverse the order of terms in each deleted series, giving the following

TABLE 2

$r \backslash c$	1	2	3	4	5
1	1				
2	1	2			
3	1	3	5		
4	1	4	9	14	
5	1	5	14	28	42

Let ${}_cT_r$ represent the number in the cth column and rth row. The numbers ${}_cT_r$ are defined for all positive integral values of c and r when $c \leqq r$ by the following relations:

$$ {}_1T_r = 1, \qquad (r = 1, 2, 3, \cdots) $$

$$ {}_cT_r = \sum_{i=c-1}^{r} {}_{c-1}T_i. $$

By use of the following identity

$$ \sum_{i=1}^{n} \prod_{j=i}^{i+x-1} j = \frac{1}{x+1} \prod_{k=n}^{n+x} k, $$

the following formula can be proved by induction:

$$ {}_cT_r = \frac{r-c+2}{r+1}\binom{r+c-1}{r}. $$

When $c = r$, we have

$$ {}_rT_r = \frac{2}{r+1}\binom{2r-1}{r}. $$

Now changing to the notation for polygons, where $r = n-2$, this becomes

$$ K_n = {}_{n-2}T_{n-2} = \frac{2}{n-1}\binom{2n-5}{n-2}. $$

II. *Partial Solution by Vladimir F. Ivanoff, San Francisco, California*

Denote the vertices of the polygon by $P_1, P_2, \cdots, P_n$ and the number of combinations of diagonals that divide the polygon into triangles by A_n. Con-

sider the side P_1P_n. In every combination, P_1P_n belongs to a certain triangle $P_1P_iP_n$ $(1<i<n)$. Therefore, taking $i=2, 3, \cdots, (n-1)$, we obtain $(n-2)$ groups of combinations, which include all possible cases.

(1). If $i=2$, we have a group with the diagonal P_2P_n always present. This group includes as many combinations as there are in the $(n-1)$-gon $P_2P_3 \cdots P_n$, i.e., A_{n-1}.

(2). If $i=3$, we have the diagonals P_3P_1 and P_3P_n always present. This group therefore includes as many combinations as there are in the $(n-2)$-gon $P_3P_4 \cdots P_n$, i.e., A_{n-2}.

(3). If $i=4$, we have always the triangle $P_1P_4P_n$ present. The quadrangle $P_1P_2P_3P_4$ and $(n-3)$-gon $P_4P_5 \cdots P_n$ are adjacent to it. The former yields A_4 combinations, and the latter A_{n-3} combinations. Therefore, the number of all combinations in this group is $A_{n-3}\cdot A_4$. For the groups $(4, 5, \cdots)$. we have, respectively, $A_{n-4}\cdot A_5$, $A_{n-5}\cdot A_6$, etc., $\cdots$. For the group $(n-2)$, the number of combinations is the same as for the first one, i.e., A_{n-1}.

Thus

$$A_n = A_{n-1}+A_{n-2}+A_{n-3}\cdot A_4+A_{n-4}\cdot A_5+\cdots+A_5\cdot A_{n-4}+A_4\cdot A_{n-3}+A_{n-2}+A_{n-1}.$$

Since $A_3=1$ and $A_4=2$, we have

$$\begin{aligned}
A_5 &= A_4 + A_3 + A_4 = 5,\\
A_6 &= A_5 + A_4 + A_4 + A_5 = 14,\\
A_7 &= A_6 + A_5 + A_4\cdot A_4 + A_5 + A_6 = 42,\\
A_8 &= A_7 + A_6 + A_5\cdot A_4 + A_4\cdot A_5 + A_6 + A_7 = 132,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Also solved by A. G. Clark.

A Note by the Proposer. The partial solution above may be completed by the usual process of finding the generating function for the A_n's. For convenience set $A_2=1$; then we may write

$$A_n = \sum_{i=2}^{n-1} A_iA_{n-i+1}.$$

Now define in a purely formal manner the generating function $f(x)$ so that

$$f(x) = \sum_{n=2}^{\infty} A_nx^{n-2}.$$

Hence

$$\begin{aligned}
[f(x)]^2 &= \sum_{n=3}^{\infty} x^{n-3} \sum_{i=2}^{n-1} A_iA_{n-i+1} = \sum_{n=3}^{\infty} A_nx^{n-3}\\
&= [f(x) - 1]x^{-1}.
\end{aligned}$$

Thus

$$f(x) = [1 - (1 - 4x)^{1/2}](2x)^{-1}$$

is the solution of the quadratic which gives $f(0)=1$. If we regard x as restricted to the values $|x| \leqq a < 4^{-1}$, then we may develop $f(x)$, as defined above, in an absolutely convergent power series. With this restriction, all the formal work above is legitimate and $f(x)$ is our generating function for the A_n's. By expansion of $f(x)$ we have

$$f(x) = \sum_{n=2}^{\infty} \frac{2}{n-1}\binom{2n-5}{n-2} x^{n-2};$$

and hence

$$A_n = \frac{2}{n-1}\binom{2n-5}{n-2}.$$

The argument for the last part of the first solution may be put in the following form. As stated ${}_cT_r$ is completely defined by the two given requirements. The values of ${}_3T_r$ and ${}_4T_r$ are easily derived in turn from these two requirements. The form of these results suggest a function of the form

$$f(c, r) = (r - c + 2)\frac{(r + c - 1) \cdots (r + 3)(r + 2)}{(c - 1)!} = \frac{(r - c + 2)}{r + 1}\binom{r + c - 1}{c - 1}.$$

In order to prove the identity $f(c, r) = {}_cT_r$, we have merely to verify that $f(c, r)$ satisfies the two requirements. Now $f(1, r) = 1$ by the usual definition of the factorial symbols. Also we easily find that $\Delta f(c, r) = f(c-1, r+1)$, where Δ applies to r alone. Hence

$$\sum_{i=c-1}^{r} f(c - 1, i) = \sum_{i=c-1}^{r} \Delta f(c, i - 1) = f(c, r) - f(c, c - 2), \quad c \geqq 2,$$
$$= f(c, r).$$

This completes the proof.

3422 [1930, 197]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Prove that the polar reciprocal of a sphere with respect to a given sphere is a quadric of revolution. Discuss the nature of this quadric.

Solution by J. H. Neelley, Carnegie Institute of Technology

Let us find the polar reciprocal of the sphere S: $x^2+y^2+z^2=a^2$ with respect to the sphere S': $(x-b)^2+y^2+z^2=r^2$. The tangent plane of S at $P_1(x_1, y_1, z_1)$ and the polar plane of its pole $P'(x', y', z')$ as to S' being identical we have

$$x_1:(x' - b) = y_1:y' = z_1:z' = a^2:(r^2 - b^2 + bx').$$

Solving for x_1, y_1, and z_1 and substituting in the relation $x_1^2+y_1^2+z_1^2=a^2$ we have the required polar reciprocal with equation

$$(a^2-b^2)x^2-2b(a^2-b^2+r^2)x+a^2(y^2+z^2)=(r^2-b^2)^2-a^2b^2.$$

This is obviously a quadric of revolution.

The three parameters a, b, and r may be varied so that the quadric is either a sphere, a circular paraboloid, a hyperboloid of one sheet, a hyperboloid of two sheets, an ellipsoid, or a cone. Of course an axis of the quadric is the line of centers of S and S' in all cases.

3424 [1930, 197]. *Proposed by Morgan Ward, California Institute of Technology.*

Let A and B be two permutable elements of an abstract group, of orders a and b respectively, and let c be the order of $C=AB=BA$. Show that m' divides c, and c divides m, where m is the L.C.M. of a and b, and m' is the product of all those prime factors of m which appear in a and b raised to different powers, every such factor being raised to the power to which it appears in m.

Solution by Wendell Holmes Langdon, Yale University

Since A and B are permutable, $(AB)^m=A^mB^m=1$, whence c is obviously a divisor of m.

Now let

$$a=fp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r},$$
$$b=fp_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$$

be the decomposition of a and b into primes, where f contains those prime factors which are raised to equal powers in a and b, and $\alpha_i\neq\beta_i$, $\alpha_i\geqq 0$, $\beta_i\geqq 0$. Let

$$\gamma_i=\max(\alpha_i,\beta_i);$$

then

$$m=fp_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}$$

and

$$m'=p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}.$$

Since c is a divisor of m, we have

$$c=f'p_1^{\delta_1}p_2^{\delta_2}\cdots p_r^{\delta_r},$$

where f' is a divisor of f, and $\delta_i\leqq\gamma_i$. In order to prove the theorem we have only to show that $\delta_i=\gamma_i$. Let us suppose that $\delta_i<\gamma_i$ for some i, say $\delta_1<\gamma_1$, and let us assume $\alpha_1>\beta_1$.

Now

$$(AB)^c=A^\sigma B^\sigma=1,\text{ where }\sigma=f'p_1^{\delta_1}\cdots p_r^{\delta_r},$$

We raise this equality to the power k, where

$$k = fp_1^{\mu}p_2^{\beta_2}p_3^{\beta_3}\cdots p_r^{\beta_r}$$

and

$$\mu = 0 \text{ if } \delta_1 \geqq \beta_1,\ \mu = \beta_1 - \delta_1 \text{ if } \delta_1 < \beta_1.$$

In the resulting equality, the exponent of B is obviously divisible by b, and we obtain

$$A^{\tau} = 1, \text{ where } \tau = p_1^{\delta_1+\mu}p_2^{\delta_2+\beta_2}\cdots p_r^{\delta_r+\beta_r}.$$

whence, a being the order of A,

$$\delta_1 + \mu \geqq \alpha_1, \text{ or } \mu \geqq \alpha_1 - \delta_1 > 0.$$

For $\mu=0$, this relation is impossible; for $\mu=\beta_1-\delta_1$, we obtain $\beta_1\geqq\alpha_1$, which is contrary to hypothesis. Thus $\delta_i=\gamma_i$ and c is divisible by m'.

Also solved by the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. H. D. Curtis, director of the Allegheny Observatory, has been appointed director of the new observatory and head of the department of astronomy at the University of Michigan.

Dr. Lena J. Hawks has been appointed head of the department of mathematics at the Georgia State Womans College.

Mr. George Karelitz, of the Westinghouse Electric and Manufacturing Company, has been appointed associate professor of mechanical engineering at Columbia University.

Assistant Professor E. G. Keller, of the University of Texas, has been appointed a mathematician of the General Electric Company.

Mr. J. H. Kusner has been appointed assistant professor of mathematics at the University of Florida.

Professor Alfred Landé, of the University of Tübingen, has been appointed professor of theoretical physics at Ohio State University.

Professor H. F. Lusk, of the College of the Pacific, has been appointed professor of engineering mathematics at Sacramento Junior College.

Edgar D. Meacham, professor of mathematics and Assistant Dean of the College of Arts and Sciences, University of Oklahoma, is spending his sabbatical year in study at the University of Chicago.

Dr. Dorothy McCoy has been appointed professor of mathematics at Belhaven College.

Mr. W. L. Porter, of Rice Institute, has been appointed professor of mathematics at the Texas Agricultural and Mechanical College.

Associate Professor J. C. Slater, of Harvard University, has been appointed head of the department of Physics at the Massachusetts Institute of Technology.

Miss Mildred E. Taylor has been appointed professor and head of the department of mathematics at Mary Baldwin College.

Dr. J. M. Thomas, assistant professor of mathematics at the University of Pennsylvania, has been appointed assistant professor of mathematics at Duke University.

Dr. L. H. Thomas, of Trinity College, has been appointed associate professor of theoretical physics at Ohio State University.

Dr. E. L. Thompson has been appointed assistant professor of mathematics at Fisk University, Nashville.

Professor R. A. Thornton, of Kittrell College, has been appointed head of the department of mathematical sciences at Talladega College.

Mr. C. E. Van Horn has been appointed head of the department of mathematics at Fisk University, Nashville.

Dr. H. S. Wall, of Northwestern University, has been promoted to an assistant professorship of mathematics.

Dr. Marie J. Weiss has been appointed assistant professor of mathematics at Sophie Newcomb College, Tulane University.

Dr. Charles Wexler has been appointed head of the department of mathematics at the Arizona State Teachers College.

Dr. G. T. Whyburn has been appointed associate in mathematics at Johns Hopkins University.

The following appointments to instructorships are announced:

University of Alabama: Mr. H. S. Thurston.

University of Arkansas: Mr. Paul Cramer.

Colorado School of Mines: Mr. Myron C. Pawley.

Long Island University: Mr. A. J. Smith.

Professor W. H. Bristol, inventor and formerly professor of mathematics at Stevens Institute of Technology, has died at the age of seventy.

THE CHAUVENET PRIZE

In the year 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period of 1929-1931.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS.
INDIANA.
IOWA.
KANSAS—Topeka, Kansas, Jan. 24.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO—Columbus, Ohio, April 2.
PHILADELPHIA—Philadelphia, Pa., Nov. 28.
ROCKY MOUNTAIN.
SOUTHERN CALIFORNIA.
TEXAS—Fort Worth, Texas, Jan.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 2, FEBRUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE FOURTEENTH ANNUAL MEETING OF THE MISSOURI SECTION

The fourteenth annual meeting of the Missouri Section of the Mathematical Association of America was held at the University of Missouri, Columbia, Mo. on Friday morning, November 28, 1930. The session was presided over by the chairman, Professor Louis Ingold, and by the vice-chairman, Professor Eugene Stephens, who took the chair while Professor Ingold presented a paper.

The meeting was followed by sessions of the American Mathematical Society on Friday afternoon and Saturday morning. The Missouri Alpha Chapter of Pi Mu Epsilon entertained with a tea on Friday afternoon, and on Friday evening a dinner was held at the Hotel Tiger.

There were present eighteen persons, including the following fifteen members of the Association: R. S. Christian, R. R. Fleet, Byron Ingold, Louis Ingold, G. H. Jamison, Paul Muehlman, W. O. Pennell, A. D. Pierson, P. R. Rider, W. H. Roever, Eugene Stephens, G. B. Sweazey, G. E. Wahlin, W. D. A. Westfall, E. Kathryn Wyant.

The following officers were chosen for the coming year: Chairman, W. H. Roever, Washington University; Vice-chairman, A. D. Pierson, Kansas City Junior College; Secretary-Treasurer, P. R. Rider, Washington University. It was decided to hold the 1931 meeting at Washington University, St. Louis, at the time of the meeting of the Missouri State Teachers Association in November.

The following program was presented:

I. *Papers*

1. "A visualization of homogeneous coordinates" by Professor W. H. Roever, Washington University.
2. "External Brocard points of a triangle" by Professor Louis Ingold, University of Missouri.

II. *A Symposium on Advanced College Courses in Algebra*

3. "Advanced algebra for the undergraduate" by Professor G. H. Jamison, Northeast Missouri State Teachers College.
4. "Algebra as an instrument of research" by Professor G. E. Wahlin, University of Missouri.
5. General discussion.

Professor Roever's paper was discussed by Professor Louis Ingold. Professor Jamison's paper led to an interesting discussion by Professors Stephens, Westfall, and Louis Ingold, and Mr. Pennell. Professor Wahlin's paper was commented on by Professor Rider. The general discussion of the symposium was participated in by Professors Fleet, Westfall, Stephens, Wahlin, and Wyant.

Abstracts of the papers follow:

1. In this paper Professor Roever showed (1) how a student may be introduced to the notion of homogeneous coordinates by citing the familiar example

of direction cosines and thus illustrating the general idea of coordinates proportional to quantities which satisfy an identical relation, and then leading up to homogeneous point and line coordinates of various kinds; and (2) how to obtain a geometric picture of such coordinates in spaces of one, two, and three dimensions which brings into evidence not only the quantities which satisfy the identical relations but also interprets geometrically these relations. Thus it is possible to set up and visualize a linear transformation of coordinates from one frame of reference to another or from one position to another with respect to a common frame of reference. Even the cases of singular transformations may be adequately illustrated by these methods.

2. Professor Ingold's paper is concerned with a point P in the plane of a triangle ABC such that the angles PBA and PBC are each equal to the supplement of the angle PAC. The cubic equation for determining the tangent of this angle is derived. If A and B are interchanged this cubic equation is unaltered though the resulting points P have different position. Two other cubic equations are obtained by interchanging A with C or B with C.

3. Professor Jamison stated that with analytic geometry and calculus as a background advanced algebra should develop analytic and interpretive power, introduce the student to the intricate notation of algebra, and give him a strong foundation in it. An intensive study of the number system with emphasis on complex numbers, the use of De Moivre's theorem, the graph and trigonometry in the solution of equations, symmetric functions, determinants with application to analytic and modern geometry and the usual chapter on theory of equations are the main topics for such a course.

4. Professor Wahlin called attention to the relations existing between various branches of mathematics. Various examples of important problems which illustrate this interrelation were given. He further called attention to those special topics of algebra which are of importance to a person wishing to do research in any branch of mathematics.

PAUL R. RIDER, *Secretary*

THE FIFTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The fifth annual meeting of the Philadelphia Section of the Mathematical Association of America was held in Bennett Hall at the University of Pennsylvania on Saturday, November, 29, 1930. Because of the unavoidable absence of Professor Miller of Swarthmore College, Professor Mitchell of the University of Pennsylvania presided at the forenoon session. Professor Wilson of Haverford College presided at the afternoon session.

The attendance was thirty-nine, including the following twenty-five members of the Association: P. A. Caris, G. G. Chambers, J. W. Clawson, Mary L. Constable, E. S. Crawley, J. E. Davis, Arnold Dresden, Tomlinson Fort,

Michael Goldberg, G. A. Harter, J. R. Kline, P. A. Knedler, H. M. Lufkin, Edith A. McDougle, D. L. McDonough, A. E. Meder, Jr., H. H. Mitchell, F. W. Owens, Helen B. Owens, C. J. Rees, George Rosengarten, J. A. Roulton, I.M. Sheffer, J. A. Shohat, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Professor Tomlinson Fort, Lehigh University; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Fort (ex-officio), Clawson and Frink. The next meeting of the Section will be held on Saturday, November 28, 1931, at the University of Pennsylvania.

The following papers were presented:

1. "On orthogonal Tchebycheff polynomials" by Dr. J. A. Shohat, University of Pennsylvania.

2. "A polar reciprocation of the complete quadrilateral" by Professor J. W. Clawson, Ursinus College.

3. "Some remarks on non-analytic functions" by Professor I. M. Sheffer, Pennsylvania State College.

4. "Almost-periodic functions" by Professor Tomlinson Fort, Lehigh University. Abstracts of the papers follow:

1. Definition, existence and uniqueness of orthogonal Tchebycheff polynomials; distribution of their roots. Examples: polynomials of Legendre, Jacobi, Laguerre, Hermite. Some minimum properties of Tchebycheff polynomials. Applications to: (a) expansion of functions in series of orthogonal polynomials; (b) mechanical quadratures; (c) polynomials in general.

2. Professor Clawson described a polar reciprocation of a complete quadrilateral and some of its related points and lines with respect to a circle having the focal (Steiner, Miquel) point of the quadrilateral for center. The quadrangle having for vertices the centers of the four circles circumscribing the triangles determined by three of the sides of the quadrilateral has for its reciprocal a second quadrilateral. The new quadrilateral is inversely similar to the original quadrilateral, and is homothetic to the inverse of the original quadrilateral with respect to the same circle; these three quadrilaterals possess a common focal point and common incentric lines. Moreover the quadrangle whose vertices are the circumcenters of the second quadrilateral is the polar reciprocal of the original quadrilateral.

3. Non-analytic functions of a complex variable are, as their name suggests, those functions for which a unique derivative does not exist (in general) at a point. Rather to each direction at a point a unique derivative exists, and (Kasner) when all directions at a point are taken the resulting derivative-locus is a circle. The property of conformality no longer holds for such functions but to each direction at a point corresponds a second direction forming a pair of dual directions, such that dual directions are transformed conformally. Especially important are the orthogonal dual pair and the self-dual directions. Dual pairs are geometrically intimately related to the Kasner circle. There is also an algebraic aspect of these functions in terms of certain matrices of the second order;

and a general process of differentiation of matrices when applied to these matrices yields known formulas for differentiation of non-analytic functions.

4. Professor Fort discussed the fundamental notions of the papers by Harald Bohr published in volumes 45, 46, and 47 of *Acta Mathematica*. He touched briefly upon work of other authors as Besicovitch, Weyl, and Wiener.

P. A. CARIS, *Secretary*

CORRELATION COEFFICIENTS AND TRANSFORMATION OF AXES

By MARIAN A. WILDER, University of Minnesota

It is well known that correlation coefficients can be interpreted as cosines of angles. One form of interpretation comes directly from the fundamental formulas of analytical geometry in space of n dimensions.[1] Another form of interpretation is based on the simplification of the exponent in the equation for the normal correlation surface.[2] This treatment in terms of reduction of the quadratic form is based on the assumption that the distribution is normal, while the first method does not require that the distribution be restricted in form at all. The purpose of this paper is to show that a calculation essentially equivalent to that involved in the reduction of the quadratic form can be carried through for the completely general distribution without assuming that it be normal.

Let $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)$ and $(z_1, z_2, \cdots, z_n)$ be three sets of n real numbers each, in which the variables are deviations from their respective arithmetic means, or such that

$$\sum x_k = \sum y_k = \sum z_k = 0.$$

(Throughout this paper the sign $\sum$ stands for a summation from $k=1$ to $k=n$.) We shall also assume that no set consists entirely of zeros, and that the three sets are not linearly dependent.

For each k, let

$$\zeta_k = z_k,$$
$$\eta_k = y_k - \beta\zeta_k,$$
$$\xi_k = x_k - \alpha_1\eta_k - \alpha_2\zeta_k,$$

where

$$\beta = \sum y_k\zeta_k/\sum\zeta_k^2, \qquad \alpha_1 = \sum x_k\eta_k/\sum\eta_k^2, \qquad \alpha_2 = \sum x_k\zeta_k/\sum\zeta_k^2.$$

---

[1] D. Jackson, *The elementary geometry of function space*, this Monthly, vol. 31 (1924), pp. 461–471; *The geometry of frequency functions*, this Monthly, vol. 31 (1925), pp. 63–73; *The Theory of Approximation*, New York (1930), pp. 154–169; *The trigonometry of correlation*, this Monthly, vol. 31 (1924), pp. 275–280.

[2] James McMahon, *Hyperspherical goniometry and its application to correlation theory for N variables*, Biometrika, vol. 15 (1923), pp. 173–208; Karl Pearson, *Excess and its relation to correlation*, Philosophical Transactions, Series A, vol. 197 (1901), pp. 18–20.

Then ξ, η, ζ satisfy the conditions $\sum \xi_k \eta_k = \sum \xi_k \zeta_k = \sum \eta_k \zeta_k = 0$, as is readily seen by substitution.

Now let

$$\begin{aligned} \lambda_k &= \xi_k (n / \textstyle\sum \xi_k^2)^{1/2}, \\ \mu_k &= \eta_k (n / \textstyle\sum \eta_k^2)^{1/2}, \\ \nu_k &= \zeta_k (n / \textstyle\sum \zeta_k^2)^{1/2}. \end{aligned}$$

Then λ, μ, ν satisfy the conditions of orthogonality and have standard deviations equal to one. We can now solve for x, y, and z in terms of λ, μ, ν, giving equations of the form

$$\begin{aligned} x_k &= a_1\lambda_k + a_2\mu_k + a_3\nu_k, \\ y_k &= \qquad\quad\; b_2\mu_k + b_3\nu_k, \\ z_k &= \qquad\qquad\qquad\;\; c_3\nu_k. \end{aligned} \tag{1}$$

It will be noted that the coefficients a_1, b_2, c_3 are positive.

If the numbers $(\lambda_k, \mu_k, \nu_k)$ are plotted as sets of coordinates with respect to a system of rectangular axes, the quantities (x_k, y_k, z_k) may be regarded as coordinates of the same points in an oblique coordinate system, with suitable scales of measurement along the axes. The equations of transformation,

$$\begin{aligned} x &= a_1\lambda + a_2\mu + a_3\nu, \\ y &= \qquad\quad b_2\mu + b_3\nu, \\ z &= \qquad\qquad\qquad c_3\nu, \end{aligned} \tag{2}$$

are adjusted to the values of the original x's, y's, and z's so that the points having the oblique coordinates (x_k, y_k, z_k) are represented by rectangular coordinates with zero correlations and unit standard deviations. We show that the angles of the oblique coordinate system then are related to the coefficients of correlation of the original x's, y's, and z's.

Specifically, we calculate the angles between the normals to the planes $x=0$, $y=0$, $z=0$. The direction cosines of the normals are

$$\begin{matrix} a_1/d_1, & a_2/d_1, & a_3/d_1; \\ 0\;\;, & b_2/d_2, & b_3/d_2; \\ 0\;\;, & 0\;\;, & c_3/d_3; \end{matrix}$$

where

$$d_1 = (a_1^2 + a_2^2 + a_3^2)^{1/2}, \; d_2 = (b_2^2 + b_3^2)^{1/2}, \; d_3 = c_3.$$

As to algebraic sign, the first three direction cosines are those of the ray from the origin toward the point with the rectangular coordinates (a_1, a_2, a_3). By the first of equations (2) the x-coordinate of this point is $a_1^2 + a_2^2 + a_3^2 > 0$. So the normal points toward the side of the plane $x=0$ where x is positive. Similarly, the other normals are on the same side of the corresponding coordinate planes

as the positive y and z axes respectively. The cosine of the angle between the planes is given by the sum of the products of the direction cosines. Hence the cosine of the angle between the planes $x=0$ and $y=0$ is $(a_2b_2+a_3b_3)/d_1d_2$; between the planes $x=0$ and $z=0$ is a_3c_3/d_1d_3; and between the planes $y=0$ and $z=0$ is b_3c_3/d_2d_3; the angle in each case being that between the normals specified. But, from equations (1), together with the conditions satisfied by the λ's, μ's, and ν's,

$$\begin{aligned}
(1/n)\sum x_k^2 &= a_1^2 + a_2^2 + a_3^2 = d_1^2,\\
(1/n)\sum y_k^2 &= b_2^2 + b_3^2 = d_2^2,\\
(1/n)\sum z_k^2 &= c_3^2 = d_3^2,\\
(1/n)\sum x_k y_k &= a_2b_2 + a_3b_3,\\
(1/n)\sum x_k z_k &= a_3c_3,\\
(1/n)\sum y_k z_k &= b_3c_3.
\end{aligned}$$

Therefore the cosines of the angles between the normals are

$$\begin{aligned}
\sum x_k y_k/(\sum x_k^2 \sum y_k^2)^{1/2} &= r_{xy},\\
\sum x_k z_k/(\sum x_k^2 \sum z_k^2)^{1/2} &= r_{xz}\\
\sum y_k z_k/(\sum y_k^2 \sum z_k^2)^{1/2} &= r_{yz}.
\end{aligned}$$

Geometric interpretations of coefficients of partial correlation, double correlation, etc., then follow as in the other treatments referred to.

These facts are perhaps rather closely suggested by the formulas of McMahon's article, but the logical development there is based throughout on the assumption that the distribution is normal.

It may be noticed that equations (1) are of the form

$$\begin{aligned}
x_k &= a_1\lambda_k + a_2\mu_k + a_3\nu_k,\\
y_k &= b_1\lambda_k + b_2\mu_k + b_3\nu_k,\\
z_k &= c_1\lambda_k + c_2\mu_k + c_3\nu_k.
\end{aligned}$$

The fact that $b_1=c_1=c_2=0$ in the particular transformation given here is immaterial. Representations of the same form, with different coefficients, are possible in terms of an infinite variety of sets of λ's, μ's, and ν's, where the new λ, μ, ν axes would be a rotation of the old ones.

By the use of a suitable notation, the method can be carried over without essential change to the case of m statistical variables and m dimensions.

A CALCULUS OF VARIATIONS PROBLEM WHOSE EXTREMALS ARE PARABOLAS

By PEARL BIERMAN and H. A. SIMMONS, Northwestern University

1. *Introduction.* The purpose of this paper is to discuss a problem which is highly illustrative of much of the elementary theory of the calculus of variations. In fact our problem is a good companion of the catenary problem in that they both illustrate essentially the same features of the calculus of variations theory. However, our problem involves *simpler algebra* than does the catenary problem, and defines *algebraically* many quantities which in the catenary problem are necessarily defined *transcendentally.* For example, in this paper conjugate points are found algebraically by solving two simultaneous quadratics and the MacNeish curve is found in a rational algebraic form.

We endeavor here to make a discussion that is essentially a parallel of the much appreciated treatment which Professor Bliss[1] has given of the catenary problem. We hope that the use of this method in connection with the problem here treated will be valuable not only to beginning students of the calculus of variations but also to other individuals who have found the algebra of this subject rather complicated and have not enjoyed a genuine appeal of the elementary portion of the calculus of variations theory.

The integral with which we are principally concerned is the case $n=1/2$ of the integral of Euler,[2]

$$I = \int_{x_1}^{x_2} y^n(1 + y'^2)^{1/2}dx,$$

and belongs therefore to a family of calculus of variations problems which includes such famous ones as the catenary $(n=1)$ and the brachistochrone $(n=-1/2)$. As indicated in footnote 2 above, the integral which we use most has received some attention, though we believe our treatment of it is the first one that describes satisfactorily the resemblance of this problem to the catenary problem.

2. *Statement of the problem.* Suppose that light travels in the xy-plane and that in this plane the index of refraction, $n(x, y)$, is directly proportional to the square root of the distance of the point (x, y) from the x-axis; then the velocity v of the light satisfies a relation of the form $v(x, y)=kn(x, y)=k/y^{1/2}$, where k is a suitably chosen constant. Further, let us call a curve $y=y(x)$ *admissible* if it has $y\geqq 0$, is continuous, and has a continuously turning tangent except possibly at a finite number of points. Our problem, then, is to find among all admissible curves through $1(x_1, y_1)$, $2(x_2, y_2)$, one along which light will pass from 1 to 2 in the shortest time. Since $ds/dt=k/y^{1/2}$, we have when $x_2>x_1$:

[1] Cf. Carus Mathematical Monograph No. 1, *Calculus of Variations*, by G. A. Bliss. We are indebted to Professor Bliss for a few suggestions concerning the present problem.

[2] Cf. Bolza's *Vorlesungen Über Variationsrechnung,* p. 145, ex. 11, and Goursat's *Cours d'Analyse Mathématique*, (1915), vol. 3, p. 555.

$$t = k\int_{x_1}^{x_2} y^{1/2}(1 + y'^2)^{1/2}dx.$$

Obviously $t>0$ and so $k>0$; we seek then to minimize the integral

$$I = \int_{x_1}^{x_2} y^{1/2}(1 + y'^2)^{1/2}dx \tag{1}$$

by properly determining the function $y=y(x)$, which satisfies the relations

$$y_1 = y(x_1),\ \ y_2 = y(x_2). \tag{2}$$

Later we shall consider the case $x_2 \geqq x_1$ and admit a general class of parametric curves.

3. *Proof that the minimizing arc is a parabola.* The integrand function in (1) is $f \equiv y^{1/2}(1+y'^2)^{1/2}$, and so $f_{y'y'} = [y^{1/2}(1+y'^2)^{-3/2}]>0$ for all points above the x-axis. Hence if we consider only admissible curves which lie in the region $y>0$, every solution of Euler's equation for our problem will have continuous curvature and will therefore not have a corner; such an arc will frequently be referred to as an *extremal.* From the expression for I, y could not be negative; the case in which $y=0$ at one or more points of the interval $x_1 \leqq x \leqq x_2$ will be discussed later. Since the curves under consideration have continuous curvature, Euler's differential equation, which must be satisfied (Cf. Bliss, p. 49),[3] has the consequence $f-y'f_{y'}=c$, or $y'=(y-c^2)^{1/2}/c$, where $c \neq 0$ is a constant of integration. Hence the extremals for our problem when $x_2>x_1$ and $y>0$ are defined by[4]

$$\pm\, 2(\alpha y - \alpha^2)^{1/2} = x - d,\ \ \alpha \equiv c^2 > 0; \tag{3}$$

or, by the equivalent of (3),

$$4\alpha(y - \alpha) = (x - d)^2. \tag{4}$$

The extremals are therefore parabolas with axes parallel to the y-axis and with the x-axis as directrix. Of (3) and (4), we shall in the future use the equation which best serves our convenience. We express the above result in

Theorem 1. If 1 and 2 are two given points in the half-plane $y>0$ and if $y=y(x)$ is an admissible arc E_{12} joining them and is a minimum time curve for our problem, then E_{12} is a single arc (without corners) of one of the parabolas (4).

4. *The number of parabolas of family* (4) *which pass through two given points.* If a parabola of (4) passes through 1 and 2, then by (2),

$$4\alpha(y_1 - \alpha) = (x_1 - d)^2,\ 4\alpha(y_2 - \alpha) = (x_2 - d)^2, \tag{5}$$

from which

---

[3] We shall use this method of referring to the first Carus Monograph.

[4] c is real in $f-y'f_{y'}=c$ since $y', f, f_{y'}$ are supposed to be real functions.

$$d = -2\alpha(y_2 - y_1)(x_2 - x_1)^{-1}\tfrac{1}{2}(x_1 + x_2). \tag{6}$$

From (5) and (6), we obtain two pairs of values for d, α. Setting $\gamma = 4y_1y_2 - (x_2-x_1)^2$, we may express the solutions for α as follows:

$$\alpha_1 = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2 + \sqrt{\gamma})^{-1}, \quad \alpha_2 = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2 - \sqrt{\gamma})^{-1}. \tag{7}$$

The corresponding values for d_1, d_2 are of course to be obtained by substitution from (7) in (6). The number of parabolas of (4) that pass through 1 and 2 then depends upon the character of γ. We express our result in

Theorem 2. The exact number of parabolas of family (4) that pass through two fixed points 1 and 2 is zero when $\gamma < 0$, one when $\gamma = 0$, and two when $\gamma > 0$.

5. *The one-parameter family of parabolas through a given point.* In the two-parameter family of extremals (3), there is a one-parameter family which passes through 1 (Cf. Bliss, p. 92) and is given by

$$y = y_1 + \tfrac{1}{4}\alpha^{-1}[(x - x_1)^2 \pm 4(x - x_1)(\alpha y_1 - \alpha^2)^{1/2}] \quad (a \neq 0). \tag{8}$$

From the manner in which (8) was obtained, it must contain all of the parabolas of (3) that pass through 1, and therefore all of them that pass through 1 and 2. Hence (8) must be satisfied by the values x_2, y_2, α_i $(i = 1, 2)$. This requires that the $-$ sign before the radical in (8) be used when $\alpha = \alpha_1$; when $\alpha = \alpha_2$, it requires the $+$ sign if $\sqrt{\gamma} > 2y_1$ and the $-$ if $\sqrt{\gamma} < 2y_1$; when $\sqrt{\gamma} = 2y_1$, the sign is immaterial. We shall refer to these facts as the sign restrictions on (8). To distinguish carefully between the properties of the parabolas of (8) that are determined by α_1 and those that are defined by α_2 is one of our important tasks.

To obtain the envelope of the curves (8), we eliminate α between (8) and the equation $y_\alpha = 0$:

$$y_\alpha = \tfrac{1}{4}\alpha^{-2}\{-(x - x_1)[(x - x_1)(\alpha y_1 - \alpha^2)^{1/2} \pm 2\alpha y_1]\} = 0 \quad (\alpha \neq 0). \tag{9}$$

Cancellation of the factor $x - x_1$ from the numerator of (9) merely loses the point $1(x_1, y_1)$ of the envelope; for all other points of the envelope, we have

$$\alpha = y_1(x - x_1)^2/[4y_1^2 + (x - x_1)^2]. \tag{10}$$

Hence α is a single-valued function of x along the envelope, and there by theorem 2, $\gamma = 0$. Using this fact in requiring that (8) be satisfied by the value of α that is given in (10), we find the equation of the sought envelope (apart from the point 1) to be

$$y = \tfrac{1}{4}y_1^{-1}(x - x_1)^2, \quad (x \neq x_1). \tag{11}$$

Where it would be trivial to mention 1 as a part of the envelope, we shall refer to (11) as the envelope G of family (8). The result which is now evident we express in

Theorem 3. The envelope of the one-parameter family of extremals (8) through the point 1 is the parabola (11). It is not one of the parabolas (8).

The last two theorems together justify

(4) of Euler's differential equation for our problem contains exactly one extremal through the point 1 in every direction. Hence family (8) has this property. We shall observe this fact below in showing how a parabola of (8) with parameter value α varies with α.

Let $x = x_2$ be a fixed value greater than x_1; substitute α_1 for α in (8); and let α_1 increase from 0 to its maximum value

$$\left\{y_1(x_2 - x_1)^2[4y_1^2 + (x_2 - x_1)^2]\right\} = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2)^{-1}$$

on G. Then y_2 decreases continuously from ∞ (when $\alpha = 0$) to $\frac{1}{4}(x_2 - x_1)^2 y_1^{-1}$ on G. While $y = y_2$ is so decreasing, the slope at 1 of the parabola with parameter value α_1 increases continuously from $-\infty$ to $-2y_1(x_2 - x_1)^{-1}$. These limiting values of y' may be found as follows. At $x = x_1$, $dy/dx = -(\alpha y_1 - \alpha^2)^{1/2}\,\alpha^{-1}$, whose limiting value as α approaches zero is $-\infty$; when $\alpha = \alpha_1 = \frac{1}{4}(x_2 - x_1)^2 (y_1 + y_2)^{-1}$, on G, the expression for y' becomes

$$-\left[\frac{y_1(x_2 - x_1)^2}{4(y_1 + y_2)} - \frac{(x_2 - x_1)^4}{16(y_1 + y_2)^2}\right]^{1/2} \div \frac{(x_2 - x_1)^2}{4(y_2 + y_2)}$$
$$= -\frac{[4y_1^2 + 4y_1y_2 - (x_2 - x_1)^2]^{1/2}}{x_2 - x_1} = -\frac{(4y_1^2 + \gamma)^{1/2}}{x_2 - x_1} = \frac{-2y_1}{x_2 - x_1}.$$

Similarly one can show that if the $-$ sign is used before the radical in (8) while $\alpha = \alpha_2$, then as α_2 increases continuously from its value on the envelope G, namely $\frac{1}{4}(x_2 - x_1)^2(y_1 + y_2)^{-1}$, to y_1, the value of y_2 will increase continuously from $\frac{1}{4}(x_2 - x_1)^2 y_1^{-1}$ to $y_1 + \frac{1}{4}(x_2 - x_1)^2 y_1^{-1}$, respectively. While $y = y_2$ is so increasing, the slope at 1 of the parabola with parameter value α_2 increases continuously from $-2y_1(x_2 - x_1)^{-1}$ to 0. Then if we use the $+$ sign before the radical of (8) and let $\alpha = \alpha_2$ decrease continuously from y_1 to 0, $y = y_2$ increases from $y_1 + \frac{1}{4}(x_2 - x_1)^2 y_1^{-1}$ to ∞, respectively. While y_2 is so increasing, the slope at 1 of the parabola with parameter value α_2 increases from 0 when $\alpha_2 = y_1$ to ∞ when $\alpha_2 = 0$.

Now, as in §6, let E', E'' be particular parabolas of (8) defined by the parameters α_1, α_2, respectively. When $\alpha = \alpha_1$ and 2 is above G, the equations (8) and (11) have a simultaneous solution (x, y) such that $x_1 < x < x_2$, as can be seen by examination of the abscissa of the point P_1 mentioned in §6. Hence E' touches G between 1 and 2. When $\alpha = \alpha_2$ and 2 is above G, equations (8) and (11) have no such solution, as can be seen from the abscissa of the point P_2 of §6. Hence E'' does not touch G between 1 and 2. If 2 is on G, (8) and (11) of course have a simultaneous solution at 2 regardless of which parameter $\alpha_i (i = 1, 2)$ is used.

We summarize the results of this section in

Theorem 5. In every direction through 1 there passes one, and only one, parabola of (8). When 2 is above G and $x_2 > x_1$, every parabola E' defined by α_1 touches G between 1 and 2, while no parabola E'' defined by α_2 so touches G.

If $x_2 > x_1$ and 2 is on G, E', and E'' coincide and touch G at 2. If $x_2 > x_1$ and 2 is below G, we have the case of Theorem 2 when $\gamma < 0$.

8. *The envelope theorem and Jacobi's condition.* To make the investigation which is next in order, we need to use the notion of a *field* and the Hilbert invariant integral. A *field* is a region of the xy-plane through every point of which there passes one, and only one, extremal of a one-parameter family of extremals all of which meet a fixed curve. If we consider only the portion of each extremal in (8) from the point 1 to the point to the right of 1 where it touches G, the region of the xy-plane for which $x > x_1$ and above G will form a field F since a portion of one of the two extremals of (8) [Cf. Cor. 1] through any point 2 above G and to the right of 1 will have been disregarded and since all of the extremals of (8) meet G. This implies that when the parameter value α in (8) is assigned and the proper sign before the radical of (8) is used, the slope function $p(x,y) = dy/dx$, obtained by differentiating (8), is uniquely defined at every point (x, y) of F. The Hilbert invariant integral for our problem is then defined by the equation

$$I^* = \int_a^b [f(y, p)dx + (dy - pdx)f_{y'}], \tag{12}$$

where a and $b > a$ are abscissas of points of $F, f(y, p) = y^{1/2}(1+p^2)^{1/2}$, y is defined by (8) when in this equation a particular α is used in accordance with our sign restrictions (Cf. §5), and dx, dy are differentials of the functions $x = x(t)$, $y = y(t)$, $t_1 \leqq t \leqq t_2$, that define in F a curve C which passes through two points with abscissas a, b and has a continuously turning tangent except possibly at a finite number of points. By adjoining the single point $1(x_1, y_1)$ to F, we form an improper field F'. After suitable assumptions[6] are made as to the values of $p(x, y)$, $\alpha(x, y)$ at 1, it is known that I^* has in F' the two fundamental properties that it has in F, namely: it is independent of the path of integration; its value along an extremal of F' from $x = a$ to $x = b > a$ is equal to $I(E_{ab})$.

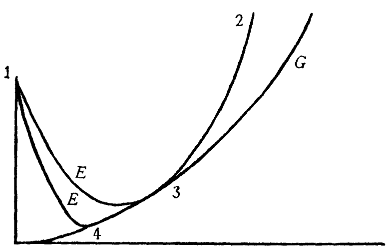


FIG. 2

Since G is a parabola and has a branch projecting to the left from any one of its points 3, between 1 and 2, our problem is one for which customary procedure (Cf. Bliss, pp. 140–141) with the integrals I and I^* in F' gives the following envelope theorem.

Theorem 6. If two parabolas E_{14} and E_{13} of family (8) touch G at 4 and 3, respectively (Fig. 2), where $x_4 < x_3$, then the time curves $E_{14} + G_{43}$ and E_{13} give equal values to I: $I(E_{14}) + I(G_{43}) = I(E_{13})$.

From Theorem 6 and the fact $f_{y'y'} > 0$ when $y > 0$, the following corollary is obvious.

---

[6] Cf. *Lectures on the Calculus of Variations*, by O. Bolza, p. 83, footnote 2.

Corollary 2. (Jacobi's condition). If a parabolic arc E_{12} is to afford a minimum time curve for our problem, the point of contact 3 of E_{12} with G must not lie on the arc E_{12}.

From Corollary 2 we know that the extremal E' mentioned in Theorem 5 can never afford a minimum time curve for our problem. In fact, since G_{43} is not an extremal arc, it is known that G_{43} of the last theorem can be replaced by an arc C_{43} such that $I(E_{14})+I(C_{43})<I(E_{13})$ [Cf. Bliss, p. 141]. Hence if our problem has a solution, it must be afforded by the curve E'' mentioned in Theorem 5. We shall investigate the minimizing properties of E'' in fields of the types which are described in Bliss, §§40 and 44.

9. *Construction of a field in a V-shaped region.* We have seen that the arc E_{12} of the parabola E'' mentioned in theorem 5 contains no point conjugate to 1. Let $d=d_2$, when $\alpha=\alpha_2$ [Cf. (6), (7)]. Then the equation of E'', taken from (4), is

$$y - \alpha_2 = \tfrac{1}{4}\alpha_2^{-1}(x - d_2)^2. \tag{13}$$

Let 0 be a point of E'' [Fig. 3] to the left of 1, but so close to 1 that the point 3 of E'' which is conjugate to 0 is still to the right of 2. The tangents to E'' at 0 and 3 meet on the x-axis at a point 4. The transformation

$$x - x_4 = \alpha_2\alpha^{-1}(X - x_4),\quad y = \alpha_2\alpha^{-1}Y, \tag{14}$$

stretches the plane along radial lines through 4 in such a way that every point (x, y) is replaced by a point (X, Y) whose distance from 4 is $\alpha\alpha_2^{-1}$ times that of (x, y). Substituting the values x, y of (14) into (13), we find that the points (x, y) of E'' are transformed into points (X, Y) which satisfy the equation

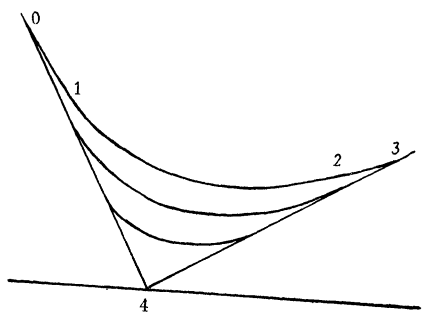


FIG. 3

$$y - \alpha = \tfrac{1}{4}\alpha^{-1}[x - x_4 + \alpha\alpha_2^{-1}(x_4 - d_2)]^2. \tag{15}$$

For a fixed value of α (15) represents a parabola of family (4) with parameters α and $d=x_4-\alpha\alpha_2^{-1}(x_4-d_2)$. Letting α vary, we see that (15) represents a one-parameter family of parabolas containing E_{03} for $\alpha=\alpha_2$ and tangent to the lines joining 4 to 0 and 3. Through each point (x, y) of the V there passes a unique extremal of family (15), and all members of the family intersect any straight line which radiates from 4 through the V. We have now constructed a field F which is analogous to the field described in Bliss, §40.

10. *Properties of the field functions.* Before we can make a sufficiency proof we must assure ourselves that certain continuity properties are possessed by

the field functions which enter into the Hilbert invariant integral (12), namely $\alpha=\alpha(x, y)$, which in (15) is the ordinate of the vertex of the parabola whose parameter value is α, and

$$p = p(x, y) = \partial y(x, \alpha)/\partial x = y'[x, \alpha(x, y)] = \tfrac{1}{2}\alpha^{-1}[x - x_4 + \alpha\alpha_2^{-1}(x_4 - d_2)],$$

which is the slope of the extremal E of (15) through the point (x, y) of F. It is obvious geometrically that α, p are uniquely defined in F. We wish to prove them *continuous* in F. If we prove the property for α, we shall have it also for p, as is seen from the definition of p. Differentiating (15) with respect to α and making a few simplifications in the result, we find

$$y_\alpha = \tfrac{1}{4}\alpha_2^{-2}[4\alpha_2^2 + (x_4 - d_2)^2] - \tfrac{1}{4}\alpha^{-2}(x - x_4)^2;\ \alpha \neq 0.$$

We assume $\alpha\neq 0$ here and thus exclude the vertex 4 of the V from consideration; this does not place an additional restriction upon our problem as we have supposed from the beginning of this paper that $y>0$ when our extremals are of the form $y=y(x)$. We see from the fact that y_α vanishes for only one admissible value of α, namely

$$[\alpha_2^2(x - x_4)^2\{4\alpha_2^2 + (x_4 - d_2)^2\}^{-1}]^{1/2},$$

and changes signs at this value, that y_α vanishes only on the boundary of V. From this fact, it can be proved by a method used in Bliss [Cf. pps. 107–108], that in the interior of V the function $\alpha(x, y)$ is continuous and has continuous partial derivatives of the first order with respect to x and y. Furthermore, by methods which are used in Bliss [pp. 60–62], it can be proved that α is continuous on the boundary of F except at the vertex. We omit these proofs because they are of the same character as the analogous proofs in Bliss.

11. *The sufficiency theorem.* That the extremal arc E_{12} of E'' is the minimizing admissible time curve $y=y(x)$ in F and joining the points 1, 2 can be proved in the same way that an analogous proof is made in Bliss [pp. 108–109]; we merely state the result:

Theorem 7. An admissible arc $y=y(x)$ on the interval $x_1\leqq x\leqq x_2$ with $x_2>x_1$, in the half-plane $y>0$, joining two fixed points 1 and 2, and affording a minimum time curve for our problem must be a single arc, without corners, of one of the parabolas (4) and must not have a point of contact with the envelope G, $y=\frac{1}{4}y_1^{-1}(x-x_1)^2$ of the one-parameter family of parabolas (8) through 1. Giving α the value α_2 of (7) and determining the corresponding value of $d=d_2$, by (6), we obtain an arc E_{12} with these two properties. If then F is one of the V-shaped regions described above, containing E_{12} in its interior and bounded by two tangents to E_{12} which meet on the x-axis, and if light travels as we have supposed, it will pass from 1 to 2 in a shorter time along E_{12} than it will along any other arc C_{12} with equations $x=x(t)$, $y=y(t)$, $t_1\leqq t\leqq t_2$, which lies in F, passes through 1, 2 and has a continuously turning tangent on t_1t_2 except possibly at a finite number of points.

This result can also be obtained by specializing a general theorem concerning *regular problems* for which the envelope analogous to G is of a particular type [Cf. Bliss, p. 161]. Indeed the conditions denoted by I, III, and IV′ in Chapter V of Bliss are both necessary and sufficient for a strong relative minimum in our problem.

Hereafter we shall refer to comparison arcs C_{12} of the type that was mentioned in Theorem 7 as *parametric arcs.*

12. *Discontinuous solutions.* The argument given in Bliss, §43, applies to our problem. We state our result in

Theorem 8. If a vertical straight line E_{12} has its upper end-point 1 in common with a parametric arc C_{13} whose length equals that of E_{12}, as shown in Fig. 4, then the time required by light of the type that we hypothesize to pass from 1 to 2 along E_{12} is less than that required to pass from 1 to 3 along C_{13} unless C_{13} coincides with E_{12}.

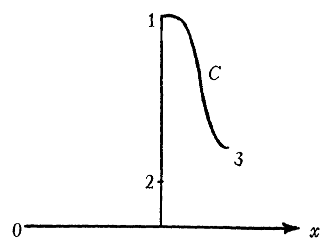


FIG. 4

Immediate consequences of Theorem 8 are the corollaries below.

Corollary 3. When 1 and 2 are in the same vertical line $x=x_1$, the segment of this line that joins 1 and 2 affords an absolute minimum time curve for our problem.

Corollary 4. If 1 and 2 are not in the same vertical line and if C_{12} is a parametric arc of length $\geqq y_1+y_2$, joining 1 and 2 and lying in the region $y\geqq 0$, the time required by light of the type that we consider to pass from 1 to 2 along the line segments 13, 34, 42, whose equations are $x=x_1$, $y=0$, $x=x_2$, respectively, is at most as great as the corresponding time over C_{12}. The arc e_{12} consisting of the segments 13, 34, 42 has a neighborhood N of the type shown in Fig. 5 in which e_{12} is an absolute minimum time curve for our problem.

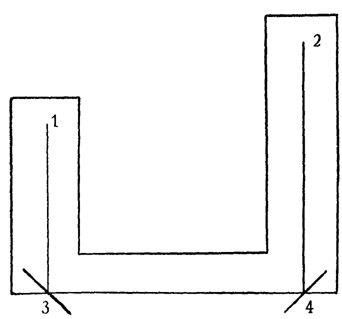


FIG. 5

In corollary 4 the time over 34 was taken to be zero since $v=ky^{-1/2}$.

The broken line e_{12} of Corollary 4 is the Goldschmidt discontinuous solution for our problem. We shall give further information about it later. It is obvious that in order to pass light from 1 to 2 along e_{12}, one would need to place at 3 and 4 [Fig. 5] mirrors inclined at angles of 135° and 45°, respectively, with the positive x-axis. These mirrors would need to be of a special character; that is, they should either have the same index of refraction as the medium in which

the light has been supposed to travel, or they should be so constructed that light would not penetrate into them to a depth $\neq 0$.

13. *The minimum in a second type of field.* The following theorem can be proved by the method which is used in Bliss, §§44–45.

Theorem 9. If E_{12} is the arc from 1 to 2 of the parabola E'' found above, joining 1 with 2 and having on it no point conjugate to 1 except possibly at 2, the time required by the light which we hypothesize to pass from 1 to 2 along E_{12} is less than the corresponding time along every other parametric arc, joining 1 and 2, and, except possibly at 2, lying entirely above the envelope G of the one parameter family of parabolas through 1, the line $x=x_1$ counting as one of the parabolas.[7]

14. *The absolute minimum.* Several further results concerning the Goldschmidt discontinuous solution of our problem are entirely analogous to results which are given in Bliss, §46, and are obtainable by the methods of that section. We merely state the results in question:

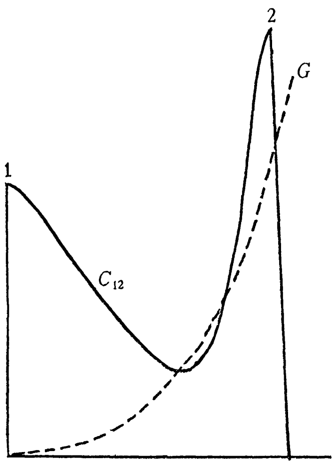


FIG. 6

Theorem 10. The time required by the light that we hypothesize to pass over every parametric arc C_{12} distinct from e_{12} and having a point 5 in common with the envelope G, as shown in Fig. 6, is greater than the time required on e_{12}.

Corollary 5. When there are fewer than two parabolas joining the points 1 and 2 the Goldschmidt solution e_{12} always furnishes an absolute minimum.

Corollary 6. When 2 is above G the parabolic arc E_{12} of Theorem 9 and e_{12} both furnish minima in sufficiently small neighborhoods, and the one over which our light passes in the shorter time provides an absolute minimum.

To distinguish the cases in which E_{12} gives the absolute minimum from those in which e_{12} gives this minimum, we shall obtain in a rational algebraic form the equation of a curve on which $I(e_{12})=I(F_{12})$ [Cf. the curve H in Fig. 7 below]. This curve will be called the MacNeish curve because it is an analog of a curve that was found by Mr. MacNeish[8] for the catenary problem.

When 2 is at 1, $I(E_{12})=0$ while

---

[7] The continuity of the extremal integral

$$I=\int_{x_1}^{x_3}[y(x,\alpha)]^{1/2}\,[1+y'^2(x,\alpha)]^{1/2}\,dx$$

when a point $3(x_3, y_3)$ varies on the interior and boundary of the field F of theorem 9 can be proved here by the method that is used in Bliss, §45.

[8] MacNeish, Annals of Mathematics, vol. 7 (1905), p. 72.

$$I(e_{12}) = 2\int_{y_1}^{y_2} [y(x, \alpha)]^{1/2} dy > 0, \quad (ds = dy),$$

so that the difference $I(E_{12}) - I(e_{12}) < 0$. For every position of 2 above or on G, except on the line $x = x_1$ where 2 was taken in §10, we have

$$I(E_{12}) - I(e_{12}) = \int_0^{s_2} y^{1/2} ds - \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}).$$

As 2 moves from 1 along a fixed parabola E of family (8), the derivative $y_2^{1/2}(1 - dy_2/ds_2)$ of the difference $I(E_{12}) - I(e_{12})$ is positive, since $|dy_2/ds_2| < 1$. When 2 is on G, $I(E_{12}) > I(e_{12})$ by theorem 10. Hence there is exactly one position 2 on the parabola E at which $I(E_{12}) = I(e_{12})$. We ask, then, what is the locus of points (x_2, y_2) which satisfy the equation

$$(16) \qquad \int_{x_1}^{x_2} [y(x, \alpha)]^{1/2} [1 + y'^2(x, \alpha)]^{1/2} dx = \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}),$$

where y is defined by (8) when $\alpha = \alpha_2$, which is the parameter value that defines the minimizing arc E_{12} of Theorem 7. Consequently the integrand of (16) is equal to

$$\alpha_2^{-3/2}[\alpha_2 y_1 + \tfrac{1}{4}(x - x_1)^2 \pm (x - x_1)(\alpha_2 y_1 - \alpha_2^2)^{1/2}],$$

the ambiguity in sign before the radical being only apparent, as we shall presently see. Using this expression for the integrand in (16) and carrying out the integration there, we find

$$(17) \; I(E_{12}) = \alpha_2^{-3/2}[\alpha_2 y_1(x_2 - x_1) + \tfrac{1}{12}(x_2 - x_1)^3 \pm \tfrac{1}{2}(x_2 - x_1)^2(\alpha_2 y_1 - \alpha_2^2)^{1/2}].$$

Now substituting in (17) for α_2 its value $\frac{1}{4}(x_2 - x_1)^2 (y_1 + y_2 - \sqrt{\gamma})^{-1}$ [Cf. (7)], we find, after simplification,

$$\begin{aligned} I(E_{12}) = 2y_1[y_1 + y_2 - \sqrt{\gamma}]^{1/2} &+ \tfrac{2}{3}(y_1 + y_2 - \sqrt{\gamma})^{3/2} \\ &\pm [y_1 + y_2 - \sqrt{\gamma}]^{1/2}[4y_1^2 - 4y_1\sqrt{\gamma} + \gamma]^{1/2}. \end{aligned}$$

Since

$$[4y_1^2 - 4y_1\sqrt{\gamma} + \gamma]^{1/2} = \sqrt{\gamma} - 2y_1 \text{ when } \sqrt{\gamma} > 2y_1,$$

and

$$= 2y_1 - \sqrt{\gamma} \text{ when } 2y_1 > \sqrt{\gamma},$$

and since the + or − sign in front of the radical of (8) is used according as $\sqrt{\gamma} > 2y_1$ or $\sqrt{\gamma} < 2y_1$, respectively, [Cf. §5], we have in all cases

$$I(E_{12}) = 2y_1[y_1 + y_2 - \sqrt{\gamma}]^{1/2} + \tfrac{2}{3}(y_1 + y_2 - \sqrt{\gamma})^{3/2} - (2y_1 - \sqrt{\gamma})[y_1 + y_2 - \sqrt{\gamma}]^{1/2}.$$

Using this result as the left member of (16), dropping the subscript 2, cancelling the factor $\frac{2}{3}$ from both members of the resulting equation, and letting $\gamma_1 = 4yy_1 - (x - x_1)^2$, we obtain

$$[y + y_1 - \sqrt{\gamma_1}]^{1/2}[y + y_1 + \tfrac{1}{2}\sqrt{\gamma_1}] = y^{3/2} + y_1^{3/2}.$$

Squaring both members of this equation, we obtain, after some simplification,

$$3(x - x_1)^2(y + y_1) - 8y^{3/2}y_1^{3/2} = \gamma_1^{3/2}.$$

Squaring this equation and simplifying again, we obtain

$$\begin{aligned}(18)\quad 9(x - x_1)^2(y + y_1)^2 - 48(y + y_1)(yy_1)^{3/2}\\ + 48y^2y_1^2 - 12yy_1(x - x_1)^2 + (x - x_1)^4 = 0.\end{aligned}$$

Transposing the second term of (18), collecting the other terms of the left member as a quadratic in $(x-x_1)^2$, and squaring the resulting equation, we obtain the rational form of the equation of the *MacNeish curve* for our problem:

$$(19)\quad \{(x - x_1)^4 + [9(y^2 + y_1^2) + 6yy_1](x - x_1)^2 + 48y^2y_1^2\}^2 = 2304(y_1 + y)^2y^3y_1^3.$$

By equating to zero the group of terms of lowest degree in the two variables $x-x_1$, y, we find that $y^3=0$ is the equation of three coincident tangents to the curve (19) at the point $(x_1, 0)$. To show that the MacNeish curve has no point except $(x_1, 0)$ in common with the envelope G, we proceed as follows. Set $y=\frac{1}{4}y_1^{-1}(x-x_1)^2$ in (19), cancel from both members of the resulting equation the factor $(x-x_1)^4$, and write $u\equiv(x-x_1)^2$. The resulting equation is

$$(9/16y_1^2)^2u^4 + (63/16y_1^2)u^3 + (179/8)u^2 + 63y_1^2u + 81y_1^2 = 0,$$

which has no positive root u.

Two more facts about the curve (19) are desired, namely that for every $x>x_1$ there is exactly one real value of y, and that y increases as $x>x_1$ increases. Solving (18) for $(x-x_1)^2$, we obtain

$$\begin{aligned}(20)\quad (x - x_1)^2 = \tfrac{1}{2}[- 9(y^2 + y_1^2) - 6yy_1 + \{[9(y^2 + y_1^2) + 6yy_1]^2\\ + 192(yy_1)^{3/2}(y + y_1 - \sqrt{y}\sqrt{y_1})\}^{1/2}],\end{aligned}$$

in which the $+$ sign has been taken before the radical because $(x-x_1)^2>0$ for values of x that are being considered. After extracting the square root of both members of (20) and using the $+$ sign before the newly introduced radical, we find that x is a single valued function of y:

$$(21)\quad x - x_1 = \left[\frac{-9(y^2+y_1^2)-6yy_1+\{9[(y^2+y_1^2)+6yy_1]^2+192(yy_1)^{3/2}(y+y_1-\sqrt{y}\sqrt{y_1})\}^{1/2}}{2}\right]^{1/2}$$

That $x>x_1$ increases with y can be shown best by differentiating equation (20) with respect to y and simplifying the result. As the proof is not necessarily simple, we give it in a few steps. Since $x-x_1>0$, $dx/dy>0$ if

$$\begin{aligned}(22)\quad [9(y^2 + y_1^2) + 6yy_1](3y + y_1) + 8y_1\sqrt{y}\sqrt{y_1}(5y + 3y_1 - 4\sqrt{y}\sqrt{y_1})\\ >(3y + y_1)\{[9(y^2 + y_1^2) + 6yy_1]^2 + 192(yy_1)^{3/2}(y + y_1 - \sqrt{y}\sqrt{y_1})\}^{1/2}.\end{aligned}$$

Since $5y+3y_1>4\sqrt{y}\sqrt{y_1}$, both members of (22) are positive. Hence, after squaring both members of (22) and making some simplifications, we obtain the following necessary and sufficient condition that $dx/dy>0$ for $x>x_1$:

$$4\sqrt{y}\sqrt{y_1}(25y^2y_1+46yy_1^2+9y_1^3-40yy_1\sqrt{y}\sqrt{y_1}-24y_1^2\sqrt{y}\sqrt{y_1})$$
$$>(3y+y_1)(-9y^3-9y^2y_1-51yy_1^2-27y_1^3+12yy_1\sqrt{y}\sqrt{y_1}+36y_1^2\sqrt{y}\sqrt{y_1}),$$

and this can be reduced to

$$\sqrt{yy_1}(64y^2y_1+64yy_1^2)+2y^2y_1^2+36yy_1^3+36y^3y_1+27y^4+27y_1^4>0$$

which is true since $y_1>0$, and $y>0$ for $x>x_1$.

Now $dy/dx=1/(dx/dy)>0$ and so (19) defines y as a single-valued (real) function of x, which increases indefinitely as x so increases since the portion of the curve (19) for which $x>x_1$ lies entirely above G (Cf. Theorem 10).

If in equation (21) we give particular values to x_1, y_1, we can obtain for x numerical values (to any desired degree of accuracy) which correspond to assigned values of y. In obtaining the table of values below, which we used in plotting the accompanying MacNeish curve and the envelope G, we took $x_1=0$, $y_1=1$. The equation of G is then $y=\frac{1}{4}x^2$.

x	.79	1.36	1.88	2.24	2.51	2.73	2.92	3.08	3.22	3.36
H: y	.5	1	2	3	4	5	6	7	8	9
G: y	.16	.46	.88	1.25	1.58	1.86	2.13	2.37	2.59	2.82

As a check on the accuracy of the equation of the MacNeish curve, we shall consider two examples.

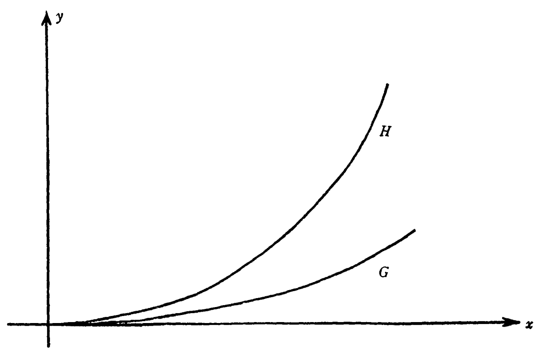


FIG. 7

Example 1. Compare $I(E_{12})$ and $I(e_{12})$ when 1 is at (0, 1) and 2 is at (1, 1).

We use the minimizing parabola of (8) through (0, 1) and (1, 1). From (7), $\alpha_2=\frac{1}{4}(2+\sqrt{3})$, and so the parabola of (8) is $y=1+(x^2-x)\,(2+\sqrt{3})^{-1}$. Consequently

$$I(E_{12}) = [2(2+\sqrt{3})^{3/2}]\int_0^1 (x^2 - x + 2 + \sqrt{3})dx$$
$$= \frac{2(2+\sqrt{3}-1/6)}{(2+\sqrt{3})^{-3/2}} < \frac{2}{3}(1+1) \equiv I(e_{12})$$

in accordance with our theory, since $2(1, 1)$ is above H [Fig. 7].

Example 2. Compare $I(E_{12})$ and $I(e_{12})$ when 1 is at $(0, 1)$ and 2 is at $(2, 2)$.

In this case $\alpha_2 = 1$ and the parabola which we desire from (8) has the equation $y = 1 + \frac{1}{4}x^2$. Hence

$$I(E_{12}) = \int_0^2 (1 + \tfrac{1}{4}x^2)dx = x + (x^3/12)\Big]_0^2 = 8/3 > \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}) = \tfrac{2}{3}(1 + 2\sqrt{2}),$$

or $I(E_{12}) > I(e_{12})$ in accordance with our theory since $2(2, 2)$ is between G and H.

Collecting the results of Theorem 10, the subsequent corollaries, and the above study of the MacNeish curve, we have

Theorem 11. When 2 is a point to the right of 1 and above the curve defined by (19), the Goldschmidt discontinuous solution e_{12}, joining 1 with 2, is a minimum time arc relative to other arcs of the parametric type (described in Theorem 7), joining 1 and 2, and lying in a sufficiently small neighborhood of e_{12}; but the absolute minimum time curve in this case is the unique parabolic arc E_{12}, joining 1 and 2, and having on it no contact point with the envelope G. When 2 is on the curve (19) and is at a point for which $x > x_1$ the times required by light of the type that we hypothesize to pass over the arc e_{12} and E_{12} are equal and are smaller than the corresponding time over every other arc of the parametric type referred to above. When 2 is between the curve (19) and the envelope G, and to the right of $x = x_1$, E_{12} provides a relative minimum time arc and e_{12} the absolute minimum. When 2 is on or below G, e_{12} is the only minimum time curve in the class of parametric arcs that we have considered, joining 1 and 2, and e_{12} furnishes the absolute minimum.

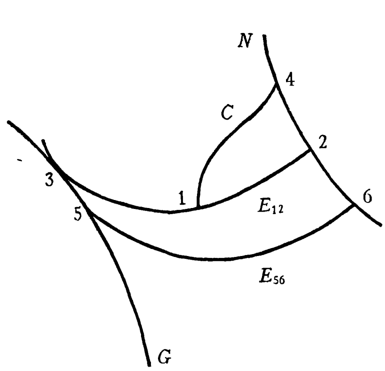


FIG. 8

15. *The case of one variable end-point.* The problem of finding the path along which light of the type that we hypothesize will pass from a given point 1 to a given curve N in the shortest time can be solved by argument of the type that is used in Bliss, §48. We merely state the result.

Theorem 12. If an admissible arc E_{12} (Cf. Fig. 8) minimizes the time of motion from a fixed point 1 to a fixed curve N, E_{12} is an arc of a parabola of the family $4(\alpha y - \alpha^2) = (x - d)^2$, intersecting N at right angles at 2, and having no point of contact with the envelope G of a one-parameter family of parabolas containing the parabola of

which E_{12} is an arc and intersecting N at right angles. Furthermore, an arc E_{12} which joins 1 to N and has these properties has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (mentioned in Theorem 7) which joins 1 to N and lies wholly in F.

The problem of finding the path along which light of the type that we hypothesize will pass from a given curve M to a given point 2 in the shortest time can be solved by the sort of argument that is used in Bliss, §32. We state the result in

Theorem 13. If an admissible arc E_{12} minimizes the time of motion from a fixed curve M to a fixed point 2 (Cf. Fig. 9) E_{12} is an arc of a parabola of the family $4(\alpha y-\alpha^2)=(x-d)^2$ such that the curve M at its intersection point 1 with E_{12} has a direction perpendicular to the tangent to E_{12} at 2, and such that E_{12} contains no point of contact with the envelope G of a one-parameter family of parabolas containing the parabola on which E_{12} is an arc and intersecting at right angles the curve N obtained by rotating M through an angle of 180° (in the xy-plane) on the center of the straight line segment joining 1 and 2. Furthermore, an arc E_{12} which joins M with 2 and has the other properties just mentioned has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (mentioned in Theorem 7) which joins M with 2 and lies entirely in F.

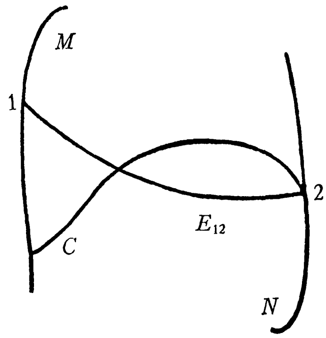


FIG. 9

16. *Geometrical construction for the focal point.* Since the extremals $4\alpha(y-\alpha)=(x-d)^2$ are of the form $y=\alpha\phi[(x-d)/\alpha]$, a well known geometrical construction for the focal point of the curve N on E_{12} (Cf. Fig. 8) can be applied. We omit the construction since it is the same as the corresponding construction for the catenary problem (Cf. Bliss, p. 125).

17. *Two variable end-points.* The problem of finding an arc joining two fixed curves and minimizing our time integral can be solved by argument of the type that is used in Bliss, §16. The result, we express in

Theorem 14. If an admissible arc E_{12} minimizes the time of motion from a given curve M to a second given curve N (Cf. Fig. 10), E_{12} is an arc of a parabola of the family $4(\alpha y-\alpha^2)=(x-d)^2$, M and N must intersect E_{12} at right angles at 1 and 2, respectively, and 1, 2 with the focal points 3, 4 of M, N, respectively, on E_{12} lie in the circular order 4312, no coincidences being allowed except that 4 may possibly fall upon 3. Further-

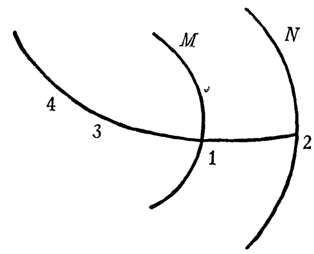


FIG. 10

more, an arc E_{12} which joins M to N and possesses the properties just mentioned has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (of Theorem 7) which joins M with N and lies entirely in F.

18. *Remarks on the corresponding problem in three dimensions.* Suppose that light travels in ordinary space of three dimensions; that x, y, z denote the rectangular coordinates of any point in this space; and that throughout the space the index of refraction $n(x, y, z)$ is directly proportional to the square root of the distance of the point (x, y, z) from the xz-plane. The velocity, v, of the light then satisfies a relation of the form $v(x, y, z) = kn(x, y, z) = k/\sqrt{y}$, where k is a suitably chosen constant. Let us call a curve $y = y(x)$, $z = z(x)$ *admissible* if it has $y \geqq 0$, is continuous, and has a continuously turning tangent except possibly at a finite number of points. Our problem then is to find among all admissible curves through two fixed points $1(x_1, y_1, z_1)$, $2(x_2, y_2, z_2)$, one along which light will pass from 1 to 2 in the shortest time.

Proceeding as in §2, we find that our problem is to minimize the integral

$$I = \int_{x_1}^{x_2} y^{1/2}[1 + y'^2 + z'^2]dx$$

by properly determining the functions $y(x)$, $z(x)$, which satisfy the relations $y_1 = y(x_1)$, $z_1 = z(x_1)$; $y_2 = y(x_2)$, $z_2 = z(x_2)$.

The integrand of the integral I is a special case of an integrand function $f(x, y, y', z, z')$ that has been used in calculus of variations theory.[9] Furthermore, in our problem $f_{y'y'}f_{z'z'} - f^2_{y'z'} = y(1+y'^2+z'^2)^{-2} > 0$ when $y > 0$, so that in the region $y > 0$ our problem is *regular*, and the extremals admitted have no corners. Hence Euler's differential equations, which must be satisfied, have the consequences[10]

$$f - y'f_{y'} - z'f_{z'} = c_1, \quad f_{z'} = c_2;$$

or

$$1 = c_1[1 + y'^2 + z'^2]^{1/2}y^{-1/2}, \quad z' = c_2[1 + y'^2 + z'^2]^{1/2}y^{-1/2}.$$

Hence $z' = c_2c_1^{-1}$, $z = c_2c_1^{-1}x + c_3$, and we have the result: *the motion takes place in a plane perpendicular to the xz-plane.*

---

[9] For example, Cf. Goursat, loc. cit., pp. 559–562.

[10] Cf. Goursat, loc. cit., p. 561.

ITALIAN CONTRIBUTIONS TO MODERN MATHEMATICS[1]

By ENRICO BOMPIANI, Rome, Italy

I accepted with pleasure the invitation of speaking to you on the Italian contributions to modern mathematics although I am perfectly aware of the difficulty of my task. It is such a wide subject that I cannot pretend to dominate it. Nor is it possible, by the nature of this report, to follow a rigorous classification of the matter from an historical viewpoint or by its content. It will therefore be sufficient to note, with some simple allusions to problems and the results, the boundaries in which such activity took place.

Before all it is necessary to define the limits of time to which I refer. I shall not speak to you of Archimedes of Syracuse nor of Zeno of Elea, whose contributions are also to be attributed to modern mathematics inasmuch as with the first we have reference to problems and methods of infinitesimal calculus, and with the second originated the logical discussion on continuity and discontinuity which precedes the modern theory of point-sets.

But it is perhaps necessary to mention here Leonardo Pisano, who, with his Liber Abbaci, brought into Italy the methods of calculation of the Arabians. The diffusion of this book in Europe, diffusion which was also a consequence of the political conditions then prevailing in Italy, gave the first impulse to the creation of our modern arithmetic and algebra.

At the beginning of the 16th century, the mathematical school of Bologna irradiated a new light on the scientific world with the discovery of the resolution of the equations of the third and fourth degree. With this discovery are connected the names of Scipione del Ferro, Ludovico Ferrari, Bombelli, Tartaglia of Brescia, Cardano of Milan. The question of the possibility of solving an algebraic equation of any order was answered by Palo Ruffini of Modena, who in the year 1799 demonstrated, before Abel, the impossibility of the algebraic solution of an equation of order greater than the fourth, giving at the same time the first germs of the theory of finite groups of substitutions, developed after him in its essential lines by Galois.

That the interest in pure arithmetic and algebra is still alive in Italy, may be proved by the researches of L. Bianchi on the modular group, on the theory of Dirichlet's and Hermitian forms, and on quadratic quaternary forms with integral rational coefficients; by the researches of Cipolla on asymptotic arithmetic; by the researches of Scorza and Cecioni on algebra.[2]

The 17th century marks the discovery of analytic geometry and of infinitesimal calculus.

The "methodus indivisibilium" of Cavalieri, which is yet referred to in modern treatises on the matter, and the numerous integrations of parabolas and

---

[1] A lecture delivered at the Fourteenth Summer Meeting of the Mathematical Association of America at Providence, R. I., on Sept. 9, 1930.

[2] See the excellent treatise of G. Scorza: *Corpi numerici e algebre* (Principato, Messina, 1922).

hyperbolas of any order by Torricelli, testify not only their genius but also that the field was ripe for the symbolical systematization of Newton and Leibniz.

To the same period belongs the discovery of the infinite algorithm, like the continued fractions (for which Cataldi gave the formulas of computation of the successive convergents and the theorem on the best approximation), the infinite series, the logarithm. Mengoli, of Bologna, had a very exact idea of the definition of limit and of the different possibilities in the summation of a series,[3] Brunacci gave (in 1804) his famous transformation which is known today as the Theorem of Brunacci-Abel and which is so useful for the evaluation of series. In modern times Cesaro, one of our most original mathematicians, extended the idea of sum to the non-convergent series; this was the first step in the effective study of non-convergent series, developed particularly by Borel. G. Sannia (at Naples) has put in evidence that the Borel method of addition is only a particular case of an infinite chain of methods requiring steadily weaker conditions for the summability of a series. More recently Picone, at Naples, has given a general treatment of all summation methods, freed from initial restrictions which limited their possible applications; his pupil, Mammana, at Cagliari, is studying at present the algebraic operations on series summed with different methods of which, notwithstanding their importance for the applications, we know very little.

I shall also mention that Cesaro's method has given some splendid results, worked out by Féjer, in the approximate representation of real functions by trigonometric polynomials.

After the remarkable discoveries of the 17th and 18th centuries a revision of the principles on which mathematical knowledge lay was necessary as a condition for its further progress. This critical current concerned itself with the analysis of the principles of geometry, at the beginning of the 19th century, and had its outlet in the construction of non-euclidean geometries (Bolyai, Lobatchewski, Gauss and Riemann). The forerunner of these geometries was the Italian Jesuit, Saccheri, who, in a book entitled "Euclidis ab omni naevo vindicatus," endeavored to demonstrate the postulate of Euclid, but in reality he gave the first exposition, coherent and elementary, of the various types of non-euclidean geometries. Beltrami, above all, contributed to their acceptance, giving a tangible interpretation of them on the surfaces of constant curvature. When the interest in these constructions (whose critical purpose was completely achieved) was cooling, the same constructions acquired a new interest for the geometry of restricted relativity in the space-time of Einstein-Minkowski.

Part of the work of Veronese (although of later date) was devoted to the revision of the principles of geometry; he gave one of the first logical arrangements of postulates and the first example of non-archimedean geometry, found again later by Hilbert. Together with the name of Veronese there are to be mentioned,

[3] Many papers by Ettore Bortolotti, relating to the development of mathematics in Italy during this period, are collected in a volume: "Studi e ricerche sulla Storia della Matematica in Italia nei secoli XVI e XVII" (Zanichelli, Bologna, 1928).

for Italy, those of Burali-Forti and of Pieri, who assumed primitive notions different from those of Veronese.

An analogous revision of the foundations of the infinitesimal calculus, started by Bolzano and Cauchy, was brought about in the second half of the 19th century in a rigorous treatment of that theory. In this accomplishment have taken part in Italy, Dini, whose work on the functions of a real variable and on the Fourier's series is known everywhere and which points out for the first time fundamental conceptions of modern analysis; Ascoli and Arzelà,[4] whose work on systems of curves has furnished the rigorous basis for the existence theorems of differential and functional equations; Peano (and his school: Vailati, Vacca, Burali-Forti, Pieri, Padoa) to whom is due a classical formulation of the principles of arithmetic and the creation of a symbolic logic which is one of the most appreciated.

To the further development of this critical current in analysis may be also connected the name of Vitali, some of whose fundamental results are now considered classical, and the most recent contributions to the problem of integration of discontinuous functions by Tonelli and by Caccioppoli, who uses the new concept of limit given by Picone.[5]

To the classical works directed to the determination of the conditions of existence and uniqueness of integrals of ordinary differential equations belong those of Volterra, of Peano (whose method was recently found again by Perron and extended to partial differential equations) of Arzelà, Nicoletti and others. The question of the uniqueness of a solution in a given interval was answered by Lipschitz assuming one condition evidently too restrictive; recently I have given a general criterion of comparison between the integrals of two differential equations from which is deduced, in particular, the theorem of uniqueness of Osgood and Tamarkine; Tonelli and Perron have demonstrated again this theorem with different methods. Recent researches in this field are due to the school of Picone, whose young pupils Cimmino, Colucci, the younger Scorza and Caccioppoli have largely extended our knowledge in this field. The last one using, without knowing it, an idea already brought out by Birkhoff and Kellogg, has established, by very simple considerations of topology of functional spaces, a theorem of existence and of uniqueness for very wide classes of functional equations.

Perhaps the most conspicuous work in the beginning of the 19th century is the analytic mechanics of Lagrangia, from Torino, who has moulded on his principles a great part of the following production in applied mathematics. It is impossible to summarize even briefly the results attained in this field, even limiting ourselves to those given in Italy.[6] I shall mention therefore only the re-

[4] Arzelà introduced, except for the name, the concept of a normal family of functions, rediscovered later by Montel.

[5] See his "Lezioni di Analisi Infinitesimale" (published by the Circolo Matematico di Catania, Catania, 1923)—Some classical theories have received a new treatment in this book.

[6] A comprehensive treatise in analytical mechanics by Levi-Civita and Amaldi has been published by the Casa Zanichelli, Bologna.

searches of Levi-Civita and of his school (Bisconcini, Signorini, Armellini) on the problem of three bodies, which have prepared the ground for the result of Sundmann. To the classic mechanics belongs also the problem of motion of a body of variable mass, either by increase or by diminution. Levi-Civita has demonstrated, on the basis of two natural hypotheses, namely independence of effects and statistical isotropy, that the application of the principles of classical mechanics leads not to the equation of Lagrange (mass times acceleration equal to force) but to the theorem of quantity of motion (rate of change of the quantity of motion, or momentum, is equal to the force) which, as is known, remains still valid in the restricted relativity theory. The conclusions of Levi-Civita were applied by one of his pupils, Vranceanu, to the problem of two bodies of variable masses. This problem was already treated by Armellini with the use of higher analytical methods: Levi-Civita has tackled again the problem using the hypothesis of adiabaticity (not necessary in the solution of Armellini) and has made important applications to astronomy and has succeeded in assigning the conditions of minimum energy also in the case of revolving bodies. This is a consequence of a theory developed by Levi-Civita of the adiabatic invariants of differential systems of Liouville, in which are framed the theorems of Gibbs, Hertz and Burgers. Geppert, under the guidance of Levi-Civita, has extended the theory of the adiabatic invariants to more general differential systems.

A recent contribution in the field of classical analytical mechanics was given by Gugino who has introduced for a material system (whose constraints are without friction and independent of the time) a new scalar, the *cinetodynamic effect;* and has demonstrated that the natural motion of a system is characterized, compared with all possible motions satisfying the same conditions, by the fact that the cinetodynamic effect is a maximum.

To the problems of applied mathematics contributed actively in Italy Betti,[7] who with his famous reciprocity theorem gave impulse to the wide and deep studies on elasticity; Beltrami, Dini, Bianchi, Arzelà, Lauricella, Cerruti, Somigliana, Marcolango, Tedone, Burgatti, Almansi, Boggio; and particularly Volterra, who extended, in papers now classical, the method of Riemann to elastic vibrating bodies. Also to Volterra is due the creation of the theory of distortions of elastic multiply-connected bodies (a theory whose consequences were clearly verified by the experiments of Trabacchi and Corbino). Further in this group of works are to be mentioned the researches of Volterra on internal cyclical motions, of Almansi on the equilibrium of sands and of Signorini on reenforced concrete.

Also to applied mathematics belong the famous paper of Levi-Civita on waves, which has given a new impulse to the study of plane hydrodynamics (by a suitable use of analytic functions) and in which have amply participated Cisotti, Signorini, Colonnetti, Finzi, Pistolesi, Masotti and many others; the rigorous solution, given by Levi-Civita, of the problem of Airy on the progressive

[7] Betti is also to be remembered as one of the founders of analysis situs: to him is due the introduction of the "Betti numbers."

waves of permanent type in straight-channels and the extension of the theory to circular channels (as is necessary for the experimental verification) given by Geppert; the study of Masotti on the motions of a perfect liquid which take place in non-plane strata; and finally the extension of the theorem of Bernoulli to homogeneous viscous liquids which is due to Lelli.

Before closing this brief review of Italian contributions to analytical mechanics and applied mathematics, it is necessary that I recall at least the names of Giorgi and of Maggi for the critical analysis of its principles; of Armellini and Burgatti for the law of distribution of planets, of Marcolongo, Boggio and Burali-Forti, who have given a symbolism of vector-analysis which is one of the most logical and convenient for the various applications.[8]

Equations similar to those of classical mechanics are found in the study of the phenomena of economics; the founder of the statistics of these phenomena was Pareto from Genoa; his work was continued in Italy, also for dynamics, by Amoroso and Scorza.

The problems of applied mathematics as well as of geometry led from the very beginning to the study of partial differential equations and of analytic functions; in fact, many of the aforementioned researches may be considered as contributions to these fields. The greatest attention to the questions of existence and of uniqueness of a solution of a partial differential equation for given boundary conditions was aroused by the famous problem of Dirichlet. Riemann believed that he had solved it by a procedure which successive criticism has proved not sufficient. With different procedures the problem was solved by Neumann, Schwarz and Poincaré. But it was Arzelà who tried first (1897) to give rigorous foundation to the idea of Riemann; an attempt which only succeeded in our century through the work of Hilbert, B. Levi, Fubini, Lebesgue, Zaremba. The method of minimizing sequences of Fubini was recently used by Courant and the school of Göttingen.

On the existence question of differential equations of the second order, and of their connection with the integral and integro-differential equations, worked Nicoletti, Severini, E. E. Levi, Picone (whose fundamental identity, which can be used in many other problems, has received from Bôcher the name of identity of Picone) Mammana, Cimmino, and Tricomi who has examined the behavior of the integrals and the conditions of uniqueness of the solution of an equation of the 2nd order with non-constant coefficients in the vicinity of its parabolic curve.

Notwithstanding the interest of these results, the problem which interests most the physicist and the technician is the numerical evaluation of the solution, the existence of which has been previously demonstrated. In this connection some results of Picone and of his school are worthy to be mentioned: the rational formulas for the perturbed motion of projectiles (taking into account the variations of the physical and dynamical condition of the atmosphere; the

[8] Reference may be made to the treatise "Analisi Vettoriale" (published by Zanichelli, Bologna).

curvature and the rotation of the earth; etc.); the formulas establishing dominating values for the solutions of any linear equation of the second order of elliptic or parabolic type (with remarkable consequences regarding the boundary conditions which may be given to determine a solution), and also for some equations of mathematical physics of order greater than the second; the evaluation of the error in the approximate calculation of the solutions of the aforementioned equations. It should be remarked that this evaluation is made to depend only upon the knowledge of approximate values of the characteristic numbers and of dominant numbers of the solution and of some of its derivatives.

From this situation arises the importance of the problem of the evaluation of the characteristic numbers to which Picone is applying successfully a newly devised method in his institute at Naples.

In the theory of harmonic functions of two variables we shall also call attention to the existence theorem for the Neumann problem in arbitrarily connected domains, given by Picone; and also to some contributions, given by Picone and G. Ascoli, to the study of singularities of these functions.

The aforementioned researches refer to the functions of real variables; we will discuss them again when dealing with the functional calculus. But first, in order to conform to the historical development, it is necessary to refer to the theory of analytic functions.

Closely related to the work of Weierstrass and Mittag-Leffler were the remarkable researches of Pincherle; but we should also remember Betti, to whom are due (before Weierstrass gave his general theorem) the developments of several entire functions in infinite products; Morera, who has inverted the famous Cauchy Theorem; and Casorati who gave the first characteristic theorem on essential singularities, a theorem which precedes the famous ones of Picard-Landau. Vivanti's theorem on the determination of a singular point on the circumference of convergence of a series with non-negative coefficients is the first of its kind, and was followed by the brilliant researches of Phragmen, Lindelöf, Mandelbrojt and recently of the young Italian, Minetti.

To the theory of some notable classes of analytic functions, like elliptic, abelian, modular, automorphic functions, have contributed Casorati, Brioschi, Pascal, D'Ovidio, Bianchi, Pincherle, Fubini, Enriques, Severi, Bagnera, De Franchis, Scorza, Comessatti, Rosati and Spampinato. The theory of algebraic functions, considered as analytic functions, is flourishing in Italy from the geometric view-point, of which I shall have occasion to speak to you later.

To the analytic functions of several complex variables, the study of which we can say has just started, although several essential results of Weierstrass, of Poincaré, of Osgood and of Carathéodory are known, important papers have been dedicated by E. E. Levi with the study of the varieties which may be boundaries of such functions, by Levi-Civita with the study of the characteristic surfaces of the equations of monogeneity, and more recently by Severi.

To another branch of analysis Italy has largely contributed: the functional calculus. Pincherle created the calculus of distributive operations, today called

linear functionals, and in his work we find many of the concepts and terms today accepted in the theory;[9] Giorgi particularly developed the symbolic calculus of Heaviside and gave valuable application of it to electrodynamical problems: he has reported on his contributions at the Toronto Congress of Mathematicians (1924).

But the functional calculus as a creative branch of analysis arises with the work of Volterra. From physical problems he drew the idea of considering functions depending not only on the values of one or more parameters, but on other functions, that is, in geometrical terminology not functions of one or more points, but functions of lines, surfaces and so on; hence the name of functions of lines replaced more recently by that of functionals.

It is impossible to refer in detail to the work of Volterra in this field, of which excellent expositions exist by Volterra himself and his school (Pérès, P. Lévy, Fantappiè). It will suffice to remember the integral and integro-differential equations which constitute important classes of functional equations, and to associate with the name of Volterra those of Fredholm, Hilbert, Picard, and to remember the more general studies of E. H. Moore and Fréchet on general analysis; and the list is necessarily too incomplete to give an idea of the enormous influence of the work of Volterra on all contemporary mathematical production.

These studies of such a general and abstract character found their most concrete application in the physical field, from which they were born, and in mathematical economics. Volterra himself gave the mathematical theory of hereditary phenomena in which the present state of a system depends not only on the present circumstances but on all its history.

Recently Volterra illustrated his theory of hereditary phenomena under considerations of energy: he showed that the work of external forces necessary to bring a system from a given state to a different state is always greater than the variation of a certain functional which depends exclusively on the present state of the system; and calculated the work dissipated by the external forces when a system returns to the initial condition.[10]

To these considerations Volterra was led by his recent researches on mathematical biology, whose consequences have been verified by the exploration of the seas.[11]

To illustrate another field in which integral and integro-differential equations show ample possibilities for practical use, I shall recall that Evans, a pupil of Volterra, and Roos have initiated with these means the study of economical phenomena in the regime of monopoly, and that F. P. Cantelli has used them largely in questions of calculus of probabilities and mathematical statistics. I shall say incidentally that an excellent treatise on the calculus of probabilities is that of Castelnuovo[12] who also contributed to the critical revision of its foundations.

---

[9] See Pincherle and Amaldi, "Le operazioni distributive" (Zanichelli, Bologna).

[10] See Atti del Congresso Internazionale dei Matematici a Bologna, 1928.

[11] See an article in "Scientia" (Bologna, Zanichelli, 1926).

[12] See the 2nd edition, two volumes, (Zanichelli, Bologna).

The researches of Volterra upon functionals relate generally to the real field, while those of Pincherle on distributive operations extend to the complex field. In the last few years Fantappiè has amply studied one class of functionals, the analytical functionals, which include the linear type of Pincherle as a particular case and which are analogous to the analytic functions. The introduction of the complex variable allows the coordination in a harmonic whole of the properties of this class of functionals. Fantappiè showed that it is possible to construct for such functionals a theory close to that of analytic functions, to give the analogues of Cauchy's Theorem and its series developments (more general then those of Volterra and Fréchet), and to study the polydromy of these functionals.

Lastly Giorgi and Fantappiè have fathomed the connections between functional analysis and the wave-mechanics.

Another branch of analysis is concerned with problems of the functional calculus: the calculus of variations whose first general treatment is due to Lagrange. Soon after his first essays Volterra had established the dependence of the calculus of variations on functional analysis. In fact, to the problems of maximum and minimum of a function correspond problems of maximum and minimum of a functional which constitute the very subject of the calculus of variations. The idea pointed out by Volterra was taken up again by Arzelà, who could not fully succeed because the functionals which presented themselves in the calculus of variations are not generally continuous. Tonelli, a pupil of Arzelà and Pincherle, was on the other hand completely successful. The Tonelli method is essentially based on the semicontinuity of such functionals, a concept analogous to that of semicontinuous functions given by Baire.

While the classical theories connect the study of the problems of variations with the differential equations of Euler, the Tonelli theory is independent of the theory of differential equations; on the contrary, the existence of the extremals satisfying given conditions is a consequence of the existence of the extremants which is demonstrated by direct proof.[13]

In the classical trend of Lagrange are to be mentioned the contributions of E. E. Levi, who gave sufficient conditions for an extremum, not depending on the knowledge of any fields of extremals, and a very simple proof of the necessity of Weierstrass's condition for the extremum of double integrals; of Burnengo who won analogous results for isoperimetrical problems; of Picone who has given new proofs, which hold also for the calculus of variations in two variables, of the Legendre and Jacobi conditions and a new suitable expression of the third variation.

Connected with the problems of the calculus of variations in many dimensions are the researches of Tonelli on the area of surfaces and the concepts of functions of two real variables absolutely continuous and with limited variations; concepts different from those already given by Vitali and Lebesgue. These concepts showed their utility in the theory of double Fourier's series, for the con-

[13] L. Tonelli, "Fundamenti di Calcolo delle Variazioni" (Zanichelli, Bologna).

vergence and uniform convergence of which Tonelli has given much simpler criteria than those already known.[14]

Lastly Caccioppoli, Nalli and Andreoli have introduced the new concept of an n-uple of functions of n variables with limited variation and of multiple integrals of Stieltjes which seem to have a great importance not only for the aforementioned questions but also for the definition of the integrals of functions of several complex variables. With a suitable extension of the concept of a function of many variables with limited variation, Caccioppoli has succeeded very recently in giving a simple analytical representation of multi-linear functionals of any degree, depending on an arbitrary number of functions (a problem which was treated by Fréchet in the case of bilinear functionals).

Let us now take a glance at the development of geometry in Italy, the country which the German geometer Klein has named the natural fatherland of geometry.

At the beginning of the 19th century arose in France, by the work of Poncelet (1821) the projective geometry, whose origin must be traced to the rules of perspective of the Italian artists of the Rinascimento.

With the rapid development of this branch of geometry in France, in Germany and in England, Italy was soon associated by the merit of Luigi Cremona-

While in the other countries of Europe this current of ideas seemed to wither with researches often sterile, in Italy it originated two branches, both very vital: the projective geometry of hyperspaces and the geometry of birational or Cremona transformations.

Projective geometry is nothing but the synthetic aspect of the invariant theory of algebraic forms with respect to the group of linear transformations; to this theory have contributed Betti, Beltrami, D'Ovidio, Bianchi, Capelli, Gerbaldi and more synthetically Bertini, Del Pezzo, Segre, Berzolari, Castelnuovo, Enriques, Severi, Scorza, Comessatti, Rosati, and many others. Some of the results of these researches have been collected in a book of Bertini, "Introduction to the projective theory of hyperspaces," of which a German translation by Duschek has recently been published. I will speak later of this branch of geometry in the differential field.

In the meantime Cremona was laying the foundations of the transformations which bear his name, which make correspond to lines higher curves (and not lines as in projective geometry). These transformations constitute a new instrument of geometrical analysis for the study of algebraic curves (or surfaces).

A very successful combination of the projective and the Cremonian trend, rich in further consequences, took place in the work of Segre and Castelnuovo, who transferred the properties which are invariant with respect to the birational transformations into projective properties of a convenient hyperspatial model; in this way projective geometry acquired a much larger domain and significance.

[14] See his treatise on Fourier's series (Zanichelli, Bologna).

The studies of Riemann on abelian integrals, transferred by Brill and Noether to the algebraic field, found in Italy a particularly favourable ground and new methods of investigation. We may say that with the work of Bertini, Segre, Castelnuovo, Enriques and Severi the theory of rational functions of two variables connected by an algebraic equation, which means geometry on an algebraic curve, is substantially finished. To the theory of algebraic correspondences between algebraic curves and to their multiple relations with algebra and analysis, remarkable, and sometimes essential contributions, were given by Scorza, Rosati, Torelli and more recently Comessatti and Chisini.

The geometrical methods, already familiar to the Italian school, proved very successful when, about the year 1890, Castelnuovo and Enriques began to study the entirely new field of algebraic surfaces, which means the theory of rational functions of three variables connected by an algebraic equation.[15] At the very beginning of their researches they obtained a surprising result. When one tries to extend to surfaces the two definitions of the genus of a curve one finds two different numbers which are invariant under birational transformations of the surface: the geometrical genus and the arithmetical genus, generally different from each other. Thus was born the distinction of surfaces into two classes: the regular surfaces (for which the two genera are equal) and the irregular ones (with different genera) which manifest quite different properties with respect to the continuous systems of algebraic curves which they contain.

In the meantime Picard and Humbert in France extended to surfaces the theory of abelian integrals which was constructed for the algebraic curves by Riemann. We owe to Castelnuovo and Severi the conclusion of this period of researches with a definite result which connects the irregularity of a surface (that means the difference between its two genera) with the number of the simple distinct abelian integrals of the second species.

Severi constructed later the theory of the basis for algebraic curves on a surface, showing that the number of independent curves which constitute the basis is exactly the same as the invariant number introduced by Picard in the theory of the integral of the third species. To the theory of the basis for manifolds of many dimensions Comessatti and Albanese have also contributed.

Let us now go to the differential geometry.

With metric differential geometry are connected from the very beginning the names of Mainardi and Codazzi, who gave the fundamental equations of the Theory of Gauss, and of Beltrami, who also gave physical applications of it.

The most prominent representative of this branch of mathematics in Italy was Luigi Bianchi, whose influence was largely spread in Italy and abroad by his famous treatise.[16] The contributions of Bianchi and of his school[17] (E. E. Levi,

[15] See a report of G. Castelnuovo in the first volume of the Atti del Congresso Internazionale dei Matematici a Bologna, 1928 (Zanichelli, Bologna).

[16] "Lezioni di Geometria Differenziale," 3rd edition (Zanichelli, Bologna).

[17] See a report of G. Fubini in Annali di Matematica (1928–29) and of G. Scorza in the Annali della Scuola Normale Superiore di Pisa (1930).

Fubini, Picone, Calapso) are so large that it is even impossible to summarize them; I will only mention the theory of transformation of pseudospherical surfaces, and his famous theorem of permutability, the asymptotic transformations of surfaces, and the theory of deformations of quadrics; and these are only a few specimens of many wonderful results.

To the geometry of Riemann, Ricci gave a new and powerful impulse, whose value has been fully appreciated only after the creation of relativity theory, with the development of an algorithm which is named the absolute differential calculus or Ricci's calculus. This method, sometimes referred to as "mathematics of relativity," was largely used, in classical and relativistic mechanics, by Ricci, Levi-Civita, Amaldi, Palatini, and many others.

An essential contribution to the geometry of Riemann spaces was the notion of parallel displacement, introduced by Levi-Civita in 1917; the richness of ideas and results contained therein has been shown in Italy by Enea Bortolotti and myself, and outside of Italy by Schouten, Weyl, Eddington, Cartan, Wirtinger, Berwald and by the American School of which I shall name only the two leaders, Eisenhart and Veblen.

In this short review of metric differential geometry, in the sense of Gauss and Riemann, mention may be made of two different extensions of it given by Vitali and myself. I considered deformations of surfaces (or manifolds) in hyperspaces which conserve not only their linear elements, but also the curvatures, of a prefixed order, of their curves: the theory of these deformations of higher species is equivalent to the invariant properties of a system of differential forms of even orders. Vitali, with a remarkable extension of Ricci's calculus, was able to construct the analogues of Riemannian geometry for metric spaces of infinitely many dimensions, the Hilbert spaces.[18]

In the last decades of the past century and in the first years of our century, there was developing a new branch of differential geometry, which now constitutes a powerful chapter of our science: projective differential geometry.

Setting aside for a while the preceding and contemporary development of differential properties in hyperspaces, I will mention first Wilczynski, who between 1901 and 1910 gave a systematic treatment of differential geometry of curves and surfaces in ordinary space.[19]

The idea of representing curves, with respect to the projective group, by differential equations, belongs to Halphen (and was developed for hyperspaces by Berzolari); Wilczynski used in the projective representation of a surface in the ordinary space two partial differential equations (whose general integral depends, as in the case of Halphen, on arbitrary constants), which represent the existence of the asymptotic curves on it.

Fubini, beginning in 1915, has re-undertaken the study of differential projective geometry of surfaces in ordinary space using certain invariant differential forms (two quadratic and one cubic). The advantage of this method is two

[18] See his "Geometria degli-spazi hilbertiani" (Zanichelli, Bologna).

[19] Or projective space of three dimensions, S_3.

fold: first, it gives a representation of the surface which is invariant with respect to the aforementioned group (or also to the more general group of projective applicabilities, discovered by Fubini); second, it allows the use of Ricci's calculus, and therefore the immediate deduction of the invariants, covariants and contravariants of the surface.

The original deduction of Fubini's forms is rather complicated; I have succeeded in showing how from the equations of Wilczynski (which express a well known elementary fact) can be deduced the Fubini forms; and more than that I gave the geometrical meaning of these forms, until then unknown.

It is not possible to give an idea of the quantity of the new facts discovered in these last years in a field which we thought familiar and well known; however the treatise of Fubini and Čech,[20] where much of the theory is presented, exempts me from further reference.

A wider and higher field of research is offered by projective differential geometry of hyperspaces. The first researches in this field were made by Del Pezzo, who, about 1886, began to study synthetically the nature of the neighborhoods, of order higher than the first, of points on the surfaces and on manifolds in hyperspaces. From them he obtained remarkable results also in the algebraic field. These researches were practically forgotten when Segre, from 1906 to 1910, with suitable analytic representation, undertook the study of surfaces whose points have as coordinates solutions of a Laplace's equation and of manifolds generated by linear spaces.

In this new field, in which almost nothing was known and everything to be done, began to work Terracini and myself; a very accurate account of our achievements in the first of fifteen years of research has been given by Terracini,[21] so that I need not refer to particular results. It seems to me on the contrary important to point out some ideas which characterize the new trend with respect to the aforementioned researches.

In the study of surfaces in ordinary space the double system of asymptotic curves has a paramount importance; in fact a surface with such a double system necessarily belongs to a space of three dimensions; that is why the general integral of Wilczynski's equations depends only on arbitrary constants.

When one tries to extend these considerations to surfaces or manifolds in hyperspaces, the necessity appears immediately of finding such system of curves, projectively determined on the surface, whose existence characterizes the dimensionality of the space to which the surface belongs. In order to do that, I introduced the quasi-asymptotic curves, which are determined by suitable conditions of intersection of the osculating spaces (of a certain order r) to the curve and the osculating spaces (of a certain order s) to the surface (for the ordinary asymptotics $s=1$, $r=2$). This was the first necessary geometric step to construct a general theory of surfaces in a space of any finite dimensionality: the analytical expression of these conditions gives the differential equations satisfied by the

[20] "Lezioni di Geometria Proiettiva Differenziale" (Zanichelli, Bologna).

[21] See his Appendix III in the treatise of Fubini and Čech.

surface. The calculation of their integrability conditions, covariants and invariants is then a matter of course and of routine; better than that is to introduce (as we have pointed out for surfaces in S_3) some invariant forms which give an invariant representation of the surface and possibly allow the use of Ricci's Calculus. Another step, which requires again geometric skilfulness, is the geometric interpretation of such invariant forms.

The preceding considerations lead, by the very statement of the problem, to systems of partial linear differential equations whose general integral depends only on arbitrary constants (whose number is determined by the dimensionality of the space to which the surface is supposed to belong).

However, it is not always necessary to fix the dimensionality when dealing with a problem in partial differential equations; on the contrary in some questions it is necessary to leave it among the unknowns. In fact the first results I obtained in freeing myself from this dimensionality were the necessary and sufficient conditions, geometrically deduced, in order that a Laplace's equation could be integrated by quadratures; a problem proposed by Darboux and which, having resisted the efforts of Goursat, was still awaiting a solution.

From this example we arrive at the most general and fertile point of view reached in this branch of geometry; we do not consider necessarily a particular group of solutions of a system of linear partial differential equations in two variables, whose general integral depends on arbitrary constants as representing the points of a surface. Any set of functions (finite or infinite in number) satisfying or not satisfying a system of differential equations, whose general integral may also depend on arbitrary functions, represents points of a surface; the dimensionality of the space (which may also not be finite) is unessential (if not determined by the given system of equations). What is on the contrary essential is the dimensionality of the spaces which contain the neighborhoods of the different orders of the points of the surface, which I call the osculating spaces, and the fact that in these spaces (always of a finite dimensionality; and not necessarily in the ambient) the projective geometry still holds. And so we reach really a projective geometry of linear partial differential equations (and not of particular types of them or of some of their solutions) and more generally of any set of linearly independent functions.

For these surfaces (or manifolds), although so general, it is possible to give a theory of point to point correspondences, as I have outlined in a communication to the International Congress of Mathematicians in Bologna (1928); and it is also possible to give a constructive meaning to the geometry of paths and to theory of projective parallel displacement, as I have shown in some lectures given in American Universities.

I have thus come to the end of this review, too extended for you perhaps, but certainly too short for the wideness of the subject. I hope that I have succeeded in giving you, even through the defiiciency of my exposition, an idea of the part that Italy takes in this truly international collaboration for the progress of human knowledge.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

In the November, 1930 issue of this Monthly, on page 493, there appeared "A simple proof of the theorem of Morley"[1] by Mr. Jacob O. Engelhardt. Since then the editor has received the following two letters which show in different ways how that proof can be simplified. Neither one of the modified proofs involves Mr. Engelhardt's Lemma 2.

I

Institut Henri Poincaré, Paris, le 30 Nov., 1930.

Monsieur et cher collègue:

La demonstration fort ingénieuse du théorème de Morley,[2] due à M. Jacob Engelhardt, me parait pouvois être encore simplifici comme il suit. Reprenous la figure de la p. 493 et posons

$$\angle\, BFD = \alpha, \quad \angle\, BDF = \gamma, \quad \angle\, CDE = \beta.$$

M. Engelhardt obtient

$$BD = 4 \sin x \sin z \sin (60° + x), \quad BF = 4 \sin x \sin z\, (60° + z).$$

On a donc

$$\frac{\sin \alpha}{\sin \gamma} = \frac{BD}{BF} = \frac{\sin (60° + x)}{\sin (60° + z)}\,.$$

D'autre part

$$\alpha + \gamma = 180° - y = 60° + x + 60° + z.$$

Par suite $\alpha = 60°+x$, $\gamma = 60°+z$. On montrerait de même que $\beta = 60°+y$. Par suite, en remarquant que $\angle BDC = 180° - y - z$.

$$\angle\, FDE = 360° - \beta - \gamma - \angle\, BDC = 60°.$$

Sentiments distingué,
ÉMILE BOREL.

II

Arizona State Teachers College, Flagstaff, Arizona, Dec. 1, 1930.

The proof of the theorem of Morley given by Mr. Jacob O. Engelhardt on page 493 of the Nov., 1930 issue of this Monthly can be simplified as follows:

He obtained the relation

$$(FD)^2 = 16 \sin^2 x \sin^2 z \big[\sin^2 (60° + x) + \sin^2 (60° + z) - 2 \sin (60° + x) \sin (60° + z) \cos y\big].$$

[1] The theorem of Morley is as follows: If three angles of a triangle be trisected, the triangle whose vertices are each the intersection of a pair of trisectors adjacent to a side is equilateral.

[2] This Monthly, vol. 37 (1930), p. 493.

The expression within the brackets is equal to $\sin^2 y$ by virtue of the law of cosines applied to the triangle whose angles[1] are y, $60°+x$, and $60°+z$ and whose circumscribed circle has a radius equal to $\frac{1}{2}$. Therefore

$$(FD)^2 = 16 \sin^2 x \sin^2 y \sin^2 z \text{ and } FD = 4 \sin x \sin y \sin z.$$

Since the expressions for DE and EF are obtainable from the expressions for FD by permuting x, y, and z, and since FD is a symmetric function of x, y, and z, it follows that $FD=DE=EF$.

W. C. RISSELMAN

MATHEMATICS CLUBS

All reports of club activities should be sent to Professor F. M. Weida, George Washington, University, Washington, D. C.

CLUB ACTIVITIES

The Mathematics Club of George Washington University, Washington, D. C.

The officers for the year 1929–1930 were: President, Dr. F. E. Johnston; Secretary, Mr. Michael Goldberg.

The programs for the year 1929-1930 were as follows:

October 21, 1929. "The packing of spheres and hyperspheres" by Mr. Michael Goldberg.
November 4. "A generalization of Riemannian geometry" by Dr. James H. Taylor.
November 18. "Divergent series" by Dr. Florence M. Mears.
December 2. "The evolution of the concept of infinity" by Dr. Tobias Dantzig.
December 16. "The geometry of rigid dynamics" by Dr. Paul Wernicke.
February 17, 1930. "Modern hydromechanics" by Dr. Edgar W. Woolard.
March 3. "Alignment charts" by Mr. Albert Wertheimer.
March 24. "Order of contact of curves" by Mr. Paul Bradt.
April 7. "The summation of series" by Mr. William J. Berry.
May 5. "Primitive roots of prime numbers" by Dr. F. E. Johnston.
June 14. Picnic at Chapel Point, Md.

(Report by Michael Goldberg)

The University of Louisville Mathematics Club, Louisville, Kentucky.

The University of Louisville Mathematics Club was organized in October, 1929. We feel that we have accomplished a great deal in creating during the first year of our existence "a greater interest in and knowledge of mathematics," which is the aim of our organization. We owe our excellent beginning to our advisers, Dr. Guy Stevenson and Dr. Walter Moore. We have an enrollment of twenty, the membership of the club being restricted to persons interested in mathematics.

The officers of the club for the year 1929–1930 were: President, Arthur Ries ('31); Vice-President, Virginia Lee Brightwell ('30); Secretary, Dorothy Fleischmann ('30); Treasurer, Helen Kline ('31); Sergeant-at-Arms, James Teller ('32).

The following programs were given at the regular bi-weekly meetings:

October 30, 1929. Election of officers.
November 15. "The art of paper cutting" by Dr. Moore.
The club was entertained by Dr. and Mrs. Moore.

[1] In this triangle the sides opposite these angles are respectively $\sin y$, $\sin(60°+x)$, $\sin(60°+z)$.

December 5. "Hyperbolic functions" by Arthur Ries.

December 19. "Mentally extracting the cube roots of numbers from one to one billion" by Dr. Sappenfield.

January 9, 1930. "Mathematical treatment of honey cells" by James Teller; Election of officers.

January 25. "Number theory" by Dr. Stevenson.

February 6. "Magic squares" by Mariam Heymann; Initiation of four new members.

February 20. "Squaring the circle" by Grace Lothman.

March 6. "The transcendence of π and e" by Dorothy Fleischmann.

March 20. "The nine point circle" by Wallace Denhard.

April 3. "Apollonius's problem" by Ruth Vogel.

April 16. "The history of mathematics" by Virginia Lee Brightwell; Picnic in Cherokee Park.

May 1. "Mathematical tricks" by George Chenault. The club was entertained by Dr. and Mrs. Stevenson.

May 15. "Number systems" by Edwin Bell; Election of officers. The club was entertained by Dr. and Mrs. Sappenfield.

May 29. Installation of officers; Picnic at Iroquois Park.

(Report by Dorothy Fleischmann)

The Mathematics Club of Hunter College of the City of New York.

The Mathematics Club of the Junior-Senior years at Hunter College held meetings at two-weekly intervals throughout the year 1929–1930. The programs of the meetings included popular topics, such as the trisection of an angle by various methods with a proof of the impossibility of the construction by means of straight-edge and compasses; the golden section and the construction of certain regular polygons; the geometric construction of a regular seventeen-sided polygon and the construction from an algebraic point of view; Non-Euclidean geometry; the fourth dimension; the construction of a straight line by means of the Peaucellier inversor. An especially snappy talk was given by the secretary of the Club on "Geometry of the compasses;" some of the members who were doing honors work in mathematics gave a discussion of "The Dedekind cut," and another member gave a talk on "The history of permutations and combinations" based on an intensive study made in connection with an advanced course in the history of mathematics.

The Club had the privilege of hearing two outside speakers. Professor W. S. Schlauch of New York University gave an address on "Mathematics as an interpreter of life," and Professor Jean Conklin, formerly of the College, gave a very timely talk on "The new planet X."

The outstanding social event of the year was the presentation of a play by the students of the Mathematics Club of the Brooklyn Branch (now a part of the Brooklyn College). It was entitled "Interlopers" and was the adaptation by a member of the faculty of a play by Professor Slaught which appeared in the March, 1928 issue of this Monthly as "The Evolution of Numbers." The large audience was fully rewarded by the lively entertainment which was provided. Besides this event, there was a social affair each semester; in the fall a play "Alice in the Wonderland of Mathematics" was given; in the spring, a feature of interest was a series of songs written by another member of the faculty. The one that proved to be the most popular was sung to the tune of "The Battle Hymn of the Republic" with a ringing chorus:

"Glory, glory to our major,
Glory, glory to our major,
Glory, glory to our major,
Its truth is marching on."

A happy day, June 16th, spent on a trip up the Hudson ended the social activities of the year.

Both the president and secretary are spending 1930–1931 at Bryn Mawr College on graduate scholarships. The officers were: President, Miriam Fassler; Vice-president, Rebecca Rosenblum; Secretary, Julia Cincotti; Treasurer, Violet Moskowitz; Publicity Manager, Muriel Rosner; Faculty Adviser, Prof. Evelyn Walker.

(Report by Lao G. Simons)

The Mathematics Club of the University of Kansas.

Officers: President, Ella Baker; Vice-President, Velt Stafford; Secretary-Treasurer, Helen Kemp; Faculty Adviser, Professor Florence Black; Social Chairmen, Ruth Pratt, Maurice Brown.

The programs for the year 1929–1930 were as follows:

October 14, 1929. Business Meeting.
October 28. Old fashioned ciphering match; refreshments.
November 11. "The left hand of learning" by Professor U. G. Mitchell.
November 25. Methods of solving equations by approximations: (1) "Newton's method" by Ella Baker; (2) "Interpolation" by Philip Bell.
December 9. (3) "Iteration" by Ralph Sickel; (4) "General discussion of the first three" by Professor Jordan.
January 13, 1930. "Interesting problems brought to the mathematics department" by Professor J. J. Wheeler.
February 10. "Div-a-lets" by Vida Dunbar.
February 24. "Magic squares" by Lenore Cummings.
March 10. "Mathematics as applied to artillery work" by Lieutenant Harry F. Myers.
March 24. "Special relativity" by Professor Stranathan.
April 14. Business meeting; Election of Officers.
April 28. "Methods of rapid calculation in business" by Mr. Bell.
May 14. Annual mathematics club picnic. (Report by Florence Black)

The Ohio Wesleyan University Chapter of Pi Mu Epsilon.

The Beta Chapter of Ohio of Pi Mu Epsilon at Ohio Wesleyan University, Delaware, Ohio held regular monthly meetings during the college year 1929–1930. Some of the subjects discussed were: "Condensers," "Plücker coordinates," and "Polar planimeters." Dr. D. C. Miller of the Case School of Applied Science, gave a very interesting lecture in May on the subject, "Sound."

(Report by Raymond F. Felts)

Pi Mu Epsilon and the White Mathematics Club of the University of Kentucky, Lexington, Kentucky.

Meetings of the two organizations alternate. The combined program for the year was the following:

November 20, 1929. "On cubic congruences" by Professor C. G. Latimer.
December 12. "Pot-pourri of mathematics" by the faculty. A meeting for freshmen.
December 19. "On interpolation in mathematics of finance" by Professor D. E. South.
January 9, 1930. "Modification of a proof by Steiner" by Mr. E. J. Canaday.
January 23. "Functions of limited variation" by Professor H. H. Downing.
February 13. "Queen Dido's problem" by Professor Elizabeth Le Stourgeon.
March 6. "Linkages" by Miss Sallie Pence.
March 20. Initiation of new members into Pi Mu Epsilon.
March 27. "On certain graphic considerations" by Professor M. C. Brown.
April 24. "De Jonquière's theory of indices" by Dean P. P. Boyd.
May 1. "Graphic construction of the roots of a quadratic equation" by Miss Alleen Lemons. "Classification of quadric surfaces by invariants and covariants" by Mr. N. B. Allison.
May 15. "An identity in theta functions" by Mr. Smith Park. In the evening the local chapter of Pi Mu Epsilon was entertained at the home of Dean P. P. Boyd. The festivities included initiation of new members.

(Report by F. Elizabeth Le Stourgeon)

The Mathematics Club of Oberlin College, Oberlin, Ohio.

The Mathematics Club of Oberlin College, founded in 1894, was reorganized after a lapse of time in 1922. Since that date, meetings have been held every two weeks of each college year.

Papers are given by students majoring in mathematics, the privilege being extended occasionally to a gifted freshman. Faculty papers are rare, but greatly appreciated. Candidates for a master's degree and undergraduate students who are candidates for honors usually present reports connected with their reading or research. Attendance at the meetings averages about thirty. The climax of the year is the annual May banquet, at which the club is honored by the presence of some mathematician of distinction. Attendance at last year's banquet was forty-five.

The officers for 1929–30 were: President, Ernest Peek ('30); Vice President, Dorothea Zilch ('30); Secretary-Treasurer, Gertrude Brockett ('30); Social Chairman, Frances Gifford ('30). The program committee consisted of Gertrude Brockett, Eugene Buell, and Professor Mary E. Sinclair, faculty adviser for the club.

The programs for the year 1929–1930 were as follows:

October 18, 1929. "Highest common factor" by Dorothea Zilch; "Related theorems about numbers" by Ernest Peek; "Problems suggested for solution" by Professor Marie Johnson.

November 1. "Families of curves of pursuit" by Emily Grace Doane; "Suggested readings in mathematics" by Miss Sinclair.

November 15. "Number domains" by Eugene Buell; "Irrational numbers" by Frances Gifford.

November 29. "Geometric constructions with compasses alone" by Talbot Harding; "Properties of digits of numbers" by Ruth Brummitt.

December 13. Christmas Party. Presentation of the mathematical play "The Evolution of Numbers" by Professor Slaught.

January 10, 1930. "Mathematics as a mode of thought" by Willard V. Quine; "Analogies in plane and spherical trigonometry" by John Insprucker.

January 17. "Large modern telescopes" by John Dudley; "Mathematical prodigies" by Evelyn Smith.

February 7. "The gyroscope" by Professor Carr,; "Congruences, modulo m" by Joseph W. Nadal.

February 21. "Rabbi Ben Ezra and the Hindu-Arabic numerals" by Faith Fitch; "Problems in maxima and minima" by Robert Lemmerman.

March 7. "Problems connected with central forces" by Ross Wilson; "Expression by series of trigonometric, hyperbolic, and exponential functions" by Ruth Roudabush.

March 21. "Actuarial mathematics" by Mary Elizabeth Schubert; "The Pythagorean theorem" by Dorothy Rainer.

April 11. "Curves related to the equilateral hyperbola" by John Hardy; Honors report by Frederick Ficken.

April 25. "Charts and graphs" by Robert Wilkins; Honors report by Gertrude Brockett.

May 7. "Theory of logarithms" by Helen Winder; "Mathematics and music" by Margaret Draeger.

May 23. Banquet. Address by Professor G. A. Bliss of the University of Chicago.

(Report by Gertrude Brockett)

CLUB TOPICS

1931 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California.

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following list of important 1931 centennial dates is presented.

B.C. 570[2] Birth of Anaximenes, prominent pupil of Thales, of Ionian school, Greek mathematician and astronomer.

[1] This Monthly, vol. 37 (1930) pp. 150–151 for a list of 1930 events and for references to previous volumes for corresponding lists from 1925 to 1929.

[2] In lists of dates, similar to the present one, which I have furnished this Monthly since 1925, occasionally events have been included which occurred prior to the beginning of the Christian era. For example in the 1930 list, 570 B.C. was given for the birth of Anaximenes, which is repeated in

B.C. 470 Birth of Hippocrates of Chios, one of the greatest Greek geometers, celebrated for his study of the quadrature of lunes.
370 Death of Democritus of Abdera, Thracian geometer.
70 Geminus of Rhodes, Greek historian of mathematics, flourished.
A.D. 1631. Publication of William Oughtred's *Clavis Mathematicae*, systematic textbook on arithmetic containing practically all known on the subject. Introduction in it of the symbol $\times$ for multiplication.
1631. Thomas Harriot, English mathematician, introduces symbols $<$ and $>$ for "is less than" and "is greater than."
1631. Death of Briggs, who was largely responsible for the invention, computation, and popularization of common logarithms.
1731. Birth of Cavendish, mathematical physicist, who determined the density of the earth by comparison with two lead balls.
1731. Publication of Alexis Clairaut's *Recherches sur les courbes a double courbure*, which secured his admission to the Paris Academy of Sciences when still under legal age.
1731. Death of Brook Taylor, discoverer of Taylor's theorem for the expansion of $f(x+h)$.
1831. Birth of Richard Dedekind, German mathematician celebrated for his work on number theory.
1831. Birth of Amedee Mannheim who designed the Mannheim slide rule.
1831. Birth of James Clerk Maxwell who revolutionized electro-magnetism and established mathematically the electro-magnetic theory of light.
1831. Birth of P. G. Tait, professor at the University of Edinburgh, who formulated the law of the flight of the golf ball after one of his sons had become a brilliant golfer, and who made important discoveries in physics and quaternions.
1831. Discovery by the French mathematician Cauchy of the general theorem which reveals the number of roots, real or complex, which lie within a given contour.
1831. Publication of Plücker's *Analytisch-Geometrische Entwicklungen*, containing improved abbreviated notation.
1831. Death of Sophie Germain, Parisian mathematician, best known for her work on the theory of elastic surfaces.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York.

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematical Introduction to Economics. By Griffith C. Evans. New York, McGraw-Hill Book Co., 1930. xi+177 pages. $3.00

Many writers on economic theory have used mathematical ideas, a few have

---

this year's list. Mr. Edward S. Smith in "A Letter to the Editor" (This Monthly, vol. 37, 1930, pp. 371–72) points out that I have been in error by one year with reference to all dates B.C. He is entirely right and I am very glad to be corrected. The error, which is easily made, is due to the fact that the first year of the Christian era was number *one*, not *zero*. The civilized world has almost universally made the same error in celebrating the bimillenium of Virgil's birth (Oct. 15, 70 B.C.) in 1930 instead of in 1931. See a note by the author, "Virgil Bimillennium Celebrated Prematurely" in *School and Society*, vol. 32 (December 13, 1930), pp. 805–806.

used mathematical notation, and a much smaller number have used mathematics more advanced than the most elementary parts of calculus. Of the last group, Professor Evans was the earliest, with an article in this Monthly[1] for February, 1924, on "The Dynamics of Monopoly," using the calculus of variations; at least he was the earliest if we except certain allusions to the higher mathematics required by statistics, H. L. Moore's work with Fourier series, and the mention in Irving Fisher's doctoral thesis of integrability conditions. However Professor Evans is now writing for those who know no more than calculus. Differential equations and the calculus of variations are used, but the reader is not frightened away by these formidable names, for they do not appear in the book until the fundamental ideas back of them have been insinuated into the mind in the form of economic problems.

Such classical problems as the bartering of nuts for apples give way in this book to questions arising more directly in a manufacturing age. Instead of beginning with supply and demand functions, the first chapter, whose general subject is monopoly, begins with cost and demand functions. Exchange between two individuals and utility are dealt with in the twelfth and eleventh chapters.

Chapter 2 brings out the principle that economic laws should be in forms invariant under changes of units. Chapter 3 deals with competition and what the author terms "cooperation" among a finite number of producers. In Chapter 4 offer and demand depend upon the rate of change of the price as well as upon the price itself; in the final section demand depends upon a sum or integral of past prices. Enter, *sine nomine*, differential and integral equations. Chapter 5 deals with changes in cost and demand functions, and particularly with those changes caused by taxation, while diversified costs, tariffs, and rent form the subject of the next chapter. Further chapters treat of foreign exchange, interest, the equation of exchange, price indices, business cycles, general concepts, production. Finally there are two chapters of more advanced material, with the calculus of variations entering at first anonymously and in a very special problem; a good bibliography of mathematical economics, and an appendix discussing certain further ideas such as the "compartment" theory of economics which Professor Evans set forth in the Proceedings of the National Academy of Sciences in 1925.

When the quantity purchased is written as a function of all past prices as well as of the current one, would it not be well to consider also situations in which the price which buyers are willing to pay is a function of past and current quantities consumed? Indeed the latter situation seems the more widespread, in view of the persistence of habits, particularly food habits, and the inertia produced by the existence of capital equipment involved in utilization. The population of China, though starving, buys immensely more rice than wheat even when the rice costs $2\frac{1}{2}$ times as much per calorie.

In dealing with n competitors the text follows tradition in assuming that

[1] Vol. 31 (1924), pp. 77–83.

all sell at the same price, though they produce different quantities. It would be illuminating to bring out the duality of price with quantity by writing the n prices as functions of the n quantities with non-vanishing Jacobian. This same scheme could be used in treating correlated demand and joint costs. The idea that there must be one single price for all sellers, of whatever sort and wherever located, merely because their products go by the same generic name, is untenable. A little considered but very important feature of competition has been brought out elsewhere by the reviewer: with reference to each seller there are groups of buyers who will deal with him in preference to others, in spite of a moderate price difference, which may vary continuously among buyers. Transportation costs and many other factors varying among buyers supplement price in affecting their decisions. The usual conception of a market as a point, in which there can be only one price at a time, needs to be extended to the notion of a market as something like an area. The result will be a more elegant as well as a more realistic theory, since prices will enter in a manner symmetrical with quantities, and will have a vital bearing on the stability of a competitive situation.

Edgeworth's paradoxical discovery that a tax on an article controlled by a monopolist who also controls a substitute commodity may lead him to reduce both prices, besides paying the tax, might very well be brought out in a course in mathematical economics. This is one proposition of economic importance which everyone can appreciate but which cannot be proved, apparently, without the use of formulae; moreover it is very suggestive of further developments.

Will this or some similar book ever come to replace the bulky "Elements of Economics" now used as sophomore texts? Not, certainly, for the majority of those who now peruse these wordy treatises. But can Professor Evans' book replace such works in the liberal education of the intelligent mathematician, who is bored even as a sophomore by Economics 1 and demands stronger meat? Perhaps a compromise will be worked out. Perhaps some day we shall have university courses in "Economics for Mathematicians," as well as in "Mathematics for Economists."

Any university could immensely improve the average quality of its graduates in economics by requiring that they must all master Professor Evans' book, solving simultaneously the difficulty about overcrowding. This simple and effective remedy for two evils will hardly be taken seriously at the present time. But it is clear that this book and other recent work in mathematical economics by Roos, Schultz, Moore and others will help to bring nearer the time when calculus will be considered an indispensable part of a liberal education.

HAROLD HOTELLING

Algebraic Equations. An introduction to the Theories of Lagrange and Galois. By Edgar Dehn. Columbia University Press, 1930. xi+200 pages. $4.25.

A number of modern expositions of the theory of equations from the group stand-point have given the reviewer the impression that their authors essayed

to present that elegant theory with a minimum number of words, with the result that the student who approaches the subject for the first time soon finds himself lost in a maze of new concepts from which he may extricate himself if he has sufficient time and pugnacity and knows how to devise illustrations which will fix these concepts in mind. The reader who has had this impression will welcome Dr. Dehn's book because of the slowness with which the theory is developed, the attention which is paid to details, the repetition of difficult and important concepts, and the care which the author takes not merely to prove theorems but to give the student a perspective on the subject as a whole.

No less than five chapters are devoted to the theory of permutations and permutation groups in order that the student may properly cultivate the technique necessary for a comprehension of the theory. If the author had only included a list of exercises, little more could have been asked. Although Lagrange's theory is included in Galois' theory, we are glad to observe that the author has devoted two chapters to Lagrange's work, explaining the weaknesses of Lagrange's theory in detail without diminishing the tribute which he pays to Lagrange's genius. The reviewer agrees with the author that a knowledge of Lagrange's theory is necessary for a proper appreciation of Galois' theory.

We regret to observe that the text is not free from defects. On page 24 the author states in passing that the relation

$$(stu \cdots)^2 = s^2t^2u^2 \cdots$$

implies that every pair of the substitutions $s, t, u, \cdots$, must be commutative. This is not true, as is shown by the example: $s=(123)$, $t=(12)$, $u=(132)$.

On page 112, in considering a lemma which is useful in proving the Jordan-Holder theorem, the author states that if N and N' are maximal invariant subgroups of G, then $G=NN'$, implying by this notation that every element of G is expressible as a product of an element of N by an element of N'. This should be proved.

A more serious gap occurs on page 170 where the author proves that, if the group of an equation is reduced by an adjunction to the domain of rationality, the same reduction can be effected by adjoining a root of a resolvent equation, and infers that a solvable equation has a solvable group. The gap may be bridged by proving that the group of the binomial equation $x^n=a$, in a field containing all the nth roots of unity, is a cyclic group, and then invoking the theorems of §75.

The student who expects to consult other works on the same subject must be warned that Dr. Dehn's definitions of certain technical terms differ from those of other writers. The algebraic operations, according to Dr. Dehn, include the rational operations and the extractions of roots; most algebraists consider the process of solving an algebraic equation or of adjoining a root of an equation to a field an algebraic operation. Consistent with this definition is his definition (page 131) of a relative algebraic domain, which conflicts with the

usage of this phrase by writers on algebraic numbers. The phrase "class of permutations (page 118) is translated literally from the German; we wish Dr. Dehn had used the recognized English equivalent "complete set of conjugates."

We note that the author defines (page 27) a permutation group as a set of permutations which leave a function unaltered, and then reconciles this definition with the more familiar one. Consistent with this definition he employs the idea of functions belonging to a group in considering Lagrange's theorem (page 30) and conjugate groups (page 33). This procedure may be defensible on pedagogical grounds, but many will object on the general principle that a proof of a theorem should, if possible, involve only essential concepts.

LOUIS WEISNER

Neuere Methode und Ergebnisse in der Hydrodynamik (Band I, *Mathematik und ihre Anwendungen in Monographien und Lehrbüchern*). By C. W. Oseen. E. Hilb, Leipzig, 1927. Akademische Verlagsgesellschaft M.B.H.

In this volume Oseen assembles the researches on fluid dynamics which date from the year 1910. No small part of these important results are due to Oseen and his pupils, Faxén and Zeilon, and published in Scandinavian journals. The results in the main concern motions of viscous fluids together with the limiting case of zero viscosity. It is found for example, that the results in this limiting case differ widely from the theory of perfect fluids but are more in agreement with the facts of experiment and observation, than the theoretical results based on ideal frictionless fluids.

The author first undertakes to investigate the linear system of partial differential equations which are obtained by neglecting the second degree terms in the general equations of motion of a viscous but incompressible fluid. These equations are valid for slow motions and are important for the field of colloidal chemistry. By this means the author resolves Whitehead's paradox. Lamb has also resolved Stokes' paradox, by Oseen's methods.

The ultimate objective of the author was to pass in complete fashion from the case of viscosity, to the limiting case of zero viscosity. This end has not been reached for the general system. For the linear case, the passage to the limit has been achieved. The results thus obtained are substantially in agreement with the results for actual fluids.

It appears then that the so-called paradox of D'Alembert that in the steady motion of a perfect fluid there is no pressure exerted against an immersed body, rests on a faulty passage to the limit. Zeilon showed more than this negative result. In Part III his results describe quantitatively the resistance and pressure on the head end of the immersed body.

In the first chapter of the present work, the hydrodynamic differential equations are derived, first on the assumption that the first derivatives of the pressure function and velocity components with respect to the position coordinates, and the accelerations, are continuous. The impulse theorem is used. Later the fundamental system of equations is derived in the form of integro-differential

equations directly from mechanical principles and assuming merely integrability of the above quantities. At the end of the first section, the special cases of steady motion with respect to a fixed reference system and motion with uniform velocity are discussed.

In the second section, the method of solution to be used later is illustrated by means of the generalized Poisson differential equation. For the interior problem a power series solution in a parameter is obtained from an infinite system of Poisson equations. Green's theorem and Green's functions are used. For the exterior problem a characteristic difficulty is exhibited arising from the fact that the power series is not convergent in the neighborhood of the zero value of the parameter.

In the third section, a generalized Green's theorem is used for the Stokes form of the system of differential equations (linear approximation). The fundamental solutions are found and by means of these the complete set of equations for steady motion are transformed into integro-differential equations.

In the fourth section, the generalized Stokes equation is solved and the fifth section gives the solution of the most general linear system.

In section seven, the first application of the integro-differential equations is made. The problem is that of determining the motion of a viscous but incompressible fluid, filling all space, when the initial velocity is known. The following result is obtained: if the motion is regular (defined in a certain specific sense) at $t=T$, then there exists a positive number $\tau>0$ such that it is regular in the time interval $T<t<T+\tau$. However, the motion may cease to be regular and become turbulent at some later time.

Section eight, the last of Part I, deals with the simplest examples of turbulence in a viscous fluid. The two problems considered are (1) a single straight line vortex with rotational symmetry and (2) the interaction between two such line vortices. The solution for the first problem is obtained in exact form immediately from the integro-differential equations, and is only a little more complicated than the Helmholtz solution for the ideal fluid.

Part II treats the boundary value problems proper of hydrodynamics. In section 9, an exact solution is given of the linear Stokes equation for steady motion subject to the conditions that the velocity vanishes at infinity and takes on prescribed values on the surface of a sphere. Lamp had already obtained a series solution, but here the solution is given in closed form. For special cases, more convenient and direct methods can be used than employing the general solution. The Stokes formula for the pressure on a sphere, moving with constant speed in a fluid, can be obtained, such as raindrops through the air, fine particles suspended in a liquid, etc. At the end of section 9, the solution for the Stokes equation subject to prescribed boundary values on a given plane is obtained.

Section 10 yields an approach to the general hydrodynamical problem of nonsteady motions. From the results obtained at this point, a solution of the Prob-

lem of Boussinesq is obtained on the non-steady motion of a small sphere in a viscous fluid.

Section 11 reproduces the theory of the steady motion of an ellipsoid in a viscous fluid due to Overbeck. This is given in preparation for later work.

Section 11 points out that even though the use of the Stokes equations cannot be justified in general, the results for the motion of a sphere for a sufficiently small neighborhood of the sphere are in substantial agreement with the data of colloidal chemistry.

Section 12 is a summary of the investigation of H. A. Lorentz on the steady motion of a sphere in a viscous fluid which is bounded by a plane wall.

Section 13 investigates the motion of a sphere between two parallel plane walls.

Section 14 is a report on the vortex interaction of two spheres moving in a viscous fluid. The results are due to Smoluchowski, Faxén, Dahl, Stimson, and Jeffery.

Section 15 points out that the paradoxes of Stokes and Whitehead are resolved if we note as in Section 11 that the Stokes equations are not approximations for the general equations.

The extended or generalized Stokes equations are obtained by Oseen by retaining those terms of the general system which should not be neglected. These equations and the accompanying boundary value problems are solved in the second half of part II.

Section 16 solves the steady motion of a sphere for a viscous fluid by means of the extended Stokes equation. The solution thus obtained differs in essential character from that of Stokes. The chief difference is that while Stokes' solution has fore and aft symmetry the solution of Oseen has decided dissymmetry in the form of a tail of vortices behind the sphere. A new term is obtained as a second approximation proportional to the square of the velocity. In addition a second approximation is obtained for the motion near the sphere.

Section 17 deals with the motion of a circular cylinder related to the work of Lamb, Barstow, Cone, and Lang.

In Section 18, the ellipsoid is considered and Overbeck's results appear as a first approximation. In the second approximation terms appear in the expression for pressure proportional to the square of the velocity. For the elliptic cylinder results due to Harrison are stated without proof.

Section 19 contains Faxén's extensions of Lorentz's work given in section 12 and in section 20, Faxén's extension of Ladenburg's work on motion of a sphere in a tube is given.

Section 21 treats the same problem as in Section 14 from the point of view of the extended Stokes' equation.

Part III contains by far the most important results, viz: the passage to the limit of zero viscosity, carrying over the solution for the extended Stokes equation.

Section 23 carries this through for a thin plate. In Section 24 this is extended

to a more general surface. In Section 25, the same problem is carried through for a solid body.

In Section 26, the solutions of certain Fredholm integral equations are used to obtain new results for the problem of steady motion of solid bodies.

Sections 27–30 make use of a method due to Hilbert to solve the steady motion of a general cylinder, circular cylinder and flat plate. These results are of fundamental importance in the computation of the lift and drag forces of aeronautics.

Finally in Section 31 by means of elliptic functions the motion of a circular plate is developed.

In addition to the valuable results themselves it is of interest to realize the even greater use which is being made in various branches of mathematical physics of boundary value problems in ordinary and partial differential equations.

H. J. ETTLINGER

Intermediate Mechanics. By D. Humphrey. Longmans, Green and Company, London and New York, 1930. $4.20.

Applied Mechanics. By Norman C. Riggs. The Macmillan Company, New York, 1930. $3.75.

These two books, although written for apparently different purposes, have many common points of merit, so that they may be discussed together. While "Applied Mechanics" is designed primarily for students of engineering, particularly those who are interested only in the technical aspects of engineering, it should make an especial appeal to that relatively small group of engineers who desire to extend their knowledge beyond the mere "rule of thumb" into the deeper foundations and developments of the general theory. On the other hand, "Intermediate Mechanics," intended for students of physics, might also be used to advantage by the engineers.

Both books are elementary in the sense that they begin by developing the concepts and postulates peculiar to mechanics. The treatment of vectors in "Applied Mechanics" requires a knowledge of geometry alone. Velocity, acceleration, and force require the notion of a limit and, in the case of these books, it is so well presented that the student should have but little difficulty in understanding them. They are particularly attractive from the fact that clear and precise definitions are formulated mathematically whereby applications can be made directly to the problems.

Such topics as D'Alembert's principle and systems of conservative forces, ideas which form a nucleus for more advanced work, are presented in an intuitive manner, with, however, sufficient analytical background so as not to fail to impress the student. The selection of problems in both texts is to be commended. In "Applied Mechanics" a large number of problems, illustrative of those met with in technical practice, are given. "Intermediate Mechanics" contains a large number of interesting examples most of which are genuine illustrations of mechanical principles rather than mere exercises in algebra or trigo-

nometry. In the latter chapters of both texts many problems are considered which are to be found in a first course of differential equations. They are treated with enough detail so that such a course need not be required as a prerequisite.

Mechanics is not an easy subject and its inherent difficulties makes many problems desirable. In these texts the exercises have been wisely chosen or devised and the reviewer feels that the space devoted by the authors to illustrative examples will be well rewarded by the encouragement it offers to the student.

S. B. LITTAUER

Advanced Algebra. By Palmer H. Graham and F. Wallace John. Prentice Hall, Inc., New York. 1930. 257 pages. $1.85.

Perhaps the first question which arises concerning any new book in the field of college algebra is what new or added feature does the book in question possess which the many others we already have do not. This book by Graham and John is designed, the authors say, to meet the needs of the ordinary freshman class, and has been written throughout with the view of enabling the student to use the book with as little help from the instructor as possible.

The first six chapters are devoted to what the authors say may be considered review material. One of these chapters is on logarithms. It seems to the reviewer that the study of logarithms is of sufficient value to warrant a more important place in the book. A somewhat similar situation exists concerning variation, which is considered only incidentally in the study of linear equations. The solution of equations is taken up in succeeding chapters. In this sequence, however, the authors meet with difficulty in proving the remainder theorem. They state, on page 104, that $x^n - r^n$ is divisible by $x-r$, without having mentioned it or proved it previously in the book. Preceding the second proof of the theorem that an equation $f(x)=0$ of the nth degree can not have more than n roots the authors make this statement, on p. 110: "In case all the roots are different, we can also prove the theorem as follows." It seems to the reviewer that in order to prove the theorem by the method employed, it is not necessary to specify that all roots are different for the proof would still hold even though some of the roots were equal.

In discussing the solution of a system of homogenous linear equations, it seems that the student might have some difficulty in grasping the significance of the indeterminate form 0/0. And since the desired result may be obtained in a more natural manner, it would be preferable to avoid using this form. The authors give the two symbols for permutation, ${}_nP_r$ and ${}^nP_r$, specifying they will use ${}_nP_r$. A similar specification is neglected for the symbols of combinations ${}_nC_r$ and ${}^nC_r$, for they use the former in the discussion of combinations and the latter in the chapter on the binomial theorem.

Since the text has been written with the idea in view of having the student use it with as little help as possible from the instructor, it would seem that the inclusion of an index and answers to some of the problems would be desirable.

There is a goodly number and wide variation of problems, apparently chosen with care. The explanation of all topics is given in a clear and concise manner, and simply enough for the average freshman to comprehend. The authors religiously follow their plan of presenting concrete examples before giving the formal proof of a theorem. A very commendable feature is a tabulation, at the end of the book, of definitions used in the text. The historical notes given at the end of several chapters, will be found interesting and helpful by student and instructor. The physical make-up of the book is attractive. On the whole, the reviewer believes the book to be quite teachable, and a text from which one can obtain good results.

H. M. HOSDORF

Trigonometry. By A. R. Crathorne and E. B. Lytle. Henry Holt and Company, 1930. Text: ix+199 pages; tables: xvi+95 pages. $1.96.

This text in trigonometry, although conventional with respect to its content, presents its material in such a concise and straightforward manner that its appearance is not only justified but welcomed. The authors have made a consistent and successful attempt to make their work sound pedagogically, rigid mathematically, and interesting to the student. The material of the text is nicely coördinated with the traditional course in college algebra; and at the same time it gives a good foundation in trigonometric analysis as required for the subsequent study of mathematics.

In numerous instances, after formulas are stated algebraically, they are translated or put into words to make the meaning clearer to the student. At the beginning of the chapter on logarithms is given a discussion of approximate numbers and errors. Since considerable computation is required in a course in trigonometry this explanation is sorely needed by the average freshman. The text first defines the trigonometric functions for acute angles, and makes the extensions later. This procedure helps to prevent a beginner from being overwhelmed at the start by more new material than he can assimilate.

The book offers a large and well varied set of problems, many of the practical examples being illustrated by figures. Answers are given to odd problems only, thereby satisfying all demands in this respect. In both problems and answers angles are expressed to the nearest tenth of minute, seconds being avoided. This is probably the preferable treatment if not the most common.

Although the book occasionally deviates from the straight and narrow path to make an interesting observation, the reviewer feels that the text could be improved by further references to the historical development of the subject and items of human interest. The preface contains an outline of a trigonometry course in thirty assignments. Even though this may not meet the needs of a particular institution, it at least offers a helpful suggestion.

The physical make-up of the book is quite pleasing. The binding, figures, and arrangement upon the page are all excellent. The book is convenient in size (5×8 instead of the larger and more common 6×9). However the smaller

page necessitates smaller type, which is objectionable in the tables, and somewhat so in the statements of the problems. A class of thirty students who had just completed a course of trigonometry examined the book and was very favorably impressed by it. Their only criticism concerned the size of type, mentioned above.

The book appears to be a polished product, and its many points of excellence will undoubtedly make it one of the most popular college texts in trigonometry.

WAYNE DANCER

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3475. *Proposed by Emma T. Lehmer, Brown University.*

Let $F(x_1, x_2, \cdots, x_n)$ be any integral, symmetric function of an even number, n, of variables and of degree less than n. Further let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be all the odd numbers less than $2n$ and $\mu_1, \mu_2, \cdots, \mu_n$ be all the numbers prime to 6 and less than $3n$. Finally let

$$\cos(\pi\lambda_\gamma/n) = \alpha_\gamma, \quad \cos(2\pi\mu_\gamma/(3n)) = \beta_\gamma \quad (\gamma = 1, \cdots, n).$$

Prove that

$$F(\alpha_1, \alpha_2 \cdots, \alpha_n) = F(\beta_1, \beta_2, \cdots, \beta_n).$$

3476. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given the cubic, $x^3+ax+b=0$, with rational roots α, β, γ; show that the equation $\alpha y^2+\beta y+\gamma=0$ also has rational roots.

3477. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If two points, harmonically separated by the centers of two unequal circles (spheres), are diametrically opposite on a circle (sphere) coaxial with the given circles (spheres), these two points are the centers of similitude of the two given circles (spheres).

Note: This is the converse of the property of the circle (sphere) of similitude. (See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 196.)

3478. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

A segment M_0M_n of unit length is divided into equal parts by points $M_1, M_2, \cdots, M_{n-1}$. Another segment M_0C of length $\sqrt{(p/n)}$ (p being a given constant) is drawn perpendicular to M_0M_n, and C is joined with the points of division. M_1M_2' is drawn parallel and equal to CM_2, $M_2'M_3'$ parallel and equal to CM_3, and finally $M_{n-1}'M_n'$ parallel and equal to CM_n. Find the limit of the locus of the vertices of this broken line as $n\to\infty$, and consequently $M_0C\to 0$.

UNSOLVED PROBLEMS

Solutions are desired for the following problems, which are the remaining unsolved problems proposed in 1913.

191(187) [1913, 196; 1919, 214]. *Proposed by L. E. Dickson.*

Find an amicable number triple by solving one of the equations (other than the last) in this Monthly, vol. 20 (1913), p. 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

279(274) [1913, 222; 1919, 214]. *Proposed by W. W. Landis, Dickinson College.*

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

196(192) [1913, 223; 1919, 214]. *Proposed by Charles Macauley, Chicago, Ill.*

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column: all the b's in the same column; all the c's in the same column, etc.

348 [1913, 312; 1919, 268]. *Proposed by E. L. Dodd, University of Texas.*

Let $(x_1, x_2, \cdots, x_n)$ be a point in n dimensions lying in the "sphere" S defined by

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1.$$

Let T be that part of S defined by a set of n linear homogeneous inequalities with non-vanishing determinant; thus:

$$a_ix_1 + b_ix_2 + \cdots + k_ix_n \geqq 0, \quad i = 1, 2, \cdots, n.$$

Find the value of

$$\frac{\int_T \cdots \int dx_1 \cdots dx_n}{\int_S \cdots \int dx_1 \cdots dx_n};$$

in other words, find the magnitude of a "solid angle" in n dimensions, with the "sphere" as unit solid angle.

Note:—This problem was discussed and left unsolved by Schläfli in the Quarterly Journal of Mathematics for 1858, 1860, 1867. *Editor.*

202 [1913, 313; 1919, 312]. *Proposed by A. R. Schweitzer, Chicago, Ill.*

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example such a system is

12,	23,	34,	45,	56,	67,	78,	89,	91,
13,	24,	35,	46,	57,	68,	79,	81,	92,
14,	25,	36,	47,	58,	69,	71,	82,	93,
51,	62,	73,	84,	95,	16,	27,	38,	49.

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set, (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set, (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating n-ads for $n>4$ is obvious.

SOLUTIONS

2844 [1920, 326]. *Proposed by the late J. L. Riley, Stephenville, Texas.*

Decompose into simple fractions the number 6099380351/1271808720 (Gauss, *Disq. Arith.*, *Werke*, vol. 1, pp. 386–387).

Solution by Otto Dunkel, Washington University.

The fraction may be written

$$f = 4 + \frac{1012145471}{1271808720}.$$

The denominator is the product of the relatively prime factors $59\cdot 47\cdot 49\cdot 16\cdot 13\cdot 9\cdot 5 = 59\cdot 21556080$. By Euclid's method it will be found that

$$1012145471 = 52 \cdot 21556080 - 1843571 \cdot 59,$$

and hence

$$f = 4 + \frac{52}{59} - \frac{1843571}{21556080} \cdot$$

We now reduce the last fraction on the right in the same manner. The successive reductions may be made by use of the following equalities:

$$\begin{aligned} 1843571 &= 107533 \cdot 47 - 7 \cdot 458640, \\ 107533 &= 27 \cdot 9360 - 2963 \cdot 49 \\ 2963 &= 368 \cdot 16 - 5 \cdot 585, \\ 368 &= 11 \cdot 13 + 5 \cdot 45, \\ 11 &= 4 \cdot 5 - 9. \end{aligned}$$

The result after a slight reduction is

$$f = 1 + \frac{52}{59} + \frac{7}{47} + \frac{22}{49} + \frac{11}{16} + \frac{5}{13} + \frac{4}{9} + \frac{4}{5} \cdot$$

3420 [1930, 196]. *Proposed by the late G. B. M. Zerr.*

Let $x=f(t)$ be the equation giving the horizontal distance of a projectile in a resisting medium in terms of the time t. Prove that the vertical distance is given by

$$y = -gf(t)\int \frac{dt}{f'(t)} + g\int \frac{f(t)}{f'(t)}dt + Af(t) + B,$$

where A and B are constants.

Solution by William Hoover, Columbus, Ohio.

Let ds/dt be the total velocity of the projectile at any instant t, and let the resistance be $k(ds/dt)^n$, where k and n are positive constants. The equations of motion are then

$$\ddot{x} + k(ds/dt)^n dx/ds = \ddot{x} + k(ds/dt)^{n-1}\dot{x} = 0, \tag{1}$$

and

$$\ddot{y} + k(ds/dt)^{n-1}\dot{y} + g = 0. \tag{2}$$

Then (1) is satisfied by $x=f(t)=f$; and hence $\ddot{f}+k(ds/dt)^{n-1}\dot{f}=0$. Solving this last equation for $k(ds/dt)^{n-1}$ and setting its value in (2), we have

$$\ddot{y} - (\ddot{f}/\dot{f})\dot{y} + g = 0. \tag{3}$$

In order to solve this last equation, set $\dot{y}=u\dot{f}$; and then (3) becomes $\dot{u}+(g/\dot{f})$

$=0$. The integration of this equation gives $u=-g\int(dt/\dot{f})+A$, and we have then $\dot{y}=-g\dot{f}\int(dt/\dot{f})+A\dot{f}$. Finally, we obtain

$$y = -g\int \dot{f}\left[\int (dt/\dot{f})\right]dt + Af + B.$$

Integrating by parts the first term on the right, we have the required result.

3425 [1930, 260]. *Proposed by Pauline Sperry, University of California.*

Cauchy's linear equation $\sum_{i=0}^{n} c_i x^{n-i} D^{n-i} y = X(x)$, where the c_i are constants, is reduced to an equation with constant coefficients by the substitution $x=e^z$. Then $x^k D^k y = \sum_{i=0}^{k-1} a_i^{(k)} D^{k-i} y$, where $Dy=dy/dz$. Show that the $a_i^{(k)}$ build a sort of Pascal's triangle with k elements in the kth row, in which any element of the $k+1$st row may be obtained from those in the kth row by taking the two numbers diagonally above it to the left and right and subtracting k times the former from the latter with the understanding that if either of the diagonals takes one outside the triangle the corresponding term is zero.

Solution by Eugene Stephens, Washington University.

The proof of the proposition in the problem may be obtained from the known symbolic derivative formula

$$x^k D^k = \prod_{j=0}^{k-1} (\theta - j), \quad x = e^z, \tag{1}$$

where $D^k y = d^k y/dx^k$ and $\theta^i y = d^i y/dz^i$. A proof of this formula will be given. By direct differentiation we easily find

$$xD = \theta; \quad x^2D^2 = \theta^2 - \theta = \theta(\theta - 1). \tag{2}$$

These results suggest (1); and the proof of (1) may be obtained by mathematical induction. Taking the derivative with respect to x of both sides of (1), and multiplying the result by x, we have

$$kx^kD^k + x^{k+1}D^{k+1} = \theta \prod_{j=0}^{k-1} (\theta - j).$$

Then, by the use of (1) and by the transposition of a term, we have

$$x^{k+1}D^{k+1} = \prod_{j=0}^{k} (\theta - j),$$

and this completes the proof of (1).

The expansion of the right side of (1) may be written

$$x^kD^k = \sum_{i=1}^{i=k} a_i^{(k)} \theta^i, \quad a_k^{(k)} = 1, \quad a_1^{(k)} = (-1)^{k-1}(k-1)!. \tag{3}$$

Hence

$$x^{k+1}D^{k+1} = \sum_{i=1}^{k+1} a_i^{(k+1)}\theta^i = (\theta - k)\sum_{i=1}^{k} a_i^{(k)}\theta^i;$$

and then

$$a_i^{(k+1)} = -ka_i^{(k)} + a_{i-1}^{(k)},\ a_{k+1}^{(k+1)} = a_k^{(k)} = 1,\ a_1^{(k+1)} = -ka_1^{(k)}. \tag{4}$$

The first two rows of the triangle are given by (2); and (4) enables us to write the following rows in turn, thus:

k							
1				1			
2			1		-1		
3		1		-3		2	
4	1		-6		11		-6

If we write $|a_i^{(k)}| = C_k^{(k-i)}$, the positive integers $C_k^{(k-i)}$ are the Stirling numbers of the first kind,[1] or the factorial coefficients of rank k.

Also solved by J. P. Dalton and Mabel S. Graham.

3426 [1930, 260]. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:

$$\sum_{i=0}^{t}(-1)^i\binom{r}{i}\frac{(2r-2i-1)!}{(r-2i)!} = (r-1)!,$$

where $t = r/2$ if r is even and $t = (r-1)/2$ if r is odd.

Solution by J. P. Dalton, University of Witwatersrand, South Africa.

This elementary binomial series may be evaluated by means of the identity

$$(1-x^2)^r(1+x)^{-r} = \sum_{i=0}^{r}(-1)^i\binom{r}{i}x^{2i}\sum_{j=0}^{\infty}(-1)^j\binom{r+j-1}{j}x^j = (1-x)^r.$$

Equate the coefficients of x^r and we have the required result.

Note by the Editors: The solver also gave a solution of 3399 [1929, 543] which is the same as II [1930, 323]. The method of solution given in III [1930, 323] applies also to this problem.

3427 [1930, 260]. *Proposed by L. S. Johnston, University of Detroit.*

Let

$$S_n \equiv \tfrac{1}{2}n(n+1),\ C_{n-k} \equiv (-1)^{k+1}S_{n+1-|k|};$$

---

[1] Nielsen. *Handbuch der Theorie der Gammafunktion*, Teubner, 1906, p. 67. Numerous references to the literature of these numbers are there given. See also the article, *A note on Stirling's numbers*, by Ginsburg in this Monthly, vol. 35 (1928), pp. 77–80.

show that the equation

$$\sum_{k=-n}^{k=n} C_{n-k}x^{n+k} = 0$$

has no negative roots, and that the number of positive roots is two or zero according as n is odd or even.

Solution by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.

This equation is easily seen to be the standard type reciprocal equation

$$\sum_{j=0}^{n-1}(-1)^{n+j-1}S_{j+1}(x^j + x^{2n-j}) + (-1)^{2n-1}S_{n+1}x^n = 0.$$

It has no negative roots (degree even: alternating signs). On summing the series on the left we get

$$\frac{(-1)^{n-1}}{(1+x)^3}[1 + x^{2n+3} + (-1)^n(2n+3)(x^{n+1} + x^{n+2})] = 0.$$

The numerator has $2n+3$ roots in all. Of these, three are the triple negative root introduced in the process of summing; hence there are $2n$ positive or complex roots. By the Descartes rule the number of positive roots cannot exceed two, therefore the given equation has either two positive roots, or none. If n is even

$$-[1 + x^{2n+3} + (2n+3)(x^{n+1} + x^{n+2})]$$

can have no positive root. If n is odd

$$[1 + x^{2n+3} - (2n+3)(x^{n+1} + x^{n+2})]$$

has a positive root lying between 0 and 1. Since it has one positive root, it must have two, and no more. This completes the proof.

3429 [1930, 261]. *Proposed by Paul Wernicke, Washington, D. C.*

Given a tetrahedron $ABCD$ and a point P not on one of its edges. Draw through P three lines, each meeting two opposite edges. Express the ratios in which the latter will be divided by these points of intersection in terms of four quantities a, b, c, d, the edge AB being divided in the ratio $a:b$, etc.

Solution by Vladimir F. Ivanoff, San Francisco, California.

The planes s_1, s_2, s_3 passing through P and DA, DB, and DC, respectively, have the line PD in common. Denote the intersection of PD with the plane ABC by E. The lines EA, EB and EC are then the intersections of the plane ABC with s_1, s_2, and s_3. Denote the intersection of EA and BC by M_{bc}; the intersection of EB and AC by M_{ac}; the intersection of EC and AB by M_{ab}.

Then the lines PM_{ab}, PM_{ac} and PM_{bc} pass through opposite edges of the tetrahedron.

For the segments AM_{ab}, BM_{ab}, BM_{bc}, etc. we have

$$(1) \qquad \frac{AM_{ab}\cdot BM_{bc}\cdot CM_{ac}}{AM_{ac}\cdot BM_{ab}\cdot CM_{bc}} = 1,$$

where the ratio is considered positive when the point of division lies within the finite segment.

It we take

$$AM_{ab}:BM_{ab} = a:b$$

and

$$BM_{bc}:CM_{bc} = b:c,$$

it follows from (1) that

$$AM_{ac}:CM_{ac} = a:c.$$

If we take

$$AM_{ad}:DM_{ad} = a:d,$$

we can, by similar process, find the remaining ratios:

$$BM_{bd}:DM_{bd} = b:d$$

and

$$CM_{cd}:DM_{cd} = c:d.$$

3432 [1930, 314]. *Proposed by Vladimir F. Ivanoff, San Francisco, California,*

Given n points, $A_1, A_2, A_3, \cdots, A_n$ in a plane. Arbitrary lines, A_iA_j, are drawn (joining pairs of these points) provided that they do not intersect each other. Prove that, regardless of the relative positions of the given points the possible number of lines is $3(n-2)$ if $n \geqq 3$.

Solution by Lester R. Ford, The Rice Institute.

Take n points, $A_1, \cdots, A_n$, on a sphere and let paths joining them be drawn on the surface of the sphere. This is an equivalent problem (by stereographic projection, for instance) from the standpoint of Analysis Situs.

Let the maximum number of paths be drawn. An additional assumption is necessary. We may require either (1) that not more than one path shall be drawn between two points; or (2) that no path shall be continuously deformable into another path between the same end points without meeting some other path.

The given points, the paths connecting them, and the regions into which these paths separate the surface of the sphere may be looked upon as the vertices, edges, and faces, respectively, of a polyhedron; and Euler's formula,

$$V + F = E + 2,$$

is satisfied. Here $V=n$; and E is the required number of paths.

The faces of this polyhedron are triangular. Suppose, on the contrary, that there is a face, f, with four or more sides, and let A_i, A_j, A_k, A_l be consecutive vertices of f. Under assumption (2), a path A_iA_k or a path A_jA_l drawn in f is

an allowable path. If assumption (1) is made, either A_i and A_k or A_j and A_l may be already joined by a path lying outside f, but both pairs cannot be so joined since the two paths would cross; and there is again an allowable path in f. In either case, the hypothesis that the maximum number of paths has been drawn is violated.

Now F triangles have $3F$ sides. Since each edge forms a side of two triangles, we have $E=3F/2$, or $F=\frac{2}{3}E$. Substituting in Euler's formula:

$$n+\tfrac{2}{3}E=E+2;$$

whence

$$E=3(n-2),$$

which was to be proved.

Also solved by A. G. Clark and Paul Wernicke.

3434 [1930, 315]. *Proposed by Harry Gwinner, Baltimore, Maryland.*

Evaluate

$$\lim_{x\to 0}(1+x)^{\log x}.$$

Solution by Ralph P. Agnew, Cornell University.

On letting y represent the expression of which the limit is to be found, we have $\log y=\log x \log(1+x)$. Elementary methods suffice to show that as $x\to 0$, $x\log x\to 0$ and $(1/x)\log(1+x)\to 1$. Hence,

$$\lim_{x\to 0}\log y=\lim_{x\to 0}[x\log x]\cdot[(1/x)\log(1+x)]=0.$$

Thus, the required limit is 1.

Also solved by S. F. Bibb, A. G. Clark, D. C. Duncan, Emma M. Gibson, V. F. Ivanoff, and William Orange.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor R. C. Archibald, of the department of mathematics at Brown University, has been appointed lecturer in mathematics at Harvard University for the second semester of the present academic year.

Dr. Wilhelm Blaschke, Director of the Mathematical Seminar, University of Hamburg, will lecture at the Johns Hopkins University as James Speyer Visiting Professor of Mathematics from March 2, to May 8, 1931. He will lecture on "Topological Questions in Differential Geometry" four hours weekly and will also be available for consultation by students.

Professor Harald Bohr, of the University of Copenhagen, has been appointed professor of physics at Princeton for the second half of this academic year.

Dr. W. L. Bragg, of the University of Manchester, has been appointed lecturer in mathematics at Cornell University for the year 1930–31.

Dr. Egon S. Pearson of the Biometric Laboratory, University of London, will lecture on Mathematical Statistics during the coming summer session from June 8 to July 16, inclusive, at the University of Iowa.

Professor R. C. Archibald, of Brown University, delivered his lecture on "Mathematics Prior to the Greeks," at Rutgers University on Friday evening, December 5, 1930.

Miss Helen Calkins has been appointed professor of mathematics at the Pennsylvania College for Women, Pittsburgh.

Mr. C. A. Lovell, of the Drexel Institute, has been appointed a member of the technical staff of the Bell Telephone Laboratories.

Professor Richard Morris addressed the Mathematics group of Haverford College on November 19, 1930, speaking on "Some Modern Geometry of the Triangle with reference to the Orthopole."

Assistant Professor W. C. Risselman, of the University of Pittsburgh, has been appointed professor of mathematics at the Arizona State Teachers College, Flagstaff.

Dr. C. W. Andrews, Librarian Emeritus of the John Crerar Library, died on November 20, 1930. He had been a member of the Mathematical Association since September, 1925.

Mary E. Caster, teacher of mathematics in the Eastside High School, Patterson, N. J., died on December 7, 1930. She was a charter member of the Mathematical Association.

Major C. H. Chepmell, of Bristol, England, a member of the Mathematical Association since May, 1921, died on November 18, 1930.

J. O. Eckersley, chief engineer, Department of Markets, New York City, a charter member of the Association, died on November 11, 1930.

Associate Professor R. A. Sheets, of Denison University, died in Colorado on December 3, 1930, after an illness of nearly a year. He had been a member of the Mathematical Association since December, 1921.

Professor H. E. Trefethen, a teacher of mathematics and astronomy in Colby College for nearly twenty years, died on November 3, 1930. He was a charter member of the Mathematical Association.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 3, MARCH

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

THE FIFTEENTH ANNUAL MEETING OF THE ASSOCIATION

The fifteenth annual meeting of the Mathematical Association of America was held at Cleveland, Ohio, on Wednesday and Thursday, December 31, 1930, and January 1, 1931, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred sixty-one were in attendance at the meetings, including the following one hundred ninety-seven members of the Association:

C. R. Adams, Brown University
R. P. Agnew, National Research Fellow, Poland, Ohio
O. P. Akers, Allegheny College
J. W. Alexander, Princeton University
R. B. Allen, Kenyon College
Nola L. Anderson, Sophie Newcomb College
R. C. Archibald, Brown University
C. S. Atchison, Washington and Jefferson College
W. L. Ayres, University of Michigan

R. W. Babcock, Kansas State Agricultural College
Clara L. Bacon, Goucher College
Grace M. Bareis, Ohio State University
Walter Bartky, University of Chicago
I. A. Barnett, University of Cincinnati
M. A. Basoco, University of Nebraska
W. D. Baten, University of Michigan
P. E. Baur, Baldwin-Wallace College
H. M. Beatty, Ohio State University
Samuel Beatty, University of Toronto
E. T. Bell, California Institute of Technology
H. A. Bender, University of Akron
Suzan R. Benedict, Smith College
Theodore Bennett, University of Wisconsin
H. R. Beveridge, Monmouth College
G. D. Birkhoff, Harvard University
L. T. Black, Ashland College
Henry Blumberg, Ohio State University
M. G. Boyce, Western Reserve University
J. B. Brandeberry, University of the City of Toledo
R. W. Brink, University of Minnesota
O. E. Brown, Case School of Applied Science
R. S. Burington, Case School of Applied Science

W. D. Cairns, Oberlin College
B. H. Camp, Wesleyan University
C. C. Camp, University of Nebraska
G. A. Campbell, American Tel. and Tel. Co., New York

R. D. CARMICHAEL, University of Illinois
W. M. CARRUTH, Hamilton College
W. B. CARVER, Cornell University
E. W. CHITTENDEN, University of Iowa
E. H. CLARKE, Hiram College
A. B. COBLE, University of Illinois
ABRAHAM COHEN, Johns Hopkins University
JULIA T. COLPITTS, Iowa State College
ELIZABETH B. COWLEY, Vassar College
C. M. CRAMLET, University of Washington
A. R. CRATHORNE, University of Illinois
C. H. CURRIER, Brown University
D. R. CURTISS, Northwestern University

C. W. DANCER, University of the City of Toledo
L. A. V. DECLEENE, St. Norbert College
E. L. Dodd, University of Texas
H. L. DORWART, Williams College
B. F. DOSTAL, University of Florida
P. S. DWYER, Antioch College
L. A. DYE, Cornell University

P. D. EDWARDS, Ball State Teachers College, Muncie, Ind.
L. P. EISENHART, Princeton University
G. C. EVANS, Rice Institute

PETER FIELD, University of Michigan
B. F. FINKEL, Drury College
C. W. FOARD, Youngstown College
T. M. FOCKE, Case School of Applied Science
F. A. FORAKER, University of Pittsburgh
W. B. FORD, University of Michigan
TOMLINSON FORT, Lehigh University
T. C. FRY, Bell Telephone Laboratories, New York

W. M. GAFAFER, Johns Hopkins University
F. J. GERST, St. Louis University
R. E. GILMAN, Brown University
J. W. GLOVER, Teachers Ins. and Annuity Assn., New York
J. S. GOLD, Bucknell University
L. M. Graves, University of Chicago
LOIS W. GRIFFITHS, Northwestern University
V. G. GROVE, Michigan State College

HARRIS HANCOCK, University of Cincinnati
J. N. HART, University of Maine
W. L. HART, University of Minnesota
E. R. HEDRICK, University of California at Los Angeles

CORA B. HENNEL, Indiana University
DEBORAH M. HICKEY, Delta State Teachers College
H. C. HICKS, Carnegie Institute of Technology
T. H. HILDEBRANDT, University of Michigan
C. E. HILLE, Princeton University
E. J. HIRSCHLER, Bluffton College
B. P. HOOVER, Carnegie Institute of Technology
H. M. HOSFORD, University of Arkansas
JEWELL C. HUGHES, Hunter College
W. A. HURWITZ, Cornell University
C. A. HUTCHINSON, University of Colorado

LOUIS INGOLD, University of Missouri
M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
R. P. JOHNSON, Carnegie Institute of Technology
W. W. JOHNSON, Y.M.C.A. School of Technology, Cleveland
F. E. JOHNSTON, George Washington University
L. S. JOHNSTON, University of Detroit
F. C. JONAH, Western Reserve University
B. W. JONES, Cornell University
E. M. JUSTIN, Case School of Applied Science

H. S. KALTENBORN, Carnegie Institute of Technology
K. D. KELLY, Georgetown, Ind.
B. F. KIMBALL, University of New Hampshire
L. C. KNIGHT, College of Wooster
H. W. KUHN, Ohio State University

E. P. LANE, University of Chicago
GILLIE A. LAREW, Randolph-Macon Woman's College
C. G. LATIMER, University of Kentucky
D. H. LEAVENS, Harvard School of Business Administration
SOLOMON LEFSCHETZ, Princeton University
FLORENCE P. LEWIS, Goucher College
W. R. LONGLEY, Yale University

W. D. MACMILLAN, University of Chicago
DOROTHY MCCOY, Belhaven College
J. A. MCLAUGHLIN, St. Bonaventure's College
E. J. MCSHANE, University of Chicago
H. E. MENKE, Otterbein College
G. M. MERRIMAN, University of Cincinnati

W. M. MILLER, Marquette University
E. C. MOLINA, American Tel. and Tel. Co., New York
C. N. MOORE, University of Cincinnati
T. W. MOORE, Indiana University
W. L. MOORE, University of Louisville
C. C. MORRIS, Ohio State University
MAX MORRIS, Case School of Applied Science
PAUL MUEHLMAN, St. Louis University
F. D. MURNAGHAN, Johns Hopkins University
J. R. MUSSELMAN, Western Reserve University

J. H. NEELLEY, Carnegie Institute of Technology
J. P. NICKOL, St. Bonaventure's College
J. A. NYSWANDER, University of Michigan

RUFUS OLDENBURGER, University of Michigan
E. G. OLDS, Carnegie Institute of Technology
H. L. OLSON, Michigan State College
OYSTEIN ORE, Yale University
F. W. OWENS, Pennsylvania State College

L. J. PARADISO, Cornell University
R. S. PARK, Eastern Teachers College, Richmond, Ky.
B. C. PATTERSON, Hamilton College
T. S. PETERSON, University of Michigan
L. C. PLANT, Michigan State College
T. M. PUTNAM, University of California

G. Y. RAINICH, University of Michigan
S. E. RASOR, Ohio State University
S. W. REAVES, University of Oklahoma
MINA S. REES, Hunter College
W. T. REID, National Research Fellow, University of Chicago
C. N. REYNOLDS, Jr., West Virginia University
C. E. RHODES, University of Cincinnati
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
H. L. RIETZ, University of Iowa
C. F. ROOS, Cornell University
S. A. ROWLAND, Ohio Wesleyan University
LULU L. RUNGE, University of Nebraska

S. T. SANDERS, Louisiana State University
GEORGE SAUTÉ, Cleveland College
MAURICE SCHEIER, St. Bonaventure's College
HAZEL E. SCHOONMAKER, New Jersey College for Women
G. T. SELLEW, Knox College
H. C. SHAUB, Washington and Jefferson College
W. F. SHENTON, American University
L. S. SHIVELY, Ball State Teachers College, Munice, Ind.
C. A. SHOOK, Yale University
C. GRACE SHOVER, Connecticut College
W. G. SIMON, Western Reserve University
LAO G. SIMONS, Hunter College
H. E. SLAUGHT, University of Chicago
L. L. SMAIL, Lehigh University
SISTER M. PAULINE SMYTH, Aquinas Institute
A. E. STANILAND, University of Pittsburgh
H. E. STELSON, Kent State College
EUGENE STEPHENS, Washington University
GUY STEVENSON, University of Louisville
K. D. SWARTZEL, University of Pittsburgh
J. L. SYNGE, University of Toronto

J. D. TAMARKIN, Brown University
J. H. TAYLOR, George Washington University
J. S. TAYLOR, University of Pittsburgh
C. F. THOMAS, Case School of Applied Science
M. O. TRIPP, Wittenberg College
BIRD M. TURNER, West Virginia University

J. H. WEAVER, Ohio State University
F. M. WEIDA, George Washington University
MARIE J. WEISS, Sophie Newcomb College
E. D. WELLS, University of Pittsburgh
WILLIS WHITED, Pennsylvania State Highway Dept.
E. A. WHITMAN, Carnegie Institute of Technology
G. T. WHYBURN, John Hopkins University
R. L. WILDER, University of Michigan
K. P. WILLIAMS, Indiana University
C. O. WILLIAMSON, College of Wooster
F. L. WREN, George Peabody College

B. F. YANNEY, College of Wooster
C. H. YEATON, Oberlin College
J. W. YOUNG, Dartmouth College

B. C. ZIMMERMAN, Marquette University

The sessions of the American Association for the Advancement of Science began on Monday evening with an address in the Music Hall of the Cleveland Public Auditorium by the retiring president, Doctor Robert A. Millikan, on

"Atomic disintegration and atomic synthesis." Following this the scientists were received in the ballroom of the Public Auditorium by the general officers of the American Association with President and Mrs. R. E. Vinson of Western Reserve University and President and Mrs. W. E. Wickenden of Case School of Applied Science. Among other general sessions of the American Association were the Sigma Xi lecture on "The science of photography" by Doctor C. E. K. Mees of the Eastman Kodak Company on Tuesday evening; an illustrated lecture on "Searching out Pluto, Lowell's trans-Neptunian planet" by R. L. Putnam, trustee of the Lowell Observatory, and Doctor V. M. Slipher, director of the Lowell Observatory, on Wednesday evening; and an illustrated lecture on "Weighing the earth" by Doctor P. R. Heyl of the U. S. Bureau of Standards on Friday evening. On Tuesday afternoon the eighth Josiah Willard Gibbs lecture was given by Doctor E. B. Wilson of the Harvard School of Public Health. This address was given under the joint auspices of the American Association and the American Mathematical Society. A large audience listened with great interest as Professor Wilson spoke on "Reminiscences of Gibbs by a student and colleague."

The annual science exhibition was housed in the gymnasium of Western Reserve University in which also the general registration offices were located. The Cleveland exhibition was larger and more varied than usual and consisted of exhibits of scientific apparatus by numerous institutions and firms as well as exhibits by a considerable number of book publishers. Of special interest to mathematicians were Doctor P. R. Heyl's models of the regular four-dimensional solids.

The science meetings were for the most part held on and about the grounds of Case School of Applied Science and Western Reserve University, although some of the societies in the social sciences held their meetings in the Cleveland hotels. This concentration of meetings in the University district enabled the scientists to go readily from one program to another when this was desired. The general local arrangements for the Cleveland meetings were in charge of Doctor H. W. Mountcastle; he and his associates were very effective in organizing the plans for the Cleveland meetings and many expressions of appreciation of their work were heard.

Professor Dunham Jackson and Secretary Cairns represented the Mathematical Association on the Council of the American Association, which met Monday afternoon and each morning thereafter for the transaction of general business. Doctor C. F. Roos, assistant professor of mathematics at Cornell University, was elected Permanent Secretary of the American Association in succession to Doctor Burton E. Livingston, who has held that office since 1920; Doctor Livingston will continue as an officer of the American Association in the capacity of General Secretary. Professor E. R. Hedrick was elected vice-president and chairman of Section A for the year 1931.

The Hotel Statler was the headquarters for the mathematicians, and convenient facilities were put at their disposal for committee rooms, for small

dinners, and for the joint dinner of the mathematicians which was held Wednesday evening. At this last dinner brief speeches were made by the presiding officer, President E. R. Hedrick, and by Professors E. T. Bell, L. P. Eisenhart, and President J. W. Young. At the close of this dinner a resolution, presented by Professor Carmichael, was adopted expressing the gratitude of the mathematicians for the kindness and generosity shown by our hosts and for the effective planning which made the meetings pleasant and successful.

The American Mathematical Society held a joint session on Monday afternoon with Section K of the American Association and the American Statistical Association at which there were papers by Professor G. C. Evans on "Simple types of economic crises and cycles," by Professor Ragnar Frisch on "A method of decomposing an empirical series into its cyclical and progressive components," and by Professor Harold Hotelling on "Recent improvements in statistical inference." The Society held sessions on Tuesday and Wednesday mornings in three sections for the reading of papers. The Society's Gibbs lecture has already been mentioned.

The program of the Mathematical Association consisted of a joint session with Section A and the Society on Wednesday afternoon and two sessions on Thursday. The excellent program was prepared by a committee consisting of Professors C. C. MacDuffee, F. W. Owens, A. H. Wilson, and W. G. Simon, Chairman. President Young presided at the morning session and Professor Carmichael at the afternoon session, being replaced by Vice-President C. N. Moore for the last part of the program. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

Joint Session of the Association with the Society and Section A of the American Association

1. "Mathematics and speculation" by Professor E. T. Bell, California Institute of Technology, retiring vice-president of Section A of the American Association.
2. "Recent developments in abstract algebra" by Professor Oystein Ore, Yale University, by invitation of the Association and the Society.
3. "Axiomatical theory of dimension" by Professor Karl Menger, University of Vienna, at the request of the Mathematical Society.

1. The address by Professor Bell was published in the March, 1931 issue of the *Scientific Monthly*.

First Session of the Association

1. "Early history of the American Mathematical Monthly" by Professor B. F. Finkel, Drury College.
2. "Analysis situs as a branch of elementary geometry" by Professor J. W. Alexander, Princeton University.
3. "The equilateral hyperbola" by Professor J. R. Musselman, Western Reserve University.

1. It was the good fortune of the Mathematical Association to have Professor Benjamin F. Finkel, founder and first editor of the MATHEMATICAL MONTHLY, present a full record of its early history. Many questions have been frequently raised as to the establishment and development of the MONTHLY in the years before it became the official organ of the Mathematical Association. This authoritative record will appear in the MONTHLY later in the year.

2. It is hoped that the excellent expository paper by Professor Alexander will appear in the MONTHLY at an early date.

3. From the standpoint of analytical geometry the ellipse and hyperbola are treated without partiality in all textbooks. Yet when the special case of each is discussed we find a whole chapter devoted to the circle, while usually one paragraph suffices for the equilateral hyperbola. Why should this be? The ease with which a circle can be constructed can only account partly for it; the blame must be put upon Plato and his followers. Plato, who had such an extraordinary influence upon the geometers of his time, insisted upon permitting in construction problems those instruments only which amount, analytically, to the straight line and the specialized ellipse. It is due to him that our geometry today is one-sided.

Professor Musselman then showed that Menaechmus about the middle of the fourth century B.C., used the equilateral hyperbola in one solution of the problem of duplicating the cube. The problem of trisecting any angle can be reduced to finding the intersection of a circle and an equilateral hyperbola. The mechanical device for constructing an equilateral hyperbola, due to H. H. S. Cunynghame,* when the asymptotes or the axes are known, would be useful for this purpose.

The study of the curve analytically can be simply done by using the Cartesian form $xy=c^2$ and expressing the coordinate of a point on the curve parametrically as $x=ct$, $y=c/t$. A number of interesting theorems concerning sets of points on the curve were used to illustrate the possibilities of this curve as a subject for further study in our elementary courses. Since the strophoid and lemniscate are inverses of the equilateral hyperbola, a knowledge of the latter is helpful in studying the former curves. But the equilateral hyperbola is worth studying for its own sake.

SECOND SESSION OF THE ASSOCIATION

1. "Mathematics in Hungary" by Professor TIBOR RADÓ, Ohio State University.

2. "Remainder terms in interpolation formulas" by Professor J. F. REILLY, University of Iowa.

3. "Theoretical and statistical investigations concerning the interrelations of demand, cost of production and profit" by Professor C. F. ROOS, Cornell University.

---

* Taylor, Ancient and Modern Geometry of Conics, p. 177.

1. In order to depict the character of preparatory mathematics in the secondary and early collegiate courses in Hungary, Professor Radó described the competitive examinations for the Eötvös Prize. The prize question is always some problem which is within the grasp of these pupils as regards its subject matter, but is of such a nature that it requires special ability and is adapted to discovering prospective scholars of real worth. He mentioned that one of the prize winners of earlier years was the well-known Féjer.

2. After discussing limitations on the application of the interpolation process, pointing out the necessity of knowing the function that is satisfied by the given tabular values, referred to as the *underlying* function, and emphasizing the importance of determining the degree of accuracy of the results obtained by interpolation, Professor Reilly indicated the development of the divided difference interpolation formula with a remainder term, in which differences to order n were used, and the remainder after $(n+1)$ terms was expressed in terms of a derivative of order $(n+1)$. The divided difference formula was specialized to produce several of the common formulas, and these in turn were combined to produce others. Thus were found the formulas due to Newton, Gauss, Sterling, Bessel and Everett with the remainder term in each case. The use of the error test in determining the sign and an upper limit of the magnitude of the remainder term was discussed.

To illustrate the use of the remainder terms examples were taken from three tables: (1) a six place table of natural logarithms, (2) a ten place table of common logarithms, and (3) a table of the amounts of a unit at compound interest.

In the first table it was found that (given log 7741 and log 7742) log 7741.5 $=8.954350$, while $R_2=.0021\times10^{-6}$ if, as is commonly done, descending differences and descending factorials are employed. It was also found that (given log 438 and log 439) log $438.5=6.083359$, while $R_2=.65\times10^{-6}$, and that (given log 53 and log 54) log $53.5=3.979638$, while $42.9\times10^{-6}<R_2<44.5\times10^{-6}$ These results indicate that if linear interpolation is to be employed, it is certainly not desirable to print a table of natural logarithms to six places when the corresponding numbers are given to only three figures or two figures.

In the second table it was found that (given log 12153 and log 12154) log $12153.5=4.084\ 7013\ 649$, while $R_2=3.7\times10^{-10}$; also that (given log 95678 and log 95679) log $95678.5=4.980\ 8143\ 581$, while $R_2=.06\times10^{-10}$. This shows that in a common logarithm table where the numbers are given to five figures and the logarithms to ten places linear interpolation is satisfactory in the latter part of the table, but not in the former part, where the use of at least second order differences is necessary.

In the third table, that of the amounts of a unit at compound interest as ordinarily constructed, interpolation was far less satisfactory than in the two other tables considered. It was found that [given $(1.02)^{10}$, $(1.0225)^{10}$, $(1.025)^{10}$, $\cdots$, to seven places] $(1.02125)^{10}=1.2340989$, when linear interpolation was employed, while $-840\times10^{-7}<R_2<-824\times10^{-7}$. Also that $R_3=8\times10^{-7}$, and $R_4=-.09\times10^{-7}$, which shows the need for third order differences. Again [given

$(1.04)^{70}$, $(1.045)^{70}$, $(1.05)^{70}$, $\cdots$, to seven places] it was found necessary to use eleventh order differences to obtain the value of $(1.0525)^{70}$ to seven places.

Two suggestions were made, (a) that shorter intervals in the interest rate be used even if it were thereby necessary to lessen the period of years covered by the table, and (b) that a statement be made in the "Explanation of the Tables" indicating what degree of accuracy is obtainable by interpolating with given orders of differences in various sections of the table.

Meetings of the Board of Trustees

Ten members of the Board of Trustees were present at the Cleveland meeting.

The following forty-one persons and one institution were elected to membership on applications duly certified:

To Individual Membership

Nola Lee Anderson, Ph.D. (Missouri). Chm., Dept. of Math., Sophie Newcomb Coll., New Orleans, La.

Ida M. Baker, A.M. (Columbia). Asso. Prof., School of Educ., Western Reserve Univ., Cleveland, Ohio.

L. M. Bauer, A.B. (Oakland City Coll.). Principal, Wabash Twp. High School, Griffin, Ind.

J. C. Bay. Librarian, The John Crerar Library, Chicago, Ill.

Lois E. Bell, A.M. (Kansas). Teacher, Junior Coll., Independence, Kans.

T. A. Bickerstaff, A.M. (Mississippi). Instr., Univ. of Mississippi, University, Miss.

Archie Blake, B.S. (Chicago). Grad. student, Univ. of Chicago, Chicago, Ill.

Arthur B. Brown, Ph.D. (Harvard). Instr., Columbia Univ., New York, N. Y.

J. H. Bushey, Ph.D. (Michigan). Asst. Prof. of Math. and Insurance, Hunter College, New York, N. Y.

D. D. Butterfield, A.M. (Princeton). Instr., Phillips Exeter Acad., Exeter, N. H.

Laura E. Christman, A.M. (Wisconsin). Teacher, Senn High School, Chicago, Ill.

A. J. Cook, Ph.D. (Chicago). Asso. Prof., Univ. of Alberta, Edmonton, Canada.

Max Coral, M.S. (Chicago). Univ. of Chicago, Chicago, Ill.

Margaret L. Darragh, A.B. (Hanover Coll.). Instr., Roosevelt High School, Chicago, Ill.

Ralph Deutsch, A.M. (Columbia). Teacher, New Utrecht High School, Brooklyn, N. Y.

F. G. Dressel, M.S. (Michigan). Instr., Duke Univ., Durham, N. C.

Sarah Fahnestock, M.S. (Chicago). Head of Dept., Marymount College, Salina, Kans.

C. W. Foard, Ph.D. (Iowa). Prof., Math. and Physics, Youngstown Coll., Youngstown, Ohio.

Pauline F. Folk, A.B., B.E. (Colorado). Grad. student, Univ. of Colorado, Boulder, Colo.

D. H. Frank, B.S. (C. C. N. Y.). Teacher, George Washington High School, New York, N. Y.

F. L. Manning, M.S. (Rutgers). Asst. Prof., Ursinus Coll., Collegeville, Pa.

Sister Mary Joan, A.B. (St. Catherine's Coll.) Teacher, St. Catherine's High School, Racine, Wis.

Dorothy McCoy, Ph.D. Prof., Belhaven Coll., Jackson, Miss.

A. F. Mish, A.B. (West Virginia). Instr., High School, Grafton, West Va.

J. P. Nickol, Ph.D. (Fribourg). Prof., Physics and Math., St. Bonaventure's Coll. St. Bonaventure, N. Y.

C. O. Oakley, Ph.D. (Illinois). Asst. Prof., Brown Univ., Providence, R. I.

A. W. Rankin, Electrical Laboratories, Electric Service Supplies Co., Philadelphia, Pa.

Y. K. Roots, M.S. (New York Univ.) Prof., Math. and Physics, Findlay Coll., Findlay, Ohio.

C. H. Rowe, (Fellow of Trinity Coll., Dublin). Erasmus Smith's Prof. of Math., Univ. of Dublin, Dublin, Ireland.

J. A. SCHOUTEN, Prof. Dr., Technological Univ., Delft, Holland. (Lecturer, 1930–1931 at Harvard, M.I.T., and Princeton.)
C. L. SEARCY, C.E. (Purdue), A.M. (California). Asst. Prof., Univ. of Nevada, Reno, Nev.
C. R. SHERER, A.M. (Nebraska). Head of Dept., Texas Christian Univ., Ft. Worth, Tex.
A. E. STANILAND, A.M. (Pittsburgh). Instr., Univ. of Pittsburgh, Pittsburgh, Pa.
H. E. STELSON, Ph.D. (Univ. of Iowa). Asst. Prof., Kent State Coll., Kent, Ohio.
J. L. SYNGE, Sc.D. (Dublin). Prof., Univ. of Toronto, Toronto, Ont., Canada.
H. B. THORNTON, A.M. (Cincinnati). Head of Dept., Sumner Jr. Coll., Kansas City, Kans.
O. E. Walder, A.M. (Nebraska). Instr., South Dakota State Coll., Brookings, S. Dak.
MAUD WILLEY, A.M. (Mills Coll.). Instr., Alabama Coll., Montevallo, Ala.
ALBERTA WOLFE, M.S. (Iowa State Coll.). Instr., Western Coll. for Women, Oxford, Ohio.
H. M. YARBROUGH, Ph.D. (Indiana). Head of Dept., Western Kentucky State Teachers Coll., Bowling Green, Ky.
B. C. ZIMMERMAN, A.M. (St. Louis Univ.). Grad. student, Marquette Univ., Milwaukee, Wis.

To Institutional Membership

UNIVERSITY OF MAINE, Orono, Maine

The financial report for the year 1930 was presented by the Secretary-Treasurer. It was accepted by the Trustees, subject to the inspection of a subcommittee; on Thursday Professor Slaught for the committee on finance and Professors E. W. Chittenden and C. A. Hutchinson examined the report and the evidences of assets and declared the report satisfactory.

Professor Dunham Jackson, chairman of the Committee on Geometry, presented a report of that committee and the Trustees voted to approve the report which concerns a possible year course in plane and solid geometry, without passing judgment as to the relative merits of any particular type of course. The full report will appear in another place in the MONTHLY.

The Trustees adopted a resolution on the death of Professor Cajori. This is incorporated in the report of the annual business meeting.

The Secretary reported that Presidents Hedrick and Young had made coincident appointment of Professor E. V. Huntington to represent the Society and the Association on the Secondary Committee on Electrical Definitions of the American Standards Association.

It was voted to approve the following associate editors of the MONTHLY for the year 1931:

Elizabeth Carlson	B. F. Finkel	J. R. Musselman
N. A. Court	R. E. Gilman	H. L. Olson
Otto Dunkel	R. A. Johnson	D. E. Smith
H. S. Everett	H. W. Kuhn	F. M. Weida

It was voted to accept the invitation of the University of California at Los Angeles to meet there in conjunction with the Society in the summer of 1932.

The President and Secretary were authorized to do whatever seems desirable in the way of formal participation by the Association in the latter part of the S.P.E.E. summer school at Minneapolis next summer. It is probable that an

evening program will be held on the last Friday of the S.P.E.E. meeting and immediately before the meetings of the Mathematical Association.

Annual Business Meeting of the Association

The Secretary announced the names of those elected to membership. He reported also the deaths of the following members:

C. W. Andrews, Librarian emeritus, The John Crerar Library, Chicago, Ill. (November 20, 1930).

W. O. Beal, Chairman, Dept. of Astronomy, University of Minnesota (February 15, 1930).

G. C. Bordner, Professor of mathematics, State Teachers College, Kutztown, Pa. (May 15, 1930).

William Cain, Emeritus Professor of mathematics, University of North Carolina (December 6, 1930).

Florian Cajori, Emeritus Professor of the history of mathematics, University of California (August 14, 1930).

Mary E. Caster, Teacher of mathematics, Eastside High School, Paterson, N. J. (December 7, 1930).

Major C. H. Chepmell, of Bristol, England (November 18, 1930).

C. W. Crum, M.D., Baltimore, Md. (December 20, 1929).

J. O. Eckersley, Chief engineer, Dept. of Markets, New York City (November 11, 1930).

J. L. Markley, Professor of mathematics, University of Michigan (April 20, 1930).

E. B. Morris, Actuary, Travelers Insurance Co., Hartford, Conn. (December 19, 1929).

E. A. Pattengill, Associate professor of mathematics, Iowa State College (February 10, 1930).

R. E. Peterson, Dept. of mathematics, Pennsylvania State College (Summer, 1929).

R. A. Sheets, Associate professor of mathematics, Denison University (December 3, 1930).

J. M. Taylor, Emeritus professor of mathematics, Colgate University (July 31, 1930). (Charter member of M.A.A., resigned 1920).

H. E. Trefethen, Associate professor of mathematics and astronomy, Colby College (November 3, 1930).

The election of officers for the year 1931 resulted in the following, as reported by the tellers, Professors C. A. Hutchinson and C. H. Yeaton:

For President: E. T. Bell, 330 votes; W. C. Graustein, 255 votes.

For Vice-Presidents: Arnold Dresden, 396 votes; Tomlinson Fort, 231 votes; C. N. Moore, 305 votes; W. H. Roever, 205 votes.

For additional members of the Board of Trustees, to serve until January 1934: C. C. Camp, 218 votes; L. L. Dines, 335 votes; T. C. Fry, 382 votes; J. W.

Glover, 285 votes; F. L. Griffin, 248 votes; H. W. Kuhn, 244 votes; E. P. Lane, 318 votes; H. P. Manning, 262 votes.

The following were accordingly declared elected:

President: E. T. BELL, California Institute of Technology.

Vice-Presidents: ARNOLD DRESDEN, Swarthmore College, C. N. MOORE, University of Cincinnati.

Additional members of the Board of Trustees: L. L. DINES, University of Saskatchewan; T. C. FRY, Bell Telephone Laboratories; J. W. GLOVER, Teachers Insurance and Annuity Association; E. P. LANE, University of Chicago.

The following resolution, presented to the Trustees on Wednesday by Professor Lao G. Simons, was also presented at this meeting of the Association and adopted by rising vote:

WHEREAS: Our fellow member and Trustee, Florian Cajori, has passed away from the scene of his earthly labors;

AND WHEREAS: Professor Cajori was a Charter Member of the Mathematical Association of America and was also its President in 1917;

AND WE RECOGNIZE FURTHER: That his professional connections with great universities and his unique position as Professor Emeritus of the History of Mathematics caused him to exert an unusual influence on the body of American students;

That his active association with many learned societies, on the programs of whose meetings his name appeared at frequent intervals, made known the extent of his scholarship and the charm of his personality to a wide circle of men and women;

That his extended publications on a remarkable variety of subjects within the compass of the History of Science constitute an almost inexhaustible fund of reference material for the student and teacher:

BE IT RESOLVED: That we, the Trustees and members of the Mathematical Association of America, put on record our appreciation of the contributions of Professor Cajori to his chosen field, the History of Mathematics, and the part that he played in making known internationally the researches of American scholars.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 15, 1930

RECEIPTS			EXPENDITURES		
Balance Dec. 16, 1929		$11,221.38	Publisher's bills (Nov. '29–Oct. '30)		$5,105.05
1928 indiv. dues	29.25		President's office		52.76
1929 indiv. dues	359.20		Manager's office		30.91
1929 instit. dues	26.50		Editor-in-chief's office		517.92
1930 indiv. dues	6,500.23		Other editors' postage		5.00
1930 instit. dues	821.50		Committee on Geometry		68.49
1930 subscriptions	830.82		*Register*		542.12
Initiation fees	226.00		Secretary-Treasurer's office:		
Advertising	718.82		Postage	$337.89	
Sale *Register*	6.50		Bond	5.00	
Sale copies of MONTHLY	246.21		Safety deposit	4.00	
Sale First Carus Mon.	45.00		Office supplies	89.10	
Sale Second Carus Mon.	45.00		Express, tel., etc.	59.50	
Sale Third Carus Mon.	32.50		Ins. back copies of MONTHLY	11.00	
Sale Fourth Carus Mon.	1,025.00		Clerical work	1,705.63	
Advance sale Carus Mon.	1.25		Printing	275.95	
Fee for translation rights Carus Mon.	1.00		Library expense	223.85	

Sale Rhind Papyrus	1,545.00	
Carrying charges Papyrus	88.50	
Contributions A. B. Chace	2,375.18	
Int. Oberlin Savgs. Bk.	111.23	
Int. Peoples Bkg. Co.	103.64	
Int. Liberty Bonds	85.00	
Int. Hardy Fund	120.00	
Int. certifs. of deposit	42.34	
Int. from Genl. Endowment Fund Bonds	385.00	
Int. Chace Fund	91.25	
Int. Chauvenet Fund	25.00	
Int. from investment of current funds	150.00	16,036.92
Total 1930 receipts		$27,258.30
Total expenditures		18,753.15
Balance to the end of 1930 business		8,505.15
Received on 1931 business		731.50
Book balance Dec. 15, 1930		$9,236.65

Paid copies MONTHLY	50.19	
Des Moines meeting	90.00	
Providence meeting	82.12	
Cleveland meeting	6.00	
Refund 1930 subscription	3.60	
Paid members' *Annals*	5.00	
Typewriter	63.03	3,011.86
Annals subvention		225.00
Paid to sections from initiation fees		166.45
Transferred to Genl. Endowment		3,000.00
Cost above par of new bonds for Genl. Endowment Fund plus accrued int.		110.50
Paid B. F. Finkel int. Hardy Fund		120.00
Sustaining memb. in Amer. Math. Society		100.00
Printing Rhind Papyrus		2,375.18
Carrying charges Papyrus		124.40
Expense acct. Papyrus		36.02
Expense acct. *Bibl. Math.*		50.00
Expense acct. Carus Mons.		131.45
Award Chauvenet Prize		100.00
Honorarium Fourth Carus Mon.		300.00
Transfer to Chace Fund		1,575.76
Transfer to Carus Mon. Fund		1,004.28
Total expenditures		$18,753.15
Cash on hand		12.53
Checking account		268.41
Oberlin Savgs. Bk. acct.		2,457.00
Peoples Bkg. Co. acct.		2,021.81
Liberty Bonds		1,000.00
Iowa Ry & Light Co. 5% Bonds		3,000.00
Certif. of deposit		476.90
Bank balance Dec. 15, 1930		$9,236.65

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CHAUVENET PRIZE FUND

Balance Dec. 16, 1929		$605.00
Interest	$25.00	
Amount set aside for 1930	20.00	45.00
		$650.00
Award Dec. 1929		100.00
		$550.00
Iowa Rwy & Light Co. 5% Bond	$500.00	
Cash in bank	50.00	
Balance Dec. 15, 1930		$550.00

Carus Monograph Fund

Balance Dec. 16, 1929		$4,560.89
Receipts: Sales	$1,148.75	
Interest	182.28	1,331.03
		$5,891.92
Expenditures	131.45	
Accrued interest on new bond	4.28	135.73
		$5,756.19
Certificates of deposit	$4,693.17	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Cash in bank	63.02	
Balance Dec. 15, 1930		$5,756.19

Arnold Buffum Chace Fund

Balance Dec. 16, 1929		$1,416.00
Receipts: Sales	$1,545.00	
For carrying charges	88.50	
Interest	105.26	1,738.76
		$3,154.76
Expenditures: Expense acct. Rhind Papyrus	24.59	
Premium on bond and accrued int.	75.76	
For carrying charges	124.40	224.75
		$2,930.01
Iowa Rwy & Light Co. 5% Bond	$1,000.00	
Western United Gas and Elec Co. Bond	1,500.00	
Certificate of deposit	430.01	
Balance Dec. 15, 1930		$2,930.01

Life Membership Fund

Liability on life memberships Dec. 16, 1929	$459.20
To be transferred to current funds, surplus	12.89
Liability on life memberships as of Jan. 1, 1931	$446.31

General Endowment Fund

Balance Dec. 16, 1929		$5,000.00
Transferred from current funds in 1930		3,000.00
		$8,000.00
Liberty Bond	$1,000.00	
Land Trust Certificate	1,000.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% bonds	2,000.00	
Northwestern Elec Co. Bonds	3,000.00	
Balance Dec. 15, 1930		$8,000.00

Of the funds on hand, indicated in the first division of the financial report, $63.02 belongs to the Carus Monograph Fund (not yet transferred); $50.00 belongs to the Chauvenet Prize Fund; and $446.31 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1931. Aside from these amounts, the Carus Monograph Fund, to the amount of $4,693.17, is carried as a separate fund in the form of two certificates of deposit which bear 4%, compounded quarterly; $430.01 of the Arnold Buffum Chace Fund is similarly carried in the form of a certificate of deposit; the contribution of Professor Walter B. Ford to the Chauvenet Fund is carried as a 5% Gold Bond; and the sum of $8,000.00 is held in reserve as the General Endowment Fund, in securities as listed above.

When the accounts were closed Dec. 15, 1930, there remained on the total business for the year 1930 the following items:

BILLS RECEIVABLE
(partly estimated)

1930 individual dues	$200.00
Advertising	100.00
	$300.00

BILLS PAYABLE
(partly estimated)

Publisher's bills (Nov., Dec. 1930)	$1,150.00
President's office	70.00
Manager's office	20.00
Editor-in-chief's office	110.00
Other editors' postage	40.00
Secretary-Treasurer's office	350.00
Annals subvention	75.00
Initiation fees due to sections	750.00
Carus Monograph Fund	63.02
Chauvenet Prize Fund	50.00
Life Membership Fund	446.31
Printing catalog of library	400.00
	$3,524.33

If to the balance on 1930 business shown in the report, $8,505.15, there be added the bills receivable, $300.00, and there be subtracted the estimated bills payable, $3,524.33, there results an estimated final balance on 1930 business of approximately $5,280, which represents the accumulated surplus in current funds. This is a gratifying result for the year when compared with the corresponding figure of $6,400 of a year ago, for it will also be kept in mind that $3,000 was transferred from the current funds to the General Endowment Funds by recommendation of the Executive Committee and vote of the Trustees.

W. D. CAIRNS, *Secretary-Treasurer*

CONCERNING CERTAIN LINEAR TRANSFORMATION APPARATUS OF CRYPTOGRAPHY[1]

By LESTER S. HILL, Hunter College

1. *Introductory Note*

Of especial interest in systematic cryptography is the linear transformation:

$$(T)\qquad \begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1f}x_f + a_1, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2f}x_f + a_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \quad \cdot \\ y_f &= a_{f1}x_1 + a_{f2}x_2 + \cdots + a_{ff}x_f + a_f, \end{aligned}$$

in which f is any positive integer, and the variables x_i, y_i, as well as the coefficients a_{ij} and a_i are elements of an arbitrary field, finite or infinite. But the linear apparatus which may be profitably employed is much more extensive. To meet the demands of effective cipher construction, we must often operate in sets which do not possess full field character. The necessary operational sets are really special linear associative and commutative algebras. In the present paper, we shall call these sets *scales*. For our purposes, the transformation T must be made available in any scale.

Moreover, it is highly desirable to extend the transformation T in the sense of permitting the x_i, y_i, a_{ij}, a_i to be square matrices of arbitrary order in an arbitrary scale. This enables us to convert a sequence $x_1, x_2, \cdots, x_f$ of f matrices into another sequence $y_1, y_2, \cdots, y_f$ of f matrices. The specification of conditions under which a unique inverse transformation exists will naturally be important.

When the underlying scale is of the type of most immediate cryptographic significance, namely the type $S(n)$ discussed in Section 3, the linear transformations T may be effected with extraordinary speed and accuracy by means of a mechanical device, no calculations of any sort being required. To avoid the expense of preparing machines of different structures, one for encipherment and the other for decipherment, we employ *involutory* transformations of type T(that is to say, transformations of period 2).

It is hoped that these notes will direct attention to a fascinating, although sadly neglected, domain of applied algebra.

2. *Scales*

The word *ring*[2] has been quite generally adopted to describe any finite or infinite set R, over which operations of so-called "addition" and "multiplica-

[1] *A Note by the Editor:* This paper was presented under a different title to the American Mathematical Society at Boulder, Colorado in August, 1929. It is the second article by Professor Hill on the subject of *Cryptography* to be published in this Monthly. The first one was *Cryptography in an algebraic alphabet*, in vol. 36 (1929), pp. 306–312.

[2] See Hasse, *Höhere Algebra*, Part 1, pp. 7–9.

tion" are in any way uniquely specified, provided that: (1) R contains at least two different elements; (2) multiplication is distributive with respect to addition, and each of these two operations is associative and commutative; and (3) if α and β denote elements of R, not necessarily different, then R contains exactly one element γ such that $\alpha+\gamma=\beta$.[3]

It is readily shown that any ring R contains exactly one zero element, and we shall denote this element by 0. The zero element has the properties, the first of which is definitional and pertains only to this element, that $\alpha+0=\alpha$ and $\alpha\cdot 0=0$, where α denotes any element of R. Concerning the second of these properties, we note that there is an infinity of rings in which every product vanishes (is equal to the zero element). According to the definition given below, such rings are clearly not *scales*.

Each element α of any ring determines uniquely an element δ such that $\alpha+\delta=0$, and δ is called the "negative" of α. We write $\delta=-\alpha$, noting the obvious implication that $\alpha=-\delta$. The element γ of postulate (3) above is denoted by $\beta-\alpha$; and we observe that $\beta-\alpha=\beta+(-\alpha)$.

Let α, β, γ denote elements, not necessarily different, of a ring R. We easily see that $\alpha(-\beta)=(-\alpha)\beta=-\alpha\beta$, $(-\alpha)(-\beta)=\alpha\beta$, $\alpha(\beta-\gamma)=\alpha\beta-\alpha\gamma$; and also that each of the equations $\alpha=\beta$, $\alpha-\beta=0$, implies the other.

An element α of a ring R is a "divisor of zero" if R contains an element β different from zero ($\beta\neq 0$) such that $\alpha\beta=0$. The zero element of a ring is always a divisor of zero.

By reason of the commutativity of multiplication, a ring R can not contain more than one element ϵ such that $\epsilon\alpha=\alpha$ for every element α of R. If one such element ϵ is present, it is called the *unit element* of R, and may be conveniently denoted by 1.

In all that follows, we shall operate exclusively in those rings which we distinguish as *scales*. Hence we emphasize the definition: *A scale is a ring which contains a unit element.* If a ring R is a scale, we shall ordinarily denote it by the letter S.

Let α denote any element of a scale S. It is readily established that S can not contain more than one element β such that $\alpha\beta=1$. If one such element β is present in the scale, we call it the "reciprocal" of α, writing $\beta=1/\alpha$, and noting the implication that $\alpha=1/\beta$. An element of a scale will be classed as *regular* or *singular* according as it has, or has not, a reciprocal.

In any scale, the unit and zero elements are respectively regular and singular; and the product of two elements, not necessarily different, is regular when and only when both elements are regular. The negative and the reciprocal of a regular element are regular. *A field is a scale in which the zero is the only singular element.*

[3] When no misunderstanding can arise, we shall employ without comment the familiar terminology and notations of elementary algebra. Thus, for instance, we shall say that addition and multiplication of the elements α and β of a ring yield respectively the "sum" $\alpha+\beta$ and the "product" $\alpha\beta$.

A regular element of a scale is never a divisor of zero. In some scales, every singular element is a divisor of zero; in other scales, this is not the case.

If α is any element, and β any regular element, of a scale S, then S contains exactly one element, γ, such that $\beta\gamma=\alpha$. We write $\gamma=\alpha/\beta$, observing that, in fact, $\alpha/\beta=\alpha(1/\beta)$.

Exponential notations are easily introduced. If α is any element of a scale S, the meaning of the symbol α^n, for positive integral n, requires no comment. When n is a negative integer or zero, this symbol is defined only for the case in which α is a *regular* element of S, and the specifications in that case are: $\alpha^0=1$ (the unit element of S), $\alpha^n=(1/\alpha)^{-n}$.

It should be noted that every field is a scale, and that every scale is a ring. The following two sections will furnish examples of scales which are fields, and of scales which are not fields. There exist an infinity of rings (finite rings as well as infinite rings) which are not scales; but the present paper completely disregards such rings.

3. *Simple Examples of Scales*

It is evident that the fields of rational, real, and ordinary complex numbers furnish three examples of scales. A subscale of each of these is found in the set of all positive and negative integers and zero. This infinite subscale contains only two regular elements, namely ± 1, and only one divisor of zero, namely 0.

Of exceptional practical interest in cryptography are the finite modular scales which we here designate as of type $S(n)$. For the integer $n \geqq 2$, let $S(n)$ denote any set of n elements associated, one-to-one, with the n integers 0, 1, 2, $\cdots$, $n-1$. If the elements α, β of $S(n)$ are associated with the integers a, b, we define:

$$\alpha+\beta=\gamma, \quad \alpha\beta=\delta$$

where γ and δ are the elements of $S(n)$ associated respectively with the *remainders* obtained upon dividing, by n, the ordinary sum $a+b$, and the ordinary product ab, of integers.

With operations thus defined, modulo n, we see that $S(n)$ is a finite scale. Its regular elements are those associated with integers prime to n. When n is prime, $S(n)$ is a field.

It will be convenient to treat, as the elements of $S(n)$, the n integers 0, 1, 2, $\cdots$, $n-1$ themselves, regarded as mere marks or symbols.

For cipher construction, perhaps the most useful scales of the type $S(n)$ are those which correspond to $n=23$, 25, 26, 27, 36, 100, 101. The first and the last of these seven are, of course, fields.

We shall draw our illustrative material from $S(26)$. We tabulate here, for later reference, the regular elements of this scale, together with their reciprocals:

$S(26)$

Element:	1,	3,	5,	7,	9,	11,	15,	17,	19,	21,	23,	25
Reciprocal:	1,	9,	21,	15,	3,	19,	7,	23,	11,	5,	17,	25

The negative of the reciprocal of an element in any scale is the reciprocal of the negative. Thus we have here:

$$-21=1/(-5)=1/21=5; \quad -17=1/(-23)=1/3=9; \quad \text{etc.}$$

Operations in $S(26)$ may be further illustrated as follows: $3+11=14$; $17+12=3$; $9+17=0$, whence $-9=17$ and $-17=9$; $3(7)=21$, whence $21/3=7$ and[4] $21/7=3$; $7(15)=1$, whence $1/7=15$ and $1/15=7$; etc. The negative of any element is obvious: $-0=0$, $-1=25$, $-2=24$, $-3=23$, etc.

4. *Scales Obtained by Algebraic Extension of Other Scales*

The theory of polynomials in any *field* is so familiar a chapter of modern algebra that not very much needs to be said here concerning polynomials in an arbitrary *scale* S (polynomials with "coefficients" which are elements of S). We note only a few points of special interest. A polynomial in a scale S is conveniently distinguished as *primary* if the coefficient in the term of highest "degree" is a *regular* element of S. Each element α of S is regarded as a polynomial in S, degrees being as follows: (1) when $\alpha\neq 0$, it is a polynomial of degree zero; and (2) when $\alpha=0$, it is a polynomial of degree[5] -1. Regular and singular elements of S, regarded as polynomials in S, are classed respectively as primary and non-primary.

We note that the degree of the product of two polynomials in a scale is equal to the sum of their degrees whenever at least one of the polynomials is primary and the other is not the polynomial 0 (of degree -1). We record also this fundamental *division property*:

Let S be any scale, finite or infinite. Let P denote any polynomial, and D any *primary* polynomial, in S. There are uniquely determined two polynomials Q and R in S, the latter of degree less than the degree of D, such that $P=QD+R$.

It is convenient to designate R as the *residue of* P, *modulo* D; and to write: $R=\text{Res}\ (P, \text{mod}\ D)$. If the degree of P is less than that of D, we see at once that[6] $\text{Res}\ (P, \text{mod}\ D)=P$.

Let us now select, as a modulus, any polynomial N, in S, which is primary and of degree $n\geqq 2$. It is easy to define addition and multiplication over the set U of all polynomials in S which have degrees less than n, *in such manner that U will be closed under these operations and will constitute a scale.* We need merely specify,[7] as the sum of two polynomials A, B of U, the sum $A+B$ in S;

[4] Similarly since $17(18)=20$ we might expect to have $20/17=18$ and $20/18=17$; but such is not the case, for while $17(18)=20$ implies $20/17=18$, it does not imply $20/18=17$. The element 18 is singular, and $20/18$ is not defined.

[5] Any other negative real number would serve equally well, for our purposes, to mark the degree of the polynomial 0. We wish merely to signalize that the "degree" of this polynomial is to be regarded as less than that of any other polynomial in S.

[6] In this case, Q is the polynomial 0, and $R=P$.

[7] For the case in which the scale S is a field, this procedure is very familiar. In this case, every polynomial in S, except the polynomial 0, is primary. But to obtain a scale U which is a field, we must employ, as modulus N, a primary polynomial *irreducible* in the field S.

and as the product, the polynomial Res (AB, mod N), where AB is the product in S.

The scale U plainly contains a subset V which is a scale simply isomorphic with the scale S. The scale V consists of those polynomials of U each of which is represented by an element of S. In this sense, we may regard U as an *extension* of S.

It is evident that when the scale S is finite, consisting of k elements, the scale U will likewise be finite, consisting of k^n elements, where n is the degree of the modulus N.

For the benefit of those readers who may not be experienced in manipulations of the character here considered, we append two examples.

Example 1: Let $S=S(2)$, of which the elements[8] are 0 and 1. Let $N=x^2+x+1$. The elements of U are the four polynomials 0, 1, x, $x+1$ in $S(2)$. Denoting these elements by a, b, c, d respectively, we find that:

$$c + d = x + (x + 1) = 1 = b;\quad cd = \text{Res}\,(x^2 + x,\ \text{mod } N) = 1 = b;\ \text{etc.}$$

Since, in this case, S is a field, and N is irreducible in S, the scale U is a field. Its operation tables in full are:

Addition

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Multiplication

	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

The zero element is a, and the unit element is b. Every element is its own negative.[9]

For variety, let us select a modulus which is reducible in $S(2)$, say $N'=x^2+1$. We are led to a scale U', the operation tables of which, aside from the four products, $cc=b$, $cd=dc=d$, $dd=a$, are exactly the same as those of U. The zero and unit elements are again a and b, respectively. There is a singular element other than the zero, namely d, and U' is therefore not a field.

Example 2: Let $S=S(6)$, of which the six elements are:

$$0;\ 1 = -5;\ 2 = -4;\ 3 = -3;\ 4 = -2;\ 5 = -1.$$

In this case, S is not a field; it contains the four singular elements, 0, 2, 3, 4.

For adequate illustration, we employ three moduli $N_1=x^2-1$, $N_2=x^2-2$, $N_3=x^2-x-1$, leading to the three scales U_1, U_2, U_3, respectively. The elements of each of these scales are the thirty-six polynomials $\alpha+\beta x$ in $S(6)$, where α and β denote any elements of $S(6)$. Interesting light is thrown upon the structural

[8] See Section 3, above. We shall take, as the elements of $S(n)$, the n integers $0, 1, 2, \cdots, n-1$ themselves, adding and multiplying modulo n.

[9] This is true in $S(2)$, and in every scale obtained from $S(2)$ by algebraic extension.

relations of U_1, U_2, U_3 by the following table, which exhibits the reciprocals of all regular elements. For convenience of tabulation, $\alpha+\beta x$ is compactly indicated by[10] $\alpha\beta$.

TABLE

Element	00	10	20	30	40	50	01	11	21	31	41	51
Recip. in U_1		10				50	01					
Recip. in U_2		10				50		51				11
Recip. in U_3		10				50	51	25	31	21	55	01

Element	02	12	22	32	42	52	03	13	23	33	43	53
Recip. in U_1				32					23		43	
Recip. in U_2		52		34		12		13				53
Recip. in U_3		32		12		14		43	53		13	23

Element	04	14	24	34	44	54	05	15	25	35	45	55
Recip. in U_1				34								
Recip. in U_2		54		32		14		55				15
Recip. in U_3		52		54		34	15	05	11	45	35	41

The table is easily interpreted. Thus, for example, it shows at a glance that the element $4+2x$ is singular in each of the three scales U_1, U_2, U_3; and that the element $5+2x$ is singular in U_1, but regular in U_2 and U_3 (with the reciprocals $1+2x$ and $1+4x$, respectively). The scales U_1, U_2, U_3, although of the same order,[11] are of very different structures; in fact, they contain respectively *seven*, *fourteen*, and *twenty-four regular* elements.

From the standpoint of cryptanalysis, it is especially significant that finite scales of the same order r and of widely divergent structures may be so easily specified in this manner.[12] Two finite *fields* of the same order are well known to be algebraically identical. The present section will serve to emphasize that there is an infinity of finite *scales* in which order does not completely determine structure.

5. *Rational Manipulations in a Scale*

It is very readily argued that, with only minor and obvious reservations, all the rational operations and apparatus incidental to the solution, when unique solutions exist, of systems of linear equations in a field are applicable in any scale.

We are especially interested in noting the existence, in any scale, of a full matric algebra of familiar type. The following comments on determinants and matrices in an arbitrary (finite or infinite) scale will not be amiss.

---

[10] The reader will hardly confuse the symbol $\alpha\beta$, used only in the present example, with the product $\alpha\beta$ of elements of $S(6)$.

[11] By the "order" of a finite scale, we understand the number of elements contained in the scale.

[12] When $r=k^n$, with k and n positive integers each greater than 1.

The determinant

$$L = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \cdot & \cdot \quad \cdots \\ \cdot & \cdot \quad \cdots \\ \cdot & \cdot \quad \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix},$$

of order n, in which the a_{ij} denote elements of the scale S, has the same meaning and properties as if S were a field. We define L to be *regular* or *singular* according as its "value" is a regular or a singular element of S, where its "value" is fixed by any one of the $2n$ equal expressions in S,

$$\sum_{i=1}^{n} a_{ij}A_{ij}, \quad \sum_{j=1}^{n} a_{ij}A_{ij}, \qquad i, j = 1, 2, \cdots, n,$$

in which A_{ij} denotes the cofactor (algebraic complement) of a_{ij} in L.

Moreover, L is called the determinant of the square matrix

$$M = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \cdot & \cdot \quad \cdots \\ \cdot & \cdot \quad \cdots \\ \cdot & \cdot \quad \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix} = (a_{ij})$$

of order n in S; and M is classed as *regular* or *singular* with L.

Upon occasion, we shall regard each element of the scale S as a matrix of order $n=1$ in S. The set, $SR\{n\}$, of all square matrices of order n in S will be called a *range* of matrices in S. When no misunderstanding can arise concerning the scale S employed, the range of order n in S will be denoted simply by $R\{n\}$. It is clear that $R\{1\}$ consists of all elements of the scale S regarded as matrices of the first order in S.

When $n>1$, a non-commutative algebra may be set up in the range $R\{n\}$ of the scale S. The procedure is a very familiar one,[13] but may be briefly recalled here:

(1) If $A=(a_{ij})$ and $B=(b_{ij})$ are matrices of the range $R\{n\}$ in the scale S, we define $A=B$ when and only when $a_{ij}=b_{ij}$ in S for every pair of indices i, j; and we define addition and multiplication as follows, operations affecting elements a_{ij}, b_{ij}, c_{ij} of S being performed, of course, under the rules of S:

$$A + B = (a_{ij}) + (b_{ij}) = C = (c_{ij}), \text{ with } c_{ij} = a_{ij} + b_{ij}.$$
$$AB = (a_{ij})(b_{ij}) = D = (d_{ij}), \text{ with } d_{ij} = \sum_{q=1}^{n} a_{iq}b_{qj}.$$

[13] The procedure is familiar for the case in which the scale S is a field. When S is not a field, certain precautions must be taken, as will be indicated.

From these definitions it is easy to conclude that, if A, B, C are any matrices of the range $R\{n\}$,

$$A + B = B + A, \quad A + (B + C) = (A + B) + C,$$
$$A(BC) = (AB)C, \qquad A(B + C) = AB + AC.$$

But, in general, $AB \neq BA$.

(2) Let β be any element of the scale S. That matrix of the range $R\{n\}$ in S which has the scalar β in each place of the principal diagonal, and the scalar 0 everywhere else, may be called the *scalar matrix* of β, and may conveniently be denoted by β_n.

(3) If A is any matrix of $R\{n\}$ in S, and β is any scalar in S, each of the mixed products βA and $A\beta$ is defined to be the matrix obtained upon multiplying every element of A by β. It is evident that

$$\beta A = \beta_n A = A\beta_n = A\beta.$$

(4) The range $R\{n\}$ in S contains an unique *zero matrix* 0_n, and an unique *unit matrix* 1_n, such that if A is any matrix of the range,

$$A + 0_n = A, \; A0_n = 0_nA = 0_n, \; A1_n = 1_nA = A.$$

These special matrices are merely the scalar matrices of the scalars 0 and 1.

(5) Corresponding to any matrix A of $R\{n\}$ in S, there is exactly one matrix $B = -A$ such that $A + B = 0_n$. The matrix, $-A$, is the mixed product of the matrix A and the scalar -1.

(6) Corresponding to any matrices A and B of the range $R\{n\}$ in S, there is exactly one matrix $C = B - A$ of the range such that $A + C = B$. Clearly, $B - A = B + (-A)$.

(7) If the matrix A of $R\{n\}$ in S is *regular*, there is exactly one matrix B of the range such that $AB = 1_n$; and B also satisfies the equation $BA = 1_n$. We call B the *reciprocal* of A, and write $B = A^{-1}$. If $B = A^{-1}$, then also $A = B^{-1}$.

(8) Let the matrix A of the range $R\{n\}$ in S be *regular*, and let M be any matrix of the range. Then $R\{n\}$ contains exactly one matrix H, and exactly one matrix K, such that $AH = M = KA$. In fact, it is clear that $H = A^{-1}M$ while $K = MA^{-1}$.

(9) The reciprocal of the *regular* matrix $A = (a_{ij})$ of $R\{n\}$ in S is easily written out; it is simply:

$$A^{-1} = \begin{pmatrix} \dfrac{A_{11}}{\rho} & \cdots & \dfrac{A_{n1}}{\rho} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \dfrac{A_{1n}}{\rho} & \cdots & \dfrac{A_{nn}}{\rho} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A_{1n} & \cdots & A_{nn} \end{pmatrix},$$

where A_{ij} is the cofactor of a_{ij} in the determinant of A, and ρ is the value of that determinant as worked out in the scale S. A singular matrix has no reciprocal.

(10) If we agree to interpret the mixed sum $\beta+A=A+\beta$, where β is a scalar in S and A is a matrix of the range $R\{n\}$ in S, as the sum $\beta_n+A=A+\beta_n$ of matrices of $R\{n\}$, then in any expression like

$$c+\sum_{q=1}^{m} a_q x_q,$$

where c, a_q, x_q are matrices of $R\{n\}$, we may replace any scalar matrix by its corresponding scalar.

(11) Exponential notations will be self-explanatory. We note, however, that the symbol A^{-q}, where A is a matrix of the range $R\{n\}$ in the scale S, and q is a positive integer or zero, is not defined unless A is regular. When A is regular, $A^0=1_n$, $A^{-q}=(A^{-1})^q$.

We are now prepared to discuss a novel class of ciphers associated with the general linear transformation in the general range $R\{n\}$ of the general scale S. It will be necessary, of course, to employ only such transformations as have unique inverses. Also it will be very desirable, for practical reasons, to make easily available a large class of *involutory* transformations.

6. *Linear Transformations in the General Range*

Consider the linear transformation T_a:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1f}x_f + a_1, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_f &= a_{f1}x_1 + a_{f2}x_2 + \cdots + a_{ff}x_f + a_f, \end{aligned}$$

where f is any positive integer, and the x_i, y_i, a_{ij}, a_i are matrices of any range $R\{n\}$ in any scale S. All operations required to effect this transformation are to be performed in the range $R\{n\}$.

If $n>1$, the algebra of $R\{n\}$ is non-commutative, as explained in Section 5. The range $R\{1\}$ coincides with the scale S itself, and the algebra of this range with the (commutative) algebra of S. When $n=1$, it is clear that T_a is merely a scalar transformation, the variables and the coefficients being elements of the scale S.

In all that follows, we shall understand the range $R\{n\}$ to include the underlying scale S as the special range $R\{1\}$. The transformation T_a converts a sequence of f matrices of the general range $R\{n\}$ into another such sequence. But when $n=1$, the matrices are of order 1, and are therefore merely *scalars* (elements of the scale S).

The rectangular array of $f(f+1)$ matrices,

$$P_a = \begin{bmatrix} a_{11} \cdots a_{1f} & a_1 \\ \cdot \quad \cdot \cdot \cdot \cdot & \cdot \\ \cdot \quad \cdot \cdot \cdot \cdot & \cdot \\ a_{f1} \cdots a_{ff} & a_f \end{bmatrix},$$

will be called the *schedule* of T_a, and will be designated as $P_a = [(a_{ij}), a_i]$. The square array of f^2 matrices $M_a = (a_{ij})$ will be called the *basis* of T_a.

We denote by J the set of all transformations which can be obtained in this way, for a fixed integer f, from the range $R\{n\}$ in the scale S. Let T_a, with $P_a = [(a_{ij}), a_i]$, and T_b, with $P_b = [(b_{ij}), b_i]$, be two transformations of the set J. Applying T_a to a sequence of f matrices $x_1, x_2, \cdots, x_f$ of $R\{n\}$ in S, we obtain the sequence $y_1, y_2, \cdots, y_f$ of matrices of the same range; applying T_b to the sequence $y_1, y_2, \cdots, y_f$, we obtain the sequence $z_1, z_2, \cdots, z_f$; compactly: $T_a(x) = y$, $T_b(y) = z$. The set J evidently contains an unique transformation T_c, with $P_c = [(c_{ij}), c_i]$, such that $T_c(x) = z$. We say that T_c is the *product* T_bT_a, distinguishing this product from T_aT_b. It is quickly found that

$$(1) \qquad c_{ij} = \sum_{q=1}^{f} b_{iq}a_{qj}, \quad c_i = b_i + \sum_{q=1}^{f} b_{iq}a_q,$$

the operations required for calculation by these formulas being effected according to the algebra of the range $R\{n\}$.

It is readily shown that products of transformations in J are associative; if T_a, T_b, T_c are any transformations in J, then $T_a(T_bT_c) = (T_aT_b)T_c$.

7. *A Fundamental Lemma*

Let T_a, with the schedule $P_a = [(a_{ij}), a_i]$, be a transformation belonging to the set J considered in Section 6. We fix our attention upon the *basis* $M_a = [a_{ij}]$ of T_a. If parentheses are removed from all the f^2 matrices in the square array $[a_{ij}]$, there results a square matrix G_a in the scale S, of order $g = fn$. The matrix G_a will be called the *frame matrix* of T_a. It is evident that G_a belongs to the range $R\{g\}$ in the scale S.

The following lemma is fundamental. It may be established by a straightforward argument which will be omitted here.

Lemma: Let T_a, T_b, T_c be transformations in J; and let their frame matrices be G_a, G_b, G_c respectively. Then $G_c = G_aG_b$ if[14] $T_c = T_aT_b$. In other words, *the frame matrix of a product of transformations is the corresponding product of the frame matrices of the transformations.*

8. *Regular Transformations in J.*

We consider now the set H of all those transformations in the set J which have *regular* frame matrices. We say that H is the set of *regular transformations* in J.

[14] Two transformations of the set J are "equal" when their schedules are exactly the same.

It is clear that H contains the identical transformation, defined by the schedule $Q=[(a_{ij}), a_i]$ in which $a_{ij}=1_n(i=j)$, $a_{ij}=0_n(i\neq j)$, $a_i=0_n$ (every index i). Here, as heretofore, we employ the designations 0_n and 1_n respectively for the zero and unit matrices of the range $R\{n\}$ in S.

By an argument based upon the *lemma*, a significant theorem may now be established:

Theorem 1: If T_a is any transformation in H, there exists, in H, an unique transformation T_b such that the schedule of the product T_aT_b is Q; and Q is likewise the schedule of the product T_bT_a.

This theorem asserts that (1) any *regular* transformation T in J has an unique inverse T^{-1}, and (2) T^{-1} is regular and has T for its inverse.

Proof: We suppose, first, that T_a is any *homogeneous* transformation in H (any transformation in H with the schedule $[(a_{ij}), a_i]$ in which $a_i=0_n$, the zero matrix of the range $R\{n\}$, for every index i). The frame matrix G_a of T_a is regular, and has an unique reciprocal[15] G_a^{-1} in the range $R\{g\}$. Hence that homogeneous transformation T_b of J which has the frame matrix G_a^{-1} is regular, and lies in H. By the *lemma*, T_b is manifestly an unique inverse to T_a in the set H.

Now let T_a be any transformation in H. Let $y_i=z_i+a_i$ $(i=1, 2, \cdots, f)$, these sums being formed, of course, in the range $R\{n\}$ of matrices. Substituting in the equations of T_a, we obtain the equations of a transformation T_c in H—a transformation converting the sequence $x_1, x_2, \cdots, x_f$ into the sequence $z_1, z_2, \cdots, z_f$. Since T_c is of homogeneous type, it has an unique inverse T_c^{-1}. Replacing z_i, in the equations of T_c^{-1}, by y_i-a_i, and simplifying (by operations in the range $R\{n\}$), we determine the equations of a transformation T_a^{-1} which is the unique inverse of T_a in H.

The argument is completed by the observation that if G_a, G_b denote any two matrices, of the range $R\{g\}$, such that $G_aG_b=1_g$, the unit matrix of the range, then also $G_bG_a=1_g$ (See 7, Section 5).

We have thus a procedure for the actual determination of the equations of the inverse transformation of which the existence is asserted, the required operations being performed *in the underlying scale S itself*. As will be indicated in examples below, it is frequently possible and convenient to find the inverse transformation by elimination processes carried out *in the range $R\{n\}$*, without descending to the scale S.

The set H obviously constitutes a *group* of transformations, this group being finite if the scale S is finite.

9. *Construction of Transformations of the Group H*

The following modifications of a matrix of any range in any scale will be called *elementary*:

(1) interchanging rows and columns; (2) adding, to every element of any row (column), α times the corresponding element of another row (column),

[15] See (9), Section 5.

where α denotes any scalar (element of the underlying scale S); (3) multiplying every element of any row (column) by a *regular* scalar, and every element of another row (column) by the reciprocal of that scalar; (4) interchanging two rows (columns); (5) changing the sign of every element of a row (column).

The value of the determinant of the matrix is, of course, not changed by (1), (2), or (3); and is changed only in sign by (4) or (5).

Now consider the special matrix ${}_gI_\beta$ of the range $R\{g\}$ in the scale S. This matrix is so defined that it differs from the unit matrix 1_g of the range only at the intersection of the last (g-th) row and last column, where it has the scalar β instead of the scalar 1. Successions of elementary modifications may evidently be applied to ${}_gI_\beta$ in such manner as to alter its appearance completely, while leaving the value, β, of its determinant unchanged. Selecting, as β, any *regular* element of the scale S, we have the means of constructing, quickly and easily, a variety of regular matrices of the range $R\{g\}$. We may, of course, use any one of these as the frame matrix of a transformation in the group H.

10. *Involutory Transformations of the Group H*

Let us call a transformation T, in J, *involutory* if T^2 (that is, TT) is the identical transformation, so that the schedule of T^2 is Q. When T is involutory, the determinant of the frame matrix of T^2 evidently has the value 1 in the scale S. Since the value of the determinant of a product of square matrices is obviously the product of the values of their determinants, we conclude that the value, δ, of the determinant of the matrix of T satisfies the equation $\delta^2=1$ in S. It follows that δ is a *regular*[16] element of S, and therefore that T is regular. Hence any involutory transformation in J is regular, and lies in the group H.

The following theorem is evident:

Theorem 2: If T_1 is an involutory transformation, and T any transformation, in H, then each of the transformations TT_1T^{-1} and $T^{-1}T_1T$ is involutory, and lies[17] in H.

For many cryptanalytic purposes, the following is an involutory transformation of sufficient complexity:

$$y_i = x_i - \lambda_i\tau\left(\sum_{j=1}^{f}\lambda_j x_j + \mu\right), \tag{2}$$

where $i=1, 2, 3, \cdots, f$; and $\lambda_1, \lambda_2, \cdots, \lambda_f, \mu$ is any sequence of $f+1$ matrices selected quite arbitrarily from the range $R\{n\}$ in the scale S; provided that

$$\sigma = \sum_{i=1}^{f}\lambda_i^2$$

is a *regular* matrix, and $\tau=2\sigma^{-1}$, the symbol 2 denoting the element $1+1$ of the

[16] The equation $\delta^2=1$, in an arbitrary scale S, does not imply $\delta=\pm 1$. For example, in the scale $S(100)$, this equation has the four roots 1, 49, 51, 99 (that is to say, ± 1 and ± 49).

[17] We have just noted that every involutory transformation in J lies in the group H.

scale S (so that τ is the sum of two matrices, each equal to the reciprocal of σ).[18]

Operations required for the application of formula (2) are to be performed in the range $R\{n\}$ of matrices in the scale S. We easily verify that the formula gives the equations of a transformation which is involutory. In fact, making two applications of this transformation, we obtain:

$$\begin{aligned}
z_i &= y_i - \lambda_i\tau\left(\sum\lambda_j y_j + \mu\right) \\
&= x_i - \lambda_i\tau\left(\sum\lambda_j x_j + \mu\right) - \lambda_i\tau\left\{ \sum\lambda_j\left[x_j - \lambda_j\tau\left(\sum\lambda_q x_q + \mu\right)\right] + \mu\right\} \\
&= x_i - \lambda_i\tau \sum\lambda_j x_j - \lambda_i\tau\mu - \lambda_i\tau\left\{ \sum\lambda_j x_j - \sum\lambda_j^2\tau\left(\sum\lambda_q x_q + \mu\right) + \mu\right\} \\
&= x_i - \lambda_i\tau \sum\lambda_j x_j - \lambda_i\tau\mu - \lambda_i\tau \sum\lambda_j x_j + \lambda_i\tau\sigma\tau \sum\lambda_q x_q + \lambda_i\tau\sigma\tau\mu - \lambda_i\tau\mu \\
&= x_i - 2\lambda_i\tau \sum\lambda_j x_j - 2\lambda_i\tau\mu + 2\lambda_i\tau \sum\lambda_j x_j + 2\lambda_i\tau\mu = x_i.
\end{aligned}$$

In other words, if we denote the transformation (2) by T, we have $T(x)=y$ and $T(y)=x$, so that $T^2(x)=x$.

The reductions made in the above verification of the involutory character of the transformation (2) will be easily understood if the reader bears in mind that $\tau\sigma=2_n$, where 2_n denotes the scalar matrix, in $R\{n\}$, of the scalar $2=1+1$ of S, and may be replaced by the scalar 2. It should also be recalled that in a mixed product of matrices and scalars we may, as explained in Section 5, shift the position of a scalar factor.

11. *Notations and Procedure in Examples*

Our illustrations will be based upon the scale $S(26)$. We shall give examples of ciphers based upon linear transformations in ranges of this scale. The extension of the method to other scales will be obvious, and will not require explicit treatment.

The following particular correspondence between $S(26)$ and the letters of the English alphabet will be adopted:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
m	*j*	*d*	*x*	*a*	*h*	*o*	*u*	*c*	*z*	*q*	*e*	*t*	*y*	*f*	*w*	*g*	*i*	*v*	*s*	*k*	*p*	*l*	*r*	*n*	*b*

or, in alphabetical arrangement:

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
4	25	8	2	11	14	16	5	17	1	20	22	0	24	6	21	10	23	19	12	7	18	15	3	13	9.

Any one of the 26! possible correspondences would serve equally well. But, to be explicit, we shall employ the foregoing.

The symbol ${}_nT_f$ will designate a transformation T which is performed in the range $R\{n\}$ of the scale $S(26)$, and is defined by f equations; such a transformation will convert a sequence of f matrices of $R\{n\}$ into another such sequence.

---

[18] Sections 13 and 14 present examples of the effective use of this formula.

Let it be desired to encipher a message by means of a transformation of the type ${}_nT_f$. Let the message be: $t_1, t_2, t_3, \cdots$, the t_i denoting simply the successive letters of the message as it is written out. Replacing the t_i by their corresponding elements of $S(26)$, we obtain the scalar sequence $q_1, q_2, q_3, q_4, \cdots$.

We now partition the q-sequence into subsequences of $k=n^2f$ elements each, writing $q_1q_2, \cdots, q_{k+1}\, q_{k+2}, \cdots, q_{2k+1}\, q_{2k+2}, \cdots$. If the last subsequence is incomplete, we fill it out, in any prearranged manner, to k elements.

Each subsequence is enciphered in the same way. The encipherment is accomplished by writing the subsequence, according to any convention, as a sequence of f square matrices, each of order n, in the scale $S(26)$, and subsequently transforming this sequence $x_1, x_2, \cdots, x_f$ of matrices, through a transformation of type ${}_nT_f$, into the sequence $y_1, y_2, \cdots, y_f$ of matrices. To decipher, we apply to the sequence $y_1, y_2, \cdots, y_f$ the transformation inverse to that used in encipherment.

The cipher subsequence actually transmitted is, of course, not the sequence $y_1, y_2, \cdots, y_f$ of matrices in $S(26)$, but the corresponding sequence of n^2f letters of the alphabet.

12. *Example 1*

The determinant of the matrix

$$\begin{pmatrix} 5 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix}$$

has the value[19] 5 in $S(26)$. Hence this matrix is regular; and the transformation

$$(1) \qquad y_1 = \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix},$$

$$(2) \qquad y_2 = \begin{pmatrix} 3 & 2 \\ 9 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} x_2 + \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix},$$

of which it is the frame matrix, is regular. In this transformation, which we shall denote by T_1, the terms

$$\begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix}$$

are, of course, chosen quite arbitrarily from the matrices of the range $R\{2\}$ in $S(26)$.

We readily find that the inverse transformation, T_1^{-1}, has these equations:

[19] See Section 9.

$$(3) \qquad x_1 = \begin{pmatrix} 11 & 0 \\ 15 & 1 \end{pmatrix} y_1 + \begin{pmatrix} 0 & -6 \\ 0 & 6 \end{pmatrix} y_2 + \begin{pmatrix} -5 & 7 \\ 1 & 16 \end{pmatrix},$$

$$(4) \qquad x_2 = \begin{pmatrix} 15 & -2 \\ -7 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} y_2 + \begin{pmatrix} 11 & -1 \\ 6 & 3 \end{pmatrix},$$

there being, throughout the work in $S(26)$, two alternative expressions for each element of the scale $(21 = -5,\ 19 = -7$, etc.).

The equations (3) and (4) may be found by a simple elimination process in the range $R\{2\}$. The coefficient of x_2 in (2) is regular, and has the reciprocal[20]

$$\frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 9\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

Left-hand multiplication of each term of (2) by the product,

$$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix},$$

yields an equation which we subtract from (1), obtaining

$$(5) \qquad y_1 - \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} y_2 = \begin{pmatrix} -7 & 0 \\ 1 & 1 \end{pmatrix} x_1 + \begin{pmatrix} -9 & -3 \\ 4 & 3 \end{pmatrix}.$$

The coefficient of x_1 in (5) being regular, we easily solve for x_1 in terms of y_1 and y_2. Then (4) is deduced in obvious manner.

In this example, $n = f = 2$. Given a message for encipherment, we first arrange it in subsequences of $n^2 f = 8$ letters each, filling out the last subsequence, if necessary, by the adjunction of further letters according to the conventions of the cipher. Let the message be, for instance, SUSPEND ATTACK, so that the initial subsequence is SUSPENDA. Let it be agreed, as a cipher prearrangement, to write:

$$\begin{pmatrix} S & U \\ S & P \end{pmatrix}, \quad \begin{pmatrix} E & N \\ D & A \end{pmatrix},$$

thus determining the two matrices in $S(26)$,

$$x_1 = \begin{pmatrix} 19 & 7 \\ 19 & 21 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 11 & 24 \\ 2 & 4 \end{pmatrix},$$

by means of the correspondence adopted in Section 11. Applying the transformation T_1 to the sequence x_1, x_2 of matrices, we obtain the sequence y_1, y_2:

$$y_1 = \begin{pmatrix} 17 & 9 \\ 12 & 2 \end{pmatrix} + \begin{pmatrix} 8 & 16 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 16 & 5 \end{pmatrix},$$

$$y_2 = \begin{pmatrix} 17 & 11 \\ 15 & 11 \end{pmatrix} + \begin{pmatrix} 11 & 24 \\ 6 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 22 & 3 \end{pmatrix}.$$

[20] See (9), Section 5.

Returning to the alphabet, we replace the sequence y_1, y_2 of matrices by

$$\begin{pmatrix} M & A \\ G & H \end{pmatrix}, \quad \begin{pmatrix} A & Z \\ L & X \end{pmatrix};$$

and the enciphered form of the initial message subsequence is $M\,A\,G\,H\,A\,Z\,L\,X$. In decipherment, we apply the transformation T_1^{-1} to the sequence y_1, y_2 of matrices, obtaining again the original matrix sequence x_1, x_2, and therewith also the original message subsequence *SUSPENDA*. The same procedure is followed with each message subsequence.

13. *Example 2*

If a cipher transformation can be made involutory without an appreciable weakening of the resistance offered to cryptanalysis, it is desirable, for many reasons, that this be done. Let us construct an involutory transformation of type ${}_2T_2$, employing formula (2), Section 10. Taking

$$\lambda_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \text{ and } \mu = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix},$$

we find that

$$\sigma = \lambda_1^2 + \lambda_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 9 & -1 \\ 5 & 14 \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ 5 & 15 \end{pmatrix}$$

is regular. Thus we have:

$$\sigma^{-1} = \begin{pmatrix} 11 & -1 \\ 5 & 16 \end{pmatrix} \text{ and } \tau = 2\sigma^{-1} = \begin{pmatrix} 22 & -2 \\ 10 & 6 \end{pmatrix}.$$

It follows that:

$$-\lambda_1\tau = \lambda_1(-\tau) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 16 & 20 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 14 & 8 \end{pmatrix}$$

$$-\lambda_2\tau = \lambda_2(-\tau) = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 16 & 20 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}.$$

Therefore, by formula (2) of Section 10, the following transformation, which we shall call T_2, is involutory:

$$y_1 = x_1 + \begin{pmatrix} 4 & 2 \\ 14 & 8 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} x_1 + \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} x_2 + \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \right],$$

$$y_2 = x_2 + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} x_1 + \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} x_2 + \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \right].$$

Its equations may be expressed:

$$y_1 = \begin{pmatrix} 7 & -2 \\ -4 & -7 \end{pmatrix} x_1 + \begin{pmatrix} 10 & 0 \\ 10 & 16 \end{pmatrix} x_2 + \begin{pmatrix} -4 & 10 \\ 12 & 10 \end{pmatrix} \tag{1}$$

$$y_2 = \begin{pmatrix} 10 & 0 \\ 10 & 16 \end{pmatrix} x_1 + \begin{pmatrix} -5 & -2 \\ 10 & 5 \end{pmatrix} x_2 + \begin{pmatrix} 16 & -6 \\ 0 & 10 \end{pmatrix} \tag{2}$$

the use of negatives being avoidable if only positive signs are desired.

The equations of T_2^{-1} are exactly the same as (1), (2) except that an interchange is made of x_1 and y_1, and of x_2 and y_2.

From T_2, we easily obtain, by Theorem 2, Section 10, further involutory transformations. Thus, denoting again by T_1 the transformation given in equations (1), (2) of Section 12, we know that each of the transformations $T_1^{-1}T_2T_1$ and $T_1T_2T_1^{-1}$ is involutory. These products may be determined from formulas (1) of Section 6, or by successive applications of the factor transformations. Following the latter method, we find that the product T_2T_1 is:

$$z_1 = \begin{pmatrix} 11 & 18 \\ 17 & 13 \end{pmatrix} x_1 + \begin{pmatrix} 10 & 2 \\ 10 & 6 \end{pmatrix} x_2 + \begin{pmatrix} 15 & 13 \\ 16 & 13 \end{pmatrix},$$

$$z_2 = \begin{pmatrix} 17 & 16 \\ 11 & 10 \end{pmatrix} x_1 + \begin{pmatrix} 21 & 8 \\ 10 & 3 \end{pmatrix} x_2 + \begin{pmatrix} 14 & 6 \\ 21 & 8 \end{pmatrix};$$

and that the (involutory) product $T_1^{-1}T_2T_1$ is:

$$y_1 = \begin{pmatrix} 3 & 8 \\ 14 & 5 \end{pmatrix} x_1 + \begin{pmatrix} 24 & 4 \\ 12 & 2 \end{pmatrix} x_2 + \begin{pmatrix} 8 & 24 \\ 4 & 12 \end{pmatrix}, \tag{3}$$

$$y_2 = \begin{pmatrix} 6 & 8 \\ 12 & 14 \end{pmatrix} x_1 + \begin{pmatrix} 3 & 18 \\ 18 & 15 \end{pmatrix} x_2 + \begin{pmatrix} 6 & 14 \\ 0 & 24 \end{pmatrix}. \tag{4}$$

The cryptographic application of each of the two involutory transformations developed in this section is essentially the same as that of the non-involutory T_1 of the preceding section, and requires no separate discussion. The only difference—an important one, from the practical standpoint—is that the same transformation, with interchange of x_i and y_i, of course, is here employed for encipherment and decipherment.

14. *Example 3*

Let us determine an involutory transformation of type ${}_3T_2$. If we set:

$$\lambda_1 = \begin{pmatrix} 3 & 8 & 14 \\ 8 & 7 & 4 \\ 14 & 4 & 21 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 6 & 18 & 16 \\ 24 & 20 & 12 \\ 16 & 22 & 8 \end{pmatrix}, \quad \mu = \begin{pmatrix} 11 & 2 & 8 \\ 3 & 10 & 7 \\ 9 & 21 & 4 \end{pmatrix},$$

where

$$\sigma = \lambda_1^2 + \lambda_2^2 = \begin{pmatrix} 9 & 6 & 4 \\ 6 & 25 & 16 \\ 4 & 16 & 3 \end{pmatrix} + \begin{pmatrix} 22 & 14 & 24 \\ 10 & 4 & 18 \\ 24 & 20 & 12 \end{pmatrix} = \begin{pmatrix} 5 & 20 & 2 \\ 16 & 3 & 8 \\ 2 & 10 & 15 \end{pmatrix}$$

is regular, and μ is arbitrary, we find, by (9) of Section 5,

$$\sigma^{-1} = \frac{1}{21}\begin{pmatrix} 17 & 6 & 24 \\ 10 & 19 & 18 \\ 24 & 16 & 7 \end{pmatrix} = 5\begin{pmatrix} 17 & 6 & 24 \\ 10 & 19 & 18 \\ 24 & 16 & 7 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 16 \\ 24 & 17 & 12 \\ 16 & 2 & 9 \end{pmatrix},$$

whence

$$\tau = 2\sigma^{-1} = \begin{pmatrix} 14 & 8 & 6 \\ 22 & 8 & 24 \\ 6 & 4 & 18 \end{pmatrix}, \quad \lambda_1\tau = \begin{pmatrix} 16 & 14 & 20 \\ 4 & 6 & 2 \\ 20 & 20 & 12 \end{pmatrix}, \quad \lambda_2\tau = \begin{pmatrix} 4 & 22 & 2 \\ 16 & 10 & 8 \\ 2 & 24 & 14 \end{pmatrix}.$$

By formula (2), Section 10, the transformation

$$y_1 = x_1 - \begin{pmatrix} 16 & 14 & 20 \\ 4 & 6 & 2 \\ 20 & 20 & 12 \end{pmatrix}(\lambda_1 x_1 + \lambda_2 x_2 + \mu),$$

$$y_2 = x_2 - \begin{pmatrix} 4 & 22 & 2 \\ 16 & 10 & 8 \\ 2 & 24 & 14 \end{pmatrix}(\lambda_1 x_1 + \lambda_2 x_2 + \mu)$$

is involutory. Its equations may be simplified to be

$$(1) \qquad y_1 = \begin{pmatrix} 3 & 6 & 2 \\ 16 & 23 & 8 \\ 2 & 16 & 13 \end{pmatrix} x_1 + \begin{pmatrix} 2 & 6 & 14 \\ 8 & 24 & 4 \\ 14 & 16 & 20 \end{pmatrix} x_2 + \begin{pmatrix} 18 & 6 & 6 \\ 24 & 20 & 22 \\ 2 & 2 & 16 \end{pmatrix}$$

$$(2) \qquad y_2 = \begin{pmatrix} 18 & 14 & 22 \\ 20 & 4 & 10 \\ 22 & 20 & 24 \end{pmatrix} x_1 + \begin{pmatrix} 15 & 16 & 20 \\ 4 & 13 & 2 \\ 20 & 8 & 11 \end{pmatrix} x_2 + \begin{pmatrix} 2 & 16 & 14 \\ 8 & 12 & 4 \\ 14 & 8 & 20 \end{pmatrix}.$$

Further involutory transformations of the type ${}_3T_2$ may, of course, be obtained from this by applying Theorem 2 of Section 10 and formulas (1) of Section 6; or by applying Theorem 2 of Section 10 and the procedure outlined at the close of Section 13.

In using the involutory transformation given by the equations (1) and (2) above, we first partition our message into subsequences of eighteen letters each, since $n^2f = 18$. We fill out the last subsequence, if it is incomplete, with any prearranged letters.

Consider, for instance, the message: *HOLD OUT. SUPPORTING AIR SQUADRONS EN ROUTE.* It contains two full subsequences. Let us treat the first of these: *HOLDOUTSUPPORTINGA.* We suppose that the convention adopted in the cipher is to write:

$$x_1 = \begin{pmatrix} H & 0 & L \\ D & 0 & U \\ T & S & U \end{pmatrix} = \begin{pmatrix} 5 & 6 & 22 \\ 2 & 6 & 7 \\ 12 & 19 & 7 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} P & P & 0 \\ R & T & I \\ N & G & A \end{pmatrix} = \begin{pmatrix} 21 & 21 & 6 \\ 23 & 12 & 17 \\ 24 & 16 & 4 \end{pmatrix}$$

by means of the correspondence specified in Section 11. Substituting these matrices for x_1 and x_2 in the equations (1), (2) of the present section, we obtain:

$$y_1 = \begin{pmatrix} 25 & 14 & 18 \\ 14 & 22 & 23 \\ 16 & 17 & 13 \end{pmatrix} + \begin{pmatrix} 22 & 0 & 14 \\ 10 & 0 & 4 \\ 24 & 0 & 20 \end{pmatrix} + \begin{pmatrix} 18 & 6 & 6 \\ 24 & 20 & 22 \\ 2 & 2 & 16 \end{pmatrix} = \begin{pmatrix} 13 & 20 & 12 \\ 22 & 16 & 23 \\ 16 & 19 & 23 \end{pmatrix},$$

$$y_2 = \begin{pmatrix} 18 & 12 & 24 \\ 20 & 22 & 18 \\ 22 & 6 & 12 \end{pmatrix} + \begin{pmatrix} 19 & 21 & 0 \\ 15 & 12 & 19 \\ 10 & 16 & 14 \end{pmatrix} + \begin{pmatrix} 2 & 16 & 14 \\ 8 & 12 & 4 \\ 14 & 8 & 20 \end{pmatrix} = \begin{pmatrix} 13 & 23 & 12 \\ 17 & 20 & 15 \\ 20 & 4 & 20 \end{pmatrix}.$$

Hence the enciphered form of the subsequence is

$$\begin{pmatrix} Y & K & T \\ L & G & R \\ G & S & R \end{pmatrix}, \quad \begin{pmatrix} Y & R & T \\ I & K & W \\ K & A & K \end{pmatrix};$$

or, as it would be transmitted,[21] $Y\,K\,T\,L\,G\,R\,G\,S\,R\,Y\,R\,T\,I\,K\,W\,K\,A\,K$.

Substitution of the matrices y_1, y_2 in the same equations (1) and (2) above yields again the original matrices x_1, x_2, the equations having first been written, of course, with x_i and y_i interchanged $(i=1, 2)$.

Each message subsequence is enciphered and deciphered in the same manner.

15. *Concluding Notes*

All the transformations discussed in the foregoing pages are obviously applicable in any range $R\{n\}$ of any scale S, the special range $R\{1\}$ coinciding with S and yielding ordinary scalar transformations. Of interest in cryptography are not only the scales $S(n)$, and their various algebraic extensions, but also certain non-modular and infinite scales. In this connection, we note especially the scale S consisting of all positive and negative integers and zero in the field of rational numbers; any *regular* transformation in a range of this scale will have a frame matrix with determinant of value ± 1.

Plans have been completed for a novel type of computing machine capable of effecting the simultaneous and speedy evaluation of any desired number of

[21] When the entire message has been enciphered, it will normally be transmitted in the conventional five-letter groups: *YKTLG RGSRY* · · · ·

linear functions of any assigned sequences of elements in any scale of type $S(n)$, the linear functions having any arbitrarily selected scheme of coefficients. The machine, although originally designed for other purposes, may be used to apply very rapidly, without calculations of any sort, all transformations proposed in this paper for which the underlying scale is an $S(n)$, and even products of such transformations with widely variable ciphers of different type. From the point of view of cryptography, this circumstance lends exceptional interest to the scales $S(n)$.

Formula (2), Section 10, demands a sequence $\lambda_1, \lambda_2, \cdots, \lambda_f$ of f square matrices such that $\lambda_1^2+\lambda_2^2+\cdots+\lambda_f^2$ is a *regular* matrix. A great variety of sequences with this property can be determined very quickly, in any range of any scale, and for any positive integer f, by means of an interesting formula which will be the subject of special discussion elsewhere.

In any scale S, it is easy to set up an algebra for ranges of matrices whose elements are in turn matrices, the elements of these latter being again matrices, etc. But no important cryptographic advantages seem to arise from these further complications.

TWO FUNCTIONAL EQUATIONS WITH INTEGRAL ARGUMENTS

By PHILIP FRANKLIN, Massachusetts Institute of Technology

Professor E. T. Bell has recently indicated[1] that the general solution of the functional equations

$$f(x, n_1)f(x, n_2) = f(x, n_1 + n_2 + c), \tag{1}$$

$$f(x, n_1)f(x, n_2) = f(x, cn_1n_2), \tag{2}$$

in which the argument n is an integer $\geqq 0$, and the constant c is an integer $\geqq 0$, while x is a parameter, had a connection with the question of possible types of arithmetic; and asked what were the general solutions of these equations. We here obtain these general solutions, showing that the solution of the first involves a single function of x, while that of the second involves an enumerable number of such functions.

Theorem 1. *The general solution of* (1) *is* $[F(x)]^{n+c}$. We prove this by noting that, in consequence of (1), we have:

$$f(x, n-1)f(x, n+1) = f(x, 2n+c) = [f(x, n)]^2 \qquad (n = 1, 2, \ldots). \tag{3}$$

This shows that if $f(x, 0)=0$, $f(x, n)=0$ for all n; and also that if $f(x,0)\neq 0$, no $f(x, n)$ can vanish. In this latter case, we may rewrite (3) in the form:

$$\frac{f(x, n+1)}{f(x, n)} = \frac{f(x, n)}{f(x, n-1)} \cdots = \frac{f(x, 1)}{f(x, 0)} = F(x). \tag{4}$$

---

[1] This Monthly, vol. 37 (1930), p. 484.

This shows that

$$f(x, n) = f(x, 0)[F(x)]^n. \tag{5}$$

Finally, from (1),

$$[f(x, 0)]^2 = f(x, c) = f(x, 0)[F(x)]^c, \tag{6}$$

so that

$$f(x, 0) = [F(x)]^c, \tag{7}$$

and

$$f(x, n) = [F(x)]^{n+c}, \tag{8}$$

which is the general solution, since the excluded case reappears by putting $F(x)=0$ in (8). This is the solution given by Professor Bell.

Theorem 2. *Let c be greater than* 1, *and let*

$$c = c_1^a c_2^b \cdots c_t^k, \quad a + b \cdots + k = s, \tag{9}$$

be the factorization of c into prime factors distinct from unity. Let the range of x be divided into two parts, denoted by x_0 and x_1. For the first part, take for all n:

$$f(x_0, n) = 0. \tag{10}$$

On the second part, take t functions $f(x_1, c_i)$ such that

$$f(x_1, c_i) \neq 0, \tag{11}$$

but otherwise arbitrary, and an enumerable number of functions

$$f(x_1, p), \tag{12}$$

one for each prime distinct from 1 *and c_i, these being absolutely arbitrary. Then, if $f(x, 1)$ is any solution of*

$$[f(x, c_1)]^a [f(x, c_2)]^b \cdots [f(x, c_t)]^k = [f(x, 1)]^{s+1}, \tag{13}$$

the values of $f(x_1, m)$, where m is any composite number with factorization:

$$m = p_1^\alpha p_2^\beta \cdots p_r^\kappa, \ \alpha + \beta + \cdots + \kappa = q, \tag{14}$$

are given by:

$$f(x_1, m) = \frac{[f(x_1, p_1)]^\alpha [f(x_1, p_2)]^\beta \cdots [f(x_1, p_r)]^\kappa}{[f(x_1, 1)]^{q-1}}. \tag{15}$$

also, $f(x_1, 0)=0$, unless all the $f(x_1, m)=1$, when it may equal 1. *This gives the general solution of* (2) *in terms of the functions of* (11), (12) *and the root in* (13), *together with the specification of x_0, and the values of x for which $f(x, 0)=1$.*

When $c=1$, we separate the range as before, and retain (10). *On the range x_1, we take*

$$(16) \qquad f(x_1, 1) = 1,$$

the functions (12) *arbitrary for all primes not unity, and use* (15) *to complete the solution.*

When $c=0$, *we must have:*

$$(17) \qquad f(x_1, n) = 1,$$

which is to be combined with (10).

To prove the statements about the general case, we note that by a repeated application of (2), the right member of (13) is found to be

$$(18) \qquad f(x, c^s),$$

and by using (2) and (9), the left member has the same value and the relation (13) is proved. This shows that if any $f(x, c_i)$ vanishes, the same is true of $f(x, 1)$ for that x, and since (2) gives:

$$(19) \qquad [f(x, n)]^2 = f(x, cn^2) = f(x, 1)f(x, n^2),$$

we see that $f(x, n)$ vanishes for all n. Thus there is a range x_0 for which (10) holds, while in the complementary range x_1, (11) is satisfied, and

$$(20) \qquad f(x_1, 1) \neq 0.$$

We next use (2) to obtain

$$(21) \qquad f(x_1, m)[f(x_1, 1)]^{q-1} = f(x_1, c^{q-1}m),$$

and in view of (14), by (2) the numerator of (15) has the same value. Since (20) holds, we may divide out and establish (15). Thus all the relations on the functions $f(x, m)$ stated in the theorem are shown to be necessary consequences of (2).

To show that no further relations are imposed, we have merely to note that the value given by (15) is unchanged if unity is counted as a prime factor to any finite power, that it gives the correct value for m a prime, and with the convention just made for $m=1$, taking $1=1^\alpha$ in place of (14). Thus (15) defines $f(x_1, n)$ for all n in terms of the fundamental functions. But, if n_1, n_2 and $c\, n_1\, n_2$ be decomposed into prime factors, and the corresponding functions in (2) are evaluated by (15), it is found that (2) becomes an identity in view of (9) and (13). Thus our method gives a solution, and it is the most general one possible since no unnecessary restrictions are imposed.

When $c=1$, we have in place of (2):

$$(22) \qquad f(x, n_1)f(x, n_2) = f(x, n_1n_2), \quad c = 1.$$

This shows that

$$(23) \qquad [f(x, 1)]^2 = f(x, 1).$$

If

$$
(24)\qquad \begin{aligned} f(x, 1) &= 0, \\ f(x, n) &= f(x, 1)f(x, n) = 0, \end{aligned}
$$

and we are on the range x_0. If not, then (16) holds, and (15) is established as before, or deduced directly from (22).

When $c=0$, we have in place of (2):

$$
(25)\qquad f(x, n_1)f(x, n_2) = f(x, 0).
$$

This gives, in particular,

$$
(26)\qquad [f(x, n)]^2 = f(x, 0),
$$

which shows that if $f(x, 0)=0$, we are on the range x_0. Otherwise, the equation

$$
(27)\qquad [f(x, 0)]^2 = f(x, 0)
$$

shows that

$$
(28)\qquad f(x_1, 0) = 1,
$$

and

$$
(29)\qquad f(x, 0)f(x, n) = f(x, 0)
$$

which follows from (25), establishes (17).

Equation (29), which holds for all values of c, shows that

$$
(30)\qquad f(x, 0) = 0,
$$

unless $f(x, n)=1$ for all $n\neq 0$, when it may equal 1 by (27) which also holds for all values of c.

We note in conclusion that the whole difficulty of the present problem lies in the restriction on the range of n. If n were a continuous variable, the transformation $n=y-c$ in (1) and $n=y/c$ in (2) would reduce them to the cases $c=0$ and $c=1$ respectively, which have been extensively studied, and can be reduced to one another by a logarithmic transformation.[2]

---

[2] See Ex. 1, 2, in §38 of de la Vallée Poussin's *Cours d'Analyse*, vol. 1, Paris, 1921.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

TWO MNEMONICS

By ALAN S. HAWKESWORTH, Washington, D. C.

The following mnemonic for π is due to Sir James Hopwood Jeans:

"How I want a drink, alcoholic of course, after the heavy chapters involving
3. 1 4 1 5 9 2 6 5 3 5 8 9
quantum mechanics."
7 9

Here is one which I have worked out for $\log_{10} \pi$:

"This Logarithm employs a zero character; mantissa follows in digits
0.4 9 7 1 4 9 8 7 2 6
precisely what I now give."
9 4 1 3 4

A REMARK ON QUADRANGULAR SETS

By ALBERT A. BENNETT, Brown University

It may be of interest to teachers of projective geometry to emphasize a few of the more fundamental relations of the subject by using to a greater extent than customary, a certain common concept here discusssed.

We shall consider a certain projective figure, $Q(ABC, DEF)$, composed of six elements, A, B, C, D, E, F, and subject to the following conditions:

I. The figure $Q(ABC, DEF)$ is completely determined if any five of its elements are given.

II.
$$\begin{aligned} &Q(ABC, DEF)=Q(BCA, EFD)=Q(CAB, FDE)=\\ &Q(CBA, FED)=Q(ACB, DFE)=Q(BAC, EDF)=\\ &Q(BDF, EAC)=Q(DFB, ACE)=Q(FBD, CEA)=\\ &Q(FDB, CAE)=Q(BFD, ECA)=Q(DBF, AEC). \end{aligned}$$

1. *The transversal postulate.* The transversal postulate and its immediate corollaries state that if A, B, C, are the vertices of a non-degenerate triangle, and if D is a third distinct point of the line BC and E a third distinct point of the line CA, then there exists a unique point F as the intersection of the lines AB and DE. The six points constitute a figure, $Q(ABC, DEF)$, satisfying the two conditions stated.

2. *Desargues' theorem on perspective triangles.* If A', B', C', and A'', B'', C'' are corresponding vertices of two triangles perspective from a line, the triangles are perspective from a point. Let $A'B'$ meet $A''B''$ in F, let $B'C'$ meet $B''C''$ in D, and let $C'A'$ meet $C''A''$ in E, where by hypothesis D, E, F, are collinear

on the "axis of perspectivity." Let the lines $A'A''$, $B'B''$, meet in a point, P. The theorem states that $C'C''$ also passes through P. Let $A'A''$ meet the axis of perspectivity in A, let $B'B''$ meet it in B, and let $C'C''$ meet it in C. The theorem of Desargues is essentially equivalent to the statement that the quadrangular set of collinear points, A, B, C, D, E, F, denoted by $Q(ABC, DEF)$ is completely determined by any five of its points.

3. *Pascal's hexagon.* Given five points, A, B, C, D, E, in the plane with no three collinear, the theory of the Pascal hexagon is made the basis for a method of constructing other points of the conic determined as containing the given five points. In the theory of geometrical constructions, the most convenient form of the proposition is one freed of actual constructional details and may be stated as follows. If A, B, C, D, E be five distinct points of a non-degenerate conic, and if AD meets BE in P, then the line CP meets the conic in a determinate point F that may be constructed by the use of ruler alone. In this form, the six points, A, B, C, D, E, F, constitute a figure $Q(ABC, DEF)$ of the type mentioned, which is a quadrangular set of points on the conic.

4. *The inversive quadrilateral.* The theory of the construction of the sixth point of a quadrangular set in the complex plane (Argand-Wessel plane) is also a theory of intersections of certain circles in the inversive plane. Let A, B, C, D, E, be any given five distinct points of the inversive plane, which in the "general" case are such that no four are concyclic. We shall not stop to consider the various special cases. Draw circles, BCD, and AEC, which meet in C and in a second point, ordinarily distinct, that may be called P. The circles PAB and PDE, meet in a second point, F. This is the sixth point of the quadrangular set, $Q(ABC, DEF)$. In particular, this gives a construction for the product of complex numbers. As an example of a quadrangular set, one has[1] for any two numbers x, y, the following $Q(0, x, 1; \infty, y, xy)$. Hence we have the following construction, whose agreement with more traditional constructions is readily verified by elementary geometry. Draw a circle through the points 0, 1, and y. Draw the straight line through 1 and x. This is the circle through 1, x, and the infinite point. Let the line meet the circle again in the point, P. The circle through 0, x, and P, meets the line through y and P in the desired point xy. The fact that in this case $Q(ABC, DEF) = Q(AEC, DBF)$, for example $xy = yx$, leads to an interesting theorem on the concurrence of the four circles circumscribing the triangles of a quadrilateral.

5. *Skew lines in space.* Given five mutually skew lines, A, B, C, D, E. construct the two reguli determined respectively as meeting the three lines B, C, D, and the three lines A, C, E. Through each point P of the line C, there is a unique line m, which meets the three skew lines, B, C, D, and also a unique line n, which meets the three skew lines, A, C, E. The lines m, n, when distinct determine a plane, meeting the five given lines in five points of a quadrilateral, whose sixth point, Q, is projective with P. As P traces the line, C, Q traces in general a space

[1] Veblen and Young, Projective Geometry, vol. 1, p. 145

cubic which in special cases reduces to a line F in space. If the five lines lie in the linear congruence determined by any four of them, the discussion may be carried on as follows, by paraphrasing case 4 mentioned above, since the lines of a linear congruence may be represented by points on a non-degenerate quadric surface in three space, the geometry of which is identical with the geometry of inversion save for certain important questions of reality. Take the reguli, BCD and AEC, which meet in the line C and in a second line, ordinarily distinct, that may be called P. The reguli PAB and PDE, meet in a second line, F, which is the desired sixth element of the set.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Integralgleichungen. By Gerhard Kowalewski. Berlin, De Gruyter, 1930. 304 pages. 15 marks.

Geometrische Transformationen. By Karl Doehlemann. Second Edition, edited by Wilhelm Olbrich. Berlin, De Gruyter, 1930. 256 pages. 13 marks.

Non-Interpolating Logarithms, Cologarithms, and Antilogarithms. By Frederick W. Johnson. San Francisco, The Simplified Series Publishing Company, 1930. Price $1.60, $2.25, $4.00, according to binding.

Topology. By Solomon Lefschetz. American Mathematical Society, Colloquium Publications, Volume XII. New York, 1930. x+410 pages.

Funktionentheorie. By Konrad Knopp. I. Grundlagen der allgemeine Theorie der analytischen Funktionen. Sammlung Göschen, 668. Berlin, De Gruyter, 1930. Fourth edition. 140 pages.

Geometry of Four Dimensions. By A. R. Forsyth. Cambridge, The University Press, 1930. Volume I, xxx+468 pages; Volume II, xii+520 pages. $25.00 the set.

Accounting, Principles and Procedure. By Walter J. Goggin and James V. Toner. New York, Houghton Mifflin Company, 1930. viii+476 pages. $3.50. Laboratory Manual, 272 pages. $1.80.

The Quantum Theory. By Fritz Reiche. Translated by H. S. Hatfield and H. L. Brose. New York, E. P. Dutton, 1930. viii+280 pages. $2.10.

Numerical Mathematical Analysis. By James B. Scarborough. Baltimore, The Johns Hopkins Press, 1930. xvi+416 pages.

Algebra for Junior and Senior High Schools. By J. W. Calhoun, E. V. White, and T. McN. Simpson. Richmond, Johnson Publishing Company, 1930.

REVIEWS

Contributions to the History of Determinants, 1900–1920. By Sir Thomas Muir, D.Sc., LL.D., F.R.S., C.M.G. Blackie and Son Ltd., London and Glasgow, 1930. pp. xxiii+408.

In this sequel to his classic four volume work on the history of determinants from 1693 to 1900, Sir Thomas Muir maintains the high level of thoroughness and interest which make the earlier volumes a delight. The plan of the present book is uniform with that of its predecessors, to which it is an indispensable adjunct. The publishers have spared no pains to make the easy, open pages attractive and intelligible at a glance, and they, with the author, are to be congratulated on a fine book.

Whether the reader is interested for its own sake in the theory of determinants, or whether he uses determinants as the merest incidents in his own work, he will find much in this volume to induce him at least to browse for a spell and enjoy the author's occasional pungent remarks on the miscellaneous fare offered. As Dr. Johnson observed of the incomparable haggis, "there is much fine, confused feeding" in a mathematical history, and in a historical account of determinants the tremendous feast resembles a Chinese banquet rather than a single, well cooked dish. To those who have acquired a more sophisticated palate, trained to the dry flavor of the greater modern algebraic theories, the theory of determinants as revealed by its history may seem rather lacking in coherence and more like an assortment of tidbits than a coordinated theory. The abstract part of it all, that which alone is entitled in a modern sense to the high designation "theory," can be stated in a couple of pages; the rest is mere repetition and variation, endlessly, on a theme which is neither intricate nor subtle.

In the twenty years covered in this volume the same duplication of results by one writer after another, as was noted frequently by the author in earlier volumes, is as conspicuous as ever. Seasoned mathematicians writing on determinants appear to be as negligent as the merest beginners in acquainting themselves with what their predecessors have done. There may be some merit in establishing known theorems all on one's own, but there is more in consigning the results to the wastebasket after discovering that they have already been printed three times or more. Some of the duplications reported in this volume seem almost incredible, and could hardly have been perpetrated honestly by anyone who had taken the trouble to consult Muir's earlier works. For this reason, if no other, the complete set of Muir's histories together with his eleven supplementary papers should be in every mathematical library. More important, anyone who itches to write anything on determinants in the classical notation might profitably spend half an hour running down the cause of his distress in Muir's histories; the odds are that he will be cured.

Another caution of a different kind is rubbed in repeatedly by a reading of Muir's historical studies: it is not the recurrent which is of greater interest, but the generating function or difference equation that gave rise to the recurrent. A determinant for the nth Bernoulli number, for example, may be, and probably

is, as futile as it is pretty. Only a second and more misguided Wronski imagines he has solved the riddle of the universe when he evolves an utterly unmanageable determinant of the nth order to express the supreme law he thinks he has discovered. It makes no difference whether the determinant be symbolized in Greek or Latin characters, or in the more mysterious looking Hebrew which some specialists on determinants favor; if the determinant is of the nth order and not almost pathologically degenerate, it is no advance over the humble generating identity of which it is a pretentious disguise.

The sequence of chapters (with eleven omissions noted and treated elsewhere by the author) follows the plan of the earlier volumes so closely that there is no need to retail it here. The extremely interesting interpolated Chapter I (α), devoted to Hadamard's approximation-theorem from 1900 to 1917, raises a point of particular historical interest, as Muir insists that Lord Kelvin's name has a claim to be attached to the theorem. Among other judicial remarks, intended no doubt to adjust priority, those introducing Chapter VI (alternants), comparing the determinantal aspects of East Prussia and the State of New York, cannot fail to amuse American readers. One would think that determinants should be the last thing on earth to inspire chauvinism or envy, hatred and malice, but apparently it is not so. Let us hope that the enthusiastic investigators mentioned, neither of whom is longer writing for publication through the usual media, are not cutting one another on the Elysian fields.

Muir's longer writings are always lightened by a human touch, which may take the form of anything from a dry joke to a warm appreciation. Nor is the reader left in any doubt as to what is or is not "important." This, to the reviewer, seems to be a strategic error in writing a scientific history. What the professional user of such a history wants is the facts, and nothing but the facts, presented to him in the most concise form consistent with clarity, and without the historian's personal opinions on those facts, even in footnotes; it does not greatly matter what the "general reader" or the dilettante would prefer. If there is any reason for supposing historians of scientific subjects to be less fallible than historians in any other field, it is not evident, and the documents in the case of the older-fashioned literary type of history seem to show that the personal opinions and interpretations of historians serve no useful purpose, except to reveal to curious psychologists the prejudices of the historians.

The entire history inevitably suggests comparison with the only other one of recent times devoted exclusively to a branch of pure mathematics. This comparison may be irrelevant, but as another reviewer ("A. C. A." in *Nature*, November 29, 1930) invites it in the words "if there exists anywhere a more detailed and comprehensive history of any branch of theoretical knowledge, one would be interested to hear of it," the material for a comparison may be briefly indicated. In round numbers, Dickson takes 1600 pages to present, in minute detail, the history of the theory of numbers from Pythagoras (or before), 500 B.C., to 1923; Muir takes 2400 to tell less minutely the incomparably simpler and shorter story of determinants from Leibnitz, 1693 to 1920, or, allowing for

the difference in type, about the same space. In range and in detail there is no common measure for the two histories. Nor is there in style; Dickson's, for a serious student, has the complete efficiency of a battery of machine guns; Muir's is like the leisurely sprinkling of a shady lawn with a garden hose. Those who object to having a hail of facts shot at them without mercy may consider how far they would be likely to penetrate hostile territory, bristling with difficulties, if armed only with a garden hose or a watering pot.

As the author frequently points out, there has been but little diminution, if any, in the rate of production of writings on determinants in the twenty years covered by this volume. It may be suggested, however, that the counting of titles is not a reliable index when, as in this history, any incidental use of determinants in a paper whose main object often is far other than a contribution to determinants, is emphasized by isolation as the one fact of apparent interest. And, in passing, it appears to the reviewer that such a dislocation of secondary matter frequently gives an erroneous impression of a writer's motives in attempting to contribute to mathematical literature at all. In some instances it seems that no more relevant estimate of a paper by picking out the determinants it contains is possible than if the compiler were to emphasize, let us say, the incidential use of linear equations or algebraic division. Nor is the matter greatly helped when the historian supplements the record of secondary facts with his personal opinion on their importance or the lack of it.

One gets the impression on reading this book that the author is somewhat chagrined by the dearth of striking contributions to determinants in recent years. A possible explanation for this lack may be the simple fact—if it is one—that the theory of determinants, as a theory, has petered out. One most striking contribution to the theory does not seem to be mentioned by the author. The name of Ricci does not occur in the list of authors reported in this book or in the preceding volumes of the history. Possibly a perusal of Ricci's paper of 1899, in which the "Systems E" appear, seemingly for the first time, may suggest to future historians the year in which the *theory* of determinants expired and was buried under the rich loam of an infinitely wider and more fertile field. Those who claim that determinants are a comparatively trivial incident in the vaster and simpler theory of tensor algebra seem to have the right on their side.

A modern reading of one paper abstracted in the present volume might have suggested to the historian that the dearth of fundamental advances he deplores is due to the nature of the subject rather than to a lack of imagination on the part of the investigators. For, it seems to the reviewer, the entire significance of E. H. Moore's "Fundamental Remark Concerning Determinantal Notation, etc.," of 1900, has been completely missed by the historian. Between them, Ricci and Moore buried the theory of determinants in 1899–1900. Subsequent progress seems to show that this funeral is the fundamental development which the historian misses in his researches.

It may be too optimistic to hope that determinants will fade out of the mathematical picture in a generation; their notation alone is a thing of beauty

to those who can appreciate that sort of beauty. But it would seem to be well worth someone's trouble to write a short, accessible tract on determinants from a modern point of view, revealing their true simplicity and their strictly incidental character, as a historical pendent to Spottiswoode's paper of 1851 in which the now apparently obsolete theory was first didactically presented. The proof of the multiplication theorem by tensor algebra, for example, is a matter of two short lines, both obvious.

Whatever the future of determinants is to be, there can be no doubt that Sir Thomas Muir's history will have no serious rival in its own field. In bringing to a close this fascinating story of a subject which claimed a considerable share of the attention of the great algebraists of the nineteenth century, Sir Thomas Muir has made all who are interested in the scientific history of science his permanent debtors.

E. T. BELL

Number. By Tobias Dantzig. New York, The Macmillan Company, 1930. viii +260 pages. $3.50.

The superb Gothic towers of Quimper Cathedral are slightly marred by many stone curlicues extruding from the sides of the fine spires. While gazing with thorough enjoyment at the Cathedral, my first remark was the rather silly and unappreciative one, "I wish those spires had a shave." In undertaking this review, I rather fear that I may appear in a similar manner to emphasize slight blemishes at the expense of expressing the real appreciation that the structural soundness and vital interest of the book deserve.

There is no more comforting thought than that the human mind exploited as it often is by activities on the level of jazz and garrulous bridge can rise to the appreciation of mathematics and the other fine arts; and there is no more amazing story than that of the development, accurate analysis and successive generalizations of the concept of number. Professor Dantzig has undertaken the almost impossible task of telling this story to the mathematically untrained but highly intelligent layman, and has performed it remarkably well. To say that the trained mathematician will be even more interested in it than will the layman is not to disparage the readability of the work, but to praise its fundamental accuracy.

Chapter I, II and V, on *Fingerprints*, *The Empty Column*, and *Symbols*, deal largely with the conjectures concerning the early, and with knowledge of the later, history of the development of the number system as we know it when we leave high school. The third chapter on *Number Lore*, though occupying a somewhat isolated position in the book, gives insight into the origins of number theory, the rest of the book being motivated by the needs of analysis. The main body of the work is a description of the continued extension of the number system that has been necessitated by the enlarging demands of analysis. Even this description gives the impression of a somewhat artificial division that does

not exist in the book since much of the early chapters is devoted to the same theme as the later.

Mathematicians have successively demanded "closure," under the processes of counting, subtracting, dividing, successive approximation, and the solution of algebraic equations. Repeatedly with irrational, but not blind, faith they have played with the symbols for an enlarged but uncreated number system. Repeatedly, under the goad of conscience, they have constructed a system to go with these symbols. The process has been like the growth of a city. First the city becomes crowded, then the optimistic real estate man gives such names as "Fairview Boulevard" or "Sweet Briar Road" to some can-infested suburban wood lot, but in time the dwellers do come, the roads are created to take the names previously given, the city again becomes overcrowded and the process restarts. But this is not the whole story of the city; there must be some interested in improving the older portions. Nor is it possible, in mathematics, to anticipate all future criticisms; an ever present need compels the redefining of old concepts which, in light of new logical analysis, have become unsatisfactory, and thus we have the work of Cantor and Dedekind in the last century. This picture of enlargement of scope and intensifying of rigor Professor Dantzig has given admirably.

There are one or two places, however, in which Professor Dantzig's bias is enough different from the reviewer's to elicit a remonstrance, and there are a few places, not in the inner structure, but on the surface where I still feel that the book "needs a shave."

I can not feel that *Number* does justice to the point of view of Kronecker and Brouwer. I doubt if it is in the nature of things that mathematics will be as limited in its processes as its most rigorous critics may believe, but I likewise feel that their criticism indicates the direction in which investigation must be carried on if we are not to leave in the body mathematical many inconsistencies. The works of Peano, Frege, and Russell and the remarkable recent work of Hilbert have done perhaps as much as have the intuitionists to show that mathematics is faced with a vital problem. As a history of the progress of the mathematical frontier, *Number* shows real insight into the past difficulties and the magnitude of the past accomplishments. This is the main task of the work. In that small portion of the book which deals with the present frontier, there is something of a land agent attitude in belittling the seriousness of the task ahead in as far as that task is in the direction of increasing logical rigor.

Just an illustrative remark or two as to the desired shave! In light of the outstanding recent work in number theory by Professor Dickson, the remark "The triangular and more generally polygonal numbers are of no scientific interest" needs the razor. On page 165, the author gives a completely erroneous notion of what the phrase "well ordered" means, defining it as what is usually called "linearly ordered" plus the notion of the order being well defined. This also is a little furry. Again on pages 129 and 130, we find such remarks as "When after a thousand-year stupor, European thought shook off the effect of the sleep-

ing powders so skillfully administered by the Christian fathers, the problem of infinity was one of the first to be revived" and in speaking of the loose way in which the mathematicians dealt with infinite processes we have "yet another cause may be discerned. It should be remembered that the brilliant minds of that period were all raised on scholastic doctrine. To minds whose logic was fed on such speculations as sacrament and atonement, trinity and trans-substantiations, the validity of infinite processes was a matter of small concern." In comparison, one should read the first chapter of Whitehead's *Science and the Modern World*, for instance, "It needs but a sentence to point out how the habit of definite exact thought was implanted in the European mind by the long dominance of the scholastic logic and scholastic divinity. The habit remained after the philosophy had been repudiated, the priceless habit of looking for an exact point and of sticking to it when found."[1] In light of this far more just appraisal of that remarkable period, the Middle Ages, this crop of whiskers seems almost Bolshevistic.

I hope these minor criticisms do not too greatly detract from sensing the real appreciation the reviewer wishes to express. By all means get the book and read it. You will enjoy it. You will feel that a wonderful tale has been well told, and if once in a while you too turn barber—well, you will enjoy that also.

M. H. INGRAHAM

Gewöhnliche Differentialgleichungen. By G. Hoheisel. Sammlung Göschen, No. 920. 2nd edition, Berlin, de Gruyter & Co., 1930. 159 pp.

This small, ridiculously inexpensive book contains a surprising amount of material, excellently presented.

The first chapter, which deals with equations of the first order, explains the usual formal methods of integration, proves the existence theorem by the Picard method and discusses singular solutions and singular points. Chapter II deals with equations of higher order, particularly with linear equations. Chapter III considers boundary value problems.

J. F. RITT

A Course of Analysis, by E. G. Phillips. Cambridge University Press, 1930. viii+361 pages.

This text, written by a member of the faculty of the University College of North Wales, is based upon a course of lectures which he has given there for several years to the mathematical honours students. While all the topics treated in this book are to be found elsewhere, there appears to be no other textbook in English covering exactly the same ground. The main purpose of the book, as stated in the preface, is to give a logical connected account of the subject, starting with the definition of *number* and proceeding from that in a natural sequence of steps. The author keeps in mind throughout that the course is for

[1] A. N. Whitehead: *Science and the Modern World*, The Macmillan Co., New York, 1926, pp. 17–18.

many students preparatory to courses in applied mathematics and physics. The examples (many of them original) have been chosen mainly to illustrate the fundamental concepts, and the type of example whose solution depends on manipulative dexterity alone has been deliberately excluded by the author.

The scope of the book may best be estimated from the list of chapter titles: (1) Number; (2) Bounds and limits of sequences; (3) Limits and continuity; (4) Differential calculus; (5) Infinite series; (6) Inequalities; (7) Integral calculus; (8) Extensions and applications of the integral calculus; (9) Functions of more than one variable; (10) Implicit functions; (11) Double integrals; (12) Triple and surface integrals; (13) Power series. Comments on a few of these chapters may be appropriate.

In chapter 1 is given a brief account of the concept of *number*. The author feels that the theory of real numbers is the foundation upon which rests the whole structure of the subject of mathematical analysis. While avoiding the details of the philosophical questions involved, the author shows that the foundations of the subject of *number* lie further back than the mere assumption that the natural numbers are "known." He points out the distinction between the concept of cardinal number and that of ordinal number. In the treatment of irrational numbers, Russell's modification of Dedekind's method is used as the definition.

Throughout the book prominence is given to the concept of the differential of a function, a concept which is especially valuable in the case of functions of more than one variable. Hence, in chapter 4, the author makes a distinction between functions which are derivable (possess derivatives) and those which are differentiable (possess differentials), and in chapter 9 he makes a similar distinction between partial derivatives and differential coefficients. Likewise in chapter 7, the author points out the distinction between definite integration, which may be thought of as an analytic substitute for an area, and indefinite integration (for which he proposes the name "calculus of primitives"), which is the inverse of integration.

Topics whose treatment makes them worthy of especial mention are: the inequalities connected with the names of Hölder, Minkowski, and Jensen (chapter 6); Hermite's method of integrating rational functions (chapter 7); and area of a curved surface (chapter 12).

As the author says in the preface, "Modern analysis requires great precision of statement." While for the most part the author lives up to this ideal, the reviewer regrets that he must call attention to the following errors and defects in this book.

First there are numerous errors that indicate lack of careful editing. It is obvious that the quantity $\omega = |x-a| + |y-b| + |z-c|$, mentioned near the bottom of page 76 is *not* the length of the diagonal of the rectangular parallelopiped which has (a, b, c) and (x, y, z) for opposite vertices. In the last footnote on page 120, there is mentioned a possibility which can never occur so long as only series of positive terms are considered. In lines 17–20 of page 123, U_n means the

sum of the first n terms of a certain infinite series, but U_{n-v} does not mean the sum of the first $n-v$ terms of that series. Finally in the line following (2) on page 325, n is used in two different senses in the same equation.

A more serious error occurs in Weierstrass's theorem on page 32. The proof as given there is entirely fallacious, as may be seen by taking any sequence of points decreasing toward a sequential limiting point.

As has been stated above, the author in general distinguishes between concepts which are usually treated as identical but which are in reality different. It is unfortunate therefore that he does not emphasize the difference between the concept of the differential of the *dependent* variable and the concept of the differential of the *independent* variable. He perpetrates on page 87 the so-called "proof" that $dx=\Delta x$. All that this proof shows is that if $y=x$, then the differential of the dependent variable y is equal to Δx. Or what is the same thing, it shows that the differential of x when considered as a function of the independent variable x, is Δx. But it tells us nothing about the differential of the independent variable itself. That is an entirely different concept and requires a new definition.

Finally the author states so many theorems in a form which is false at worst and ambiguous at best, that the author seems at times to be aiming for "great carelessness of statement" rather than for "great precision of statement." An example of this carelessness is found at the bottom of page 188, where the author states: "If $f(x)$ be a continuous function and $a\leqq\xi\leqq b$, then

$$\int_a^b f(x)dx = (b - a)f(\xi)."$$

If we interpret this as most readers would, it means that the conclusion is true for *any* number ξ of the interval (a, b). But in that form the statement is false. What the author means is that "if $f(x)$ be a continuous function and (a, b) a given interval, then there exists a number ξ such that $a\leqq\xi\leqq b$ and

$$\int_a^b f(x)dx = (b - a)f(\xi)."$$

False statements similar to this one may be found also on page 103, lines 2–5; on page 104, lines 12–18; and on page 189, lines 9–10 and 14–16. A statement which might be called ambiguous occurs on page 179, in the last three lines, where the author gives another approximate form of the theorem stated above.

HARRY MERRILL GEHMAN

An Introduction to Mathematics. By Isaiah Miller. New York, F. S. Crofts and Co., 1930. xiii+297 pages.

In the preface the author states that in preparing this book he has kept in mind two types of students: first, those who will never take additional work in mathematics; and second, those who will continue the work in science or agri-

culture for advanced degrees and will doubtless desire to pursue additional courses in mathematics. He has therefore attempted to write a book basic in the fundamental principles of mathematics and at the same time has made practical applications wherever possible. Considerable attention is paid to fundamental principles and the applications are good, especially those relating to agriculture. The reviewer wishes to commend this feature of the text.

It is a book of 209 pages, not including the tables, with about the same emphasis upon algebra as is found in other books in the field of freshmen mathematics, a somewhat smaller amount of trigonometry, a chapter on statistics, one on annuities and insurance, and an introduction to some of the elementary notions of analytic geometry and calculus.

The author states that college freshmen need considerable drill on the fundamental processes of algebra before attempting a very extensive study of mathematics. This idea has been carried out by the author and hence we find the book to be elementary in character. It should be a satisfactory text to use, especially where students come to college with only one year of algebra and one year of geometry. The reviewer recommends it for students with this sort of preparation who desire to take a course in freshman mathematics.

In a book of this size, with a considerable review of high school algebra, quite a little space given to applications, and a treatment of the topics listed above, an author must of necessity omit certain phases of college algebra and trigonometry. The reviewer believes that the author is justified in his choice of subject matter according to his first purpose; namely, that of writing a book for those who will never take additional work in mathematics. It is not quite clear just how the second purpose is to be carried out, that of preparing students for analytic geometry and calculus. While there is no uniform agreement upon what topics should be stressed, it seems that a student would be considerably handicapped if he entered a course in analytic geometry, to be followed by calculus, with no previous study of such topics in algebra as equations in the quadratic form, simultaneous quadratic equations, infinite geometrical progressions, harmonic progressions, the binomial theorem for negative and fractional exponents, partial fractions, series, variation, indeterminate forms, theory of limits, determinants, and some of the elements of the theory of equations. Since applications are stressed, other topics that might be considered are the use of the base e, the slide rule, and logarithmic paper. Similar remarks are true of the sections involving trigonometry. The student in later courses in mathematics might feel the need for a knowledge of such topics as the solution of trigonometric equations, radian measure, the polar form of a complex number, DeMoivre's theorem, periodicity, and inverse trigonometric functions. From the point of view of the applications the graphs of the trigonometric functions might well be a part of the course. There will be instructors who feel that more problems in trigonometry might have been included. However, this is not a serious objection as supplementary problems can always be provided by the instructor when the need arises.

The book is printed on good paper, the type used is good, and the pages are very pleasing to the eye. The figures are well drawn. However, in the figure on page 55, the lines might better not be at an angle of 45° with the x-axis. No typographical errors of importance were noted. A good set of tables is included, occupying about 80 pages, which will be sufficient for the needs of the students.

FREDRICK WOOD

PROBLEMS AND SOLUTIONS

EDITED by B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3479. *Proposed by C. A. Rasmussen, University of Alabama.*

If B and C be two fixed points and A a variable point at which BC subtends a given angle and if E and F be the feet of the perpendiculars from B and C, respectively, to the opposite sides of the triangle ABC, determine the path of a point P, the intersection of FP with EP, if $\angle BEP$ and $\angle PEC$, respectively, remain equal to $\angle ECB$ and $\angle FBC$.

3480. *Proposed by G. W. Wishard, Norwood, Ohio.*

Prove the following theorem: *If 1 be annexed to any triangular number in the nonary scale of notation, the result will be another triangular number.* Thus, 1, 11 111, 1111, etc. *ad infinitum* are triangular numbers in the nonary system.

3481. *Proposed by J. A. Calderhead, Boston, Mass.*

Calculate the critical speed in a high speed shaft, and show that $l=6N^{-1}(3EIw^{-1})^{1/2}$ gives the safe length when N = critical speed in revolutions per minute, w = weight per unit length, E = modulus of elasticity, and I = moment of inertia.

3482. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If the circumscribed and the inscribed spheres of a tetrahedron are concentric, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

3483. *Proposed by Mabel M. Young, Wellesley College.*

A is a fixed point, B a variable point on a circle with center at O. Show that the locus of the orthocenter of triangle AOB is a strophoid.

3484 *Proposed by S. A. Corey, Des Moines, Iowa.*

Obtain a two-parameter solution of the Diophantine equations,

$$r^2 + s^2 + t^2 = 2(v^2 + w^2) = x^2 + y^2 + z^2.$$

3485. *Proposed by J. M. Feld, New York, N. Y.*

Let the sides of triangle $A_1A_2A_3$ be trisected. A_{ij} is the trisection point on a_j nearer A_i. Let P_i be the intersection of $A_{ij}A_j$ and $A_{ik}A_k$ and let Q_i be the intersection $A_{kj}A_j$ and $A_{jk}A_k$. Prove:

(1) The triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homothetic to $A_1A_2A_3$ and the common homothetic center is the centroid of $A_1A_2A_3$.

(2) $$P_1P_2/A_1A_2 = \tfrac{1}{4},\ Q_1Q_2/A_1A_2 = \tfrac{1}{5}.$$

(3) If C is the intersection of $A_{23}A_3$ and $A_{12}A_2$, and D is the intersection of $A_{13}A_3$ and $A_{32}A_2$, CD is parallel to A_2A_3.

3486. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Find a function of x such that the integral of that function between the limits 0 and x is equal to the reciprocal of the original function.

UNSOLVED PROBLEMS

Solutions are requested for the remaining five unsolved problems proposed in 1914 which are reprinted below.

205 [1914, 55]. *Proposed by E. T. Bell.*

Show that in the usual arithmetical sense the form that follows admits of composition; give the requisite transformations, and indicate how several (if not all) solutions may be found. The variables are the x_i:

$$x_0^2 + nrx_1^2 + mrx_2^2 + mnx_3^2 + mnrx_4^2 + mn^2r^2x_5^2 + nr^2m^2x_6^2 + rm^2n^2x_7^2.$$

287 [1914, 55; 1919, 312]. *Proposed by Walter H. Drane.*

While sitting in an empaled enclosure, I noticed that the spokes of the wheels of passing automobiles, when viewed through the pickets of the fence, appeared to revolve more slowly than they really did, and in some instances even appeared to be revolving in a direction opposite to that in which they were really turning. Explain this optical illusion.

353 [1914, 55; 1919, 312]. *Proposed by Richard P. Lochner.*

The center of a sphere, radius $R=5$ inches, is $a=10$ inches above the surface of a sphere, radius $r=12\frac{1}{2}$ inches. There is a point of light at $b=1$ inch horizontally from a point $c=10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

291 [1914, 122]. *Proposed by Emma M. Gibson, Springfield, Missouri.*

The time of descent, down a rough inclined plane, of a spherical shell which contains a smooth solid sphere of the same material as itself is t_1. The time of

descent, down the same plane, of a solid sphere of the same material and radius as the shell is t_2. Determine the thickness of the shell.

From Loudon's *Elementary Theory of Rigid Dynamics*, p. 188.

299 [1914, 267]. *Proposed by B. F. Finkel, Drury College.*

A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased, that it may rest with its base in the common surface of the fluids.

From Walton's *Hydrostatical Problems.*

SOLUTIONS

3430 [1930, 261] *Proposed by Richard M. Sutton, California Institute of Technology.*

It is physically possible, given a large number of unit resistances, to make any resistance p/q between two points A and B, where p and q are integers. The result may be accomplished by connecting in parallel q groups of p resisances each, requiring (pq) resistances. However, it is usually possible to accomplish the same result by a fewer number of unit resistances. The problem is: "Find the minimum number of unit resistances necessary to make a resistance p/q between two points A and B in an electric circuit, p and q being both integers."

Solution by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.

A neat problem! Its solution depends upon the self-evident facts that if x and y are integers (i) the minimum number of unit resistances required to assemble a *resistance* of x units is x *in series;* (ii) the minimum number of unit resistances required to assemble a conductance of y units, i.e., a resistance of $1/y$ units, is y *in parallel.* Consider now a resistance of p/q, where p and q are integers, prime to each other. Express p/q as a terminating continued fraction

$$\frac{p}{q} = r_1 + \frac{1}{c_1 +}\,\frac{1}{r_2 +}\,\frac{1}{c_2 +}\cdots .$$

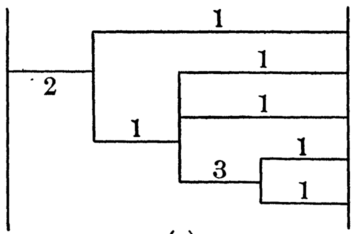
The partial quotients are resistances and conductances alternately, (hence the notation), and they determine the minimum network. The solution may be read thus:—A resistance r_1 in series with [a conductance c_1 in parallel with {a resistance r_2 in series with (a conductance c_2 in parallel with $\cdots$)}], bearing in mind that r_i signifies r_i unit resistances *in series*, while c_i signifies c_i unit resistances *in parallel.* The minimum number of unit resistances required is

$$\sum r_i + \sum c_i.$$

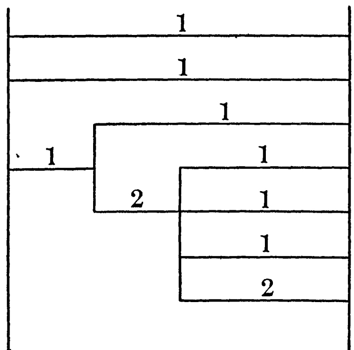
As a numerical example, resistances of 101/39 and 39/101 may be built from eleven units. For

$$(a)\ \frac{101}{39} = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2};\ \text{and}\ (b)\ \frac{39}{101} = 0 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2}.$$

Hence the following networks:



(*a*)



(*b*)

A note by the Editors. If the denominator of the fraction p/q is a composite integer, where p and q have no common factor, then the partial quotients of the single continued fraction into which p/q is developed do not always give a minimum. This may be shown by examples. Thus

$$\frac{34}{15} = 2 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3}, \qquad \text{9 units,}$$

whereas

$$\frac{34}{15} = 1 + \frac{1}{1+}\,\frac{1}{2} + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}, \qquad \text{8 units.}$$

Another example for which there are more than two decompositions is as follows:

$$\frac{31}{30} = 1 + \frac{1}{30} = \frac{1}{5} + \frac{1}{1+}\,\frac{1}{5} = \frac{1}{2} + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{7}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{1}{3} + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}\,.$$

These decompositions require 31, 11, 11, 10, 9 units, respectively.

3436 [1930, 315]. *Proposed by Eugene M. Berry, Lynchburg College.*

Let p, q, and r be concurrent lines. Through any point on p draw lines

p', a, b, c, d, such that p' is perpendicular to p and that angles ab and cd are bisected by p. Show that the line aq-dr meets the line cr-bq on p' and meets the line cq-br on p.

I. *Solution by W. Randolph Church, Hightstown, New Jersey.*

Let the elements p, q, r, and p', a, b, c, d, be given so as to satisfy the conditions as stated in the problem. With the point pp' as center, draw a convenient circle. With respect to this circle, construct the poles of all the given lines. Let them be called by the corresponding capital letters. A, B, C, D, P, and P' will be at infinity, but their directions will be definitely known. Join the poles just constructed so as to form the polars of the points of intersection of the given lines. Draw also aq-dr, cq-br, cr-bq and locate the corresponding points AQ-DR, CQ-BR, CR-BQ. From the fact that the angle between two lines is equal to the angle which the line joining their poles subtends at the center of the circle, it is immediately apparent that angle PQ-QC equals angle PR-RD and angle PQ-QA equals angle PR-RB, for it is given that angle p-c equals angle p-d and angle p-a equals angle p-b. From elementary theorems concerning congruence and parallels we can conclude that the line joining CQ-BR and AQ-DR is parallel to RQ. Now P, Q, R are collinear. That is, CQ-BR, AQ-DR and P are collinear. Consequently their polars are concurrent; that is, cq-br, aq-dr, and p are concurrent. Similarly, the line joining AQ-DR and CR-BQ is perpendicular to RQ; that is, AQ-DR, CR-BQ and P' are collinear; or finally, aq-dr, cr-bq and p' are concurrent.

Remark: The same method enables us, if desired, to show that the remaining trios of the following set are concurrent:

$$
\begin{array}{lll}
aq - br,\ cq - dr,\ p'; & aq - br,\ dq - cr,\ p'; & aq - cr,\ bq - dr,\ p'; \\
aq - cr,\ dq - br,\ p; & aq - dr,\ bq - cr,\ p'; & aq - dr,\ cq - br,\ p; \\
ar - bq,\ cr - dq,\ p'; & ar - bq,\ dr - cq,\ p'; & ar - cq,\ br - dq,\ p'; \\
ar - cq,\ dr - bq,\ p; & ar - dq,\ br - cq,\ p'; & ar - dq,\ cr - bq,\ p.
\end{array}
$$

It would be shown at the same time that the following five lines are concurrent:

$$aq - br,\ cq - dr,\ ar - bq,\ cr - dq,\ p'.$$

II. *Solution by Otto Dunkel, Washington University.*

The pencil of lines a, b, c, d, p, p', with the vertex V, is projective with the pencil b, a, d, c, p, p'. Hence the following two ranges of points are projective.

$$
\begin{array}{c}
ra,\ rb,\ rc,\ rd,\ rp,\ rp', \\
qb,\ qa,\ qd,\ qc,\ qp,\ qp';
\end{array}
\tag{1}
$$

and, since $rp \equiv qp \equiv W$, the lines joining the two points vertically above one another in (1) meet in a point U. The last pair of points shows that U lies on p'.

Also, if we select any square matrix from (1), the lines joining the diagonal points meet on a straight line. Since *ra-qa*, *rb-qb* meet in V, and *ra-qp*, *rp-qb*, in W, this line is p.

The following two ranges of points on r are projective,

$$\begin{array}{c} ra,\ rb,\ rc,\ rd,\ rp,\ rp', \\ rb,\ ra,\ rd,\ rc,\ rp,\ rp'; \end{array} \tag{2}$$

and we have two projective pencils by joining the points of the first row of (2) with qa, and those of the second with qb. The rays *qa-rp*, *qb-rp* coincide in q, and hence corresponding rays meet on a straight line. Since *qa-ra*, *qb-rb* meet in V, and *qa-rp'*, *qb-rp'* meet on p', this line is p'. Hence the pairs of lines *qa-rb*, *qb-ra*; *qa-rc*, *qb-rd*; *qa-rd*, *qb-rc* meet on p'. In a similar manner using the vertices qc and qd we find that the pairs of lines

$$qc - ra,\ qd - rb;\quad qc - rb,\ qd - ra;\quad qc - rd,\ qd - rc$$

meet also on p'.

Also solved by D. C. Duncan, Paul Wernicke, and G. A. Yanosik.

3437 [1930, 315]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

A variable circle passes through a fixed point and is tangent to a fixed circle. Prove that the diametric opposite of the fixed point on the variable circle describes a central conic of which the fixed point is a focus and the fixed circle is the auxiliary circle.

State and prove the converse of this proposition.

State the corresponding proposition for the parabola.

Solution by H. T. R. Aude, Colgate University.

Using rectangular coordinates take the center of the given circle at the origin and let a be its radius. Let A be the given point and place it at $(c, 0)$. On the variable circle of radius r which passes through A and is tangent to the given circle let $P(x, y)$ represent the diametric opposite to A. The center D of the variable circle has then the coordinates $\frac{1}{2}(x+c)$, $\frac{1}{2}y$ and with $AP=2r$ it follows that, for $c>a$, $OD=r\pm a$ according as the variable circle has external or internal contact with the given circle; if $c<a$ the variable circle is only tangent internally to the given circle and $OD=a-r$. The following two equations express the above statements for AP and OD in all cases (also for $c=a$).

$$4r^2 = (x - c)^2 + y^2, \tag{1}$$

$$4(r \pm a)^2 = (x + c)^2 + y^2. \tag{2}$$

It follows by subtraction that $2r = \pm(a^{-1}cx - a)$, and eliminating r^2 we obtain the equation,

$$a^2y^2 + (a^2 - c^2)x^2 = a^2(a^2 - c^2), \tag{3}$$

which is recognized as the equation of hyperbola or ellipse with the usual meaning for c and a, with the given point as a focus and the given circle for auxiliary circle. In case $c=a$, equation (3) becomes $y^2=0$, a limiting case of both the ellipse and the hyperbola.

The converse proposition, that all circles constructed on the focal radii of a central conic as diameter are tangent to the auxiliary circle, follows at once from the above when it is recognized that the expressions for $2r$, namely $a-a^{-1}cx$ and $a^{-1}cx-a$, respectively, are the focal radii of ellipse and hyperbola from the focus (c, 0).

For the parabola, the corresponding propositions are: If a variable circle is drawn so that it passes through a fixed point and is tangent to a fixed line, then the diametric opposite of the fixed point on the variable circle describes a parabola which has the fixed point for focus and the fixed line as tangent at its vertex. And the converse, that all circles, upon the focal radii of a parabola as diameter have for envelope the line which is tangent to the parabola at its vertex.

Also solved by Rufus Crane, William Hoover, and G. A. Yanosik.

3438 [1930, 315]. *Proposed by the late F. P. Matz.*

Solve

$$\int_0^{dy/dx} \frac{\cos w dw}{16+9\sin^2 w} = \frac{1}{12}\tan^{-1}(x).$$

Solution by Ralph P. Agnew, Cornell University.

The integrand is periodic and gives the value 0 when integrated over any interval of length 2π. Hence dy/dx may be expected to be a real multiple-valued function of x for certain values of x. On carrying out the indicated integration, we find that the given equation is equivalent to

$$(1) \qquad \tan^{-1}\left(\frac{3}{4}\sin\frac{dy}{dx}\right) = \tan^{-1} x,$$

where the left member lies between $-\pi/2$ and $\pi/2$, its algebraic sign agreeing with that of $\sin(dy/dx)$. Thus we see that if the given equation is to have a solution for dy/dx, its right member must represent the principal value of $\tan^{-1}x$. With this agreement, we find from (1) that $\sin(dy/dx)=4x/3$ and hence that

$$(2) \qquad \frac{dy}{dx} = n\pi + (-1)^n \sin^{-1}\frac{4x}{3}, \quad |x| \leqq \frac{3}{4}$$

where n may be any integer, and the second term of the right member is $\geqq -\pi/2$ and $\leqq \pi/2$. From (2) we find

$$y = n\pi x + (-1)^n\left[x\sin^{-1}\frac{4x}{3} + \frac{1}{4}(9-16x^2)^{1/2}\right] + C, \quad |x| \leqq \frac{3}{4},$$

a result which holds when n is an integer, C is a constant, and the right member of the given equation is the principal value of the arctangent.

Also solved by S. F. Bibb, D. C. Duncan, E. C. Kennedy, and H. A. Meyer.

3440 [1930, 316]. *Proposed by A. Pelletier, Montreal, Canada.*

A triangle is circumscribed about a circle. Prove that the three following lines are concurrent: (1) the line joining the points of contact of any two sides; (2) the line joining the points of intersection of these sides with the bisectors of the opposite angles; (3) the line joining the feet of the altitudes on these sides.

I. *Solution by Mabel M. Young, Wellesley College.*

Let $A_1A_2A_3$ be the given triangle and I the centre of the inscribed circle. On any side, a_k, let G_k be the point of contact with the inscribed circle, H_k the foot of the altitude from A_k and I_k the point of intersection with the internal bisector of the opposite angle. We wish to prove that the lines H_kH_j, I_kI_j, G_kG_j are concurrent.

Let the escribed circle with centre, J_k, on the internal bisector of angle $A_jA_kA_i$ touch a_k in N_k. Through the points H_k, I_k, G_k, N_k draw lines perpendicular to a_k. These cut the internal bisector through A_k in A_k, I_k, I, J_k. Since these points form a harmonic set, the range $H_kI_kG_kN_k$ is harmonic and the rays joining these points to A_k form a harmonic pencil. But in any triangle, the altitudes meet in the orthocentre; the internal bisectors meet in the incentre; the lines from the vertices to the points of contact of the inscribed circle meet in the Gergonne point; and the lines from the vertices to the internal points of contact of the escribed circles meet in the Nagel point. Let these points be denoted, respectively, by H, I, G, N. Then H, I, G, N are projected from each vertex of the triangle by four harmonic rays and accordingly lie on a conic passing through A_1, A_2, A_3. Project A_i, H, I, G from A_k and A_j. Since the six points lie on a conic the two pencils are projective and the sections by a_k and a_j are perspective. Thus $A_iH_kI_kG_k \overline{\wedge} A_iH_jI_jG_j$ and lines joining corresponding points of these ranges are concurrent in a point P_{kj}. Since N_k and N_j are corresponding points on the perspective ranges, N_kN_j also passes through P_{kj}. In any triangle there are three such points of concurrence.

II. *Solution by George A. Yanosik, New York University.*

Let the given triangle ABC be the triangle of reference for the coordinates α, β, γ, which are so chosen that the center of the inscribed circle has the coordinates $(1, 1, 1)$. Then the equation of this circle is

$$\alpha^{1/2} \cos (A/2) + \beta^{1/2} \cos (B/2) + \gamma^{1/2} \cos (C/2) = 0.$$

The equation of a line (1) of the problem is that of the polar of C with respect to this circle, or

$$\mu \equiv \alpha \cos^2 (A/2) + \beta \cos^2 (B/2) - \gamma \cos^2 (C/2) = 0. \tag{1}$$

The internal bisector of angle A has the equation $\beta-\gamma=0$; the internal bisector of B has the equation $\alpha-\gamma=0$. Hence the equation of the corresponding line (2) is

$$\nu \equiv \alpha + \beta - \gamma = 0. \tag{2}$$

The perpendicular from A to BC has the equation $\beta \cos B-\gamma \cos C=0$; the perpendicular from B to AC, the equation $\alpha \cos A-\gamma \cos C=0$. Hence the equation of the corresponding line (3) is

$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0 \text{ or } 2\mu - \nu = 0. \tag{3}$$

Hence the three lines (1), (2), (3) meet in a point.

A note by the Editors. It is not necessary to use the equation of the inscribed circle, for the equation (1) may be obtained by a simple computation from the figure. If the equation of the inscribed circle is desired, it may be easily obtained from (1), since that line is the chord of contact of the two sides $\alpha=0$, $\beta=0$. The equation of the circle must then be of the form $\alpha\beta+k\mu^2=0$. Using the two sides $\beta=0$, $\gamma=0$ in the same manner, and then equating coefficients, the equation of the circle will be obtained. This appears to be a simpler method of derivation than that given in Salmon's *Conic Sections.*

Also solved by D. C. Duncan and William Hoover.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Beginning with our January, 1931 issue the American Mathematical Monthly is to be included in the regular list of periodicals indexed in the "International Index to Periodicals" published by the H. W. Wilson Co.

Dr. Ganesh Prasad, Hardinge Professor of Higher Mathematics in Calcutta University and President of the Calcutta Mathematical Society, has given to that society an endowment for the purpose of awarding a prize and medal which is to be known as the Krishnakumari Ganesh Prasad Prize and Medal, in memory of his daughter. The award will be made every fifth year to the author of the best thesis embodying the result of original research in a topic connected with the history of Hindu mathematics before 1600 A.D. The subject of the thesis will be prescribed by the Council of the Calcutta Mathematical Society at least two years in advance.

At the Cleveland meeting of the American Association for the Advancement of Science, Assistant Professor C. F. Roos, of the department of mathematics of Cornell University, was elected permanent secretary, as successor to Professor B. E. Livingston, who becomes general secretary. Professor E. R. Hedrick was elected vice-president and chairman of Section A (Mathematics), and Professor G. C. Evans vice-president and chairman of Section K (Social and Economic Sciences).

At the Cleveland meeting of the American Association for the Advancement of Science an organization was formed for the purpose of promoting and improving the teaching of physics of college and university grade under the name of the American Association of Teachers of Physics. Regional as well as general meetings are to be held. The first general meeting will be at the Bureau of Standards in April in connection with the regular meeting of the American Physical Society. At that meeting Dr. A. W. Hull, assistant director of research of the General Electric Company, will present a paper on "The Training of Physicists for Industry."

Professor George Birkhoff, of Harvard University, has been elected Corresponding Member of the Royal Academy of Sciences of the Institute of Bologna.

Professor Wilhelm Blaschke, of the University of Hamburg, has recently delivered lectures, as visiting lecturer of the American Mathematical Society, at the following Universities and Colleges: Princeton, January 6, 7; Swarthmore, January 8; Columbia, January 14, 15; Cornell, January 22, 23; Chicago, January 27, 28; Iowa, February 4; Wisconsin, February 11; Ohio State, February 13, 14; Dartmouth, February 16, 17; Massachusetts Institute of Technology, February 19, 20, 24; Harvard, February 25; Brown, February 26. The subjects of these lectures were "Topological Questions of Differential Geometry," "Selected Topics from Differential Geometry," "Mathematical Problems of Nomography," and "The Geometrical Foundations of the Theory of Relativity." Professor Blaschke will lecture at The Johns Hopkins University as James Speyer Visiting Professor of Mathematics from March 2 to May 8, 1931, on "Topological Questions in Differential Geometry."

Professor Albert Einstein has been engaged in special research at the California Institute of Technology during the first semester of the present academic year.

Dr. R. A. Fisher, Chief Statistician of Rothamsted Agricultural Experimental Station, Harpenden, England will be in residence on the staff at the Iowa State College during the first half of the summer session, June 16 to July 24 and later will visit a number of colleges, universities and experiment stations throughout the country. In connection with Doctor Fisher's visit, Iowa State College announces a special group of summer courses covering the theory and application of statistics.

Professor Cassius J. Keyser lectured at Duke University, Durham, North Carolina, on March 12 and 13 on "Humanism and Pseudo-Humanism" and "The Human Significance of Science and Mathematics." Mr. Keyser is Professor Emeritus of Mathematical Philosophy at Columbia University.

Professor M. I. Pupin, of Columbia University, has received the decoration of the White Eagle of the First Order, conferred on him by Alexander I of Jugoslavia for outstanding services to the nation.

Professor G. B. Drummond, formerly at the Mississippi Agricultural and Mechanical College, has been appointed an assistant professor of mathematics at the Oklahoma Agricultural and Mechanical College.

Dr. Charles Hopkins has been promoted to be associate in mathematics at the University of Illinois.

Professor T. von Karman has been appointed professor of aeronautics at the California Institute of Technology.

Dr. Louis Lindsey, professor of applied mathematics at Syracuse University, has been granted leave of absence for the second semester of the present year. The time will be spent in travel and investigation in the South.

Associate Professor H. W. March, of the University of Wisconsin, has been promoted to a professorship of mathematics.

Miss Georgia E. Robinson has been appointed professor of mathematics at the Junior College, Centerville, Iowa.

Dr. J. A. Schouten, of the Delft Technical School, has been appointed professor of geometry at Princeton for the second semester of 1930–31.

Mr. C. E. Smith has been appointed assistant in the Chabot Observatory.

The following courses in mathematics are announced for the summer 1931:

Columbia University, July 6 to August 14. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: Survey of mathematics; Geometric transformations. By Professor W. B. Fite: Groups of finite order. By Professor J. F. Ritt: Differential equations. By Professor P. A. Smith: Functions of a real variable.

Cornell University, July 6 to August 14. In addition to the usual elementary work, the following advanced courses will be offered. By Professor Virgil Snyder: Teachers' course. By Professor F. R. Sharpe: Elementary differential equations, and Advanced analytic geometry. By Professor W. A. Hurwitz: Theory of equations. By Professor D. C. Gillespie: Advanced calculus. By Professor W. B. Carver: Projective geometry. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig, B. W. Jones.

University of Illinois, June 22 to August 15. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor R. D. Carmichael: Functions of a complex variable; Linear differential equations. By Professor G. A. Miller: Theory of groups; History of mathematics. By Professor J. B. Shaw: Modern algebra. By Associate Professor Lytle: Fundamental concepts of mathematics; Teachers' course. By Assistant Professor Levy: Geometric transformations. By Dr. Hopkins: Theory of equations and determinants. By Dr. Ogg: Constructive and projective geometry.

University of Iowa. First term, June 8 to July 16. In addition to courses in college algebra, trigonometry, analytic geometry and calculus, the following subjects are offered. By Miss Ruth Lane: Subject matter and teaching of mathematics. By Professor Egon S. Pearson: Statistics; Seminar in statistics. By Professor Chittenden: Survey of mathematics; Modern theories of integration; Seminar in general topology. By Mr. Craig: Theory of equations. By Assistant

Professor Woods: Differential equations; Modern analytical geometry. By Dr. Earl: Mathematics of finance; Theory of approximation. By Assistant Professor Ward: Elementary mechanics; Ordinary differential equations (real variables). By Dr. Conkwright: Projective geometry; Theory of numbers. By the Staff: Reading and Research. Second Term, July 20 to August 20. By Professor Nordgaard: The History of mathematics. By Professor Reilly: Differential equations; Interpolation; Seminar in finite summation. By Professor Chittenden: Matrices and determinants; Introduction to general topology; Seminar in general topology. By Associate Professor Wylie: Meteors. By the Staff: Reading and Research.

Johns Hopkins University, June 29 to August 7. By Professor F. D. Murnaghan: College algebra; Differential and integral calculus; Functions of a complex variable *or* Calculus of variations (according to demand).

University of Kanas. First term, June 9 to July 18. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following courses are offered: By Professor Ashton: Series; Seminar. By Professor Mitchell: Teachers course; Theory of numbers; Seminar. Second term, July 20 to August 14. By Professor Mitchell: Advanced algebra; History of mathematics; Seminar. By Professor Jordan: Advanced calculus.

University of Maine, July 6 to August 14. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teachers' course. By Assistant Professor Lucas: Theory of equations; Analytic geometry in homogeneous coordinates. By Professor Willard: Theory of functions of a complex variable.

Massachusetts Institute of Technology. First period, June 16 to July 28: Courses in calculus and differential equations covering the prescribed work of the first two years. Second period, July 29 to September 9: Courses given in first period repeated. August 10 to September 12: Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. July 6 to July 31: Courses in methods of teaching mathematics in the Junior High School and the Senior High School. June 16 to July 7: Courses in advanced calculus and theoretical aeronautics. July 8 to July 28: Course in aeronautical continued. July 6 to August 3: Differential Equations, intended primarily for army officers.

University of Minnesota. First term, June 17 to July 25. In addition to the usual elementary work the following courses will be offered: By Professor Dunham Jackson: Vector analysis. By Professor Griffith C. Evans of the Rice Institute: Mathematical theory of economics. By Associate Professor Underhill: Theory of equations. By Assistant Professor Gladys Gibbens: Differential equations. By Professors Jackson, Underhill, and Gibbens: Reading in advanced mathematics. Second term, July 27 to August 29. By Professor Raymond W. Brink: Infinite series. By Dr. R. A. Fisher, Chief Statistician, Rothamsted Experimental Station, Harpenden, England: Statistical Methods. By Professor Griffith C. Evans of the Rice Institute: Potential theory. By

Professor Raymond W. Brink and Assistant Professor Elizabeth Carlson: Reading in advanced mathematics.

Stanford University, June 18 to August 29. In addition to the usual courses in calculus and differential equations, and a beginning course in the Theory of groups of permutations, the following courses will be given by Professor Edmund Landau (of the University of Gottingen): Foundations of arithmetic; Selected topics from the theory of functions.

Syracuse University. In addition to the usual courses in mathematics through the calculus, the following courses are offered: By Professor F. F. Decker: The teaching of algebra and geometry in secondary schools; and either the introduction to modern algebra or the theory of groups. By Professor A. D. Campbell: Analytic projective geometry or the general theory of the functions of a complex variable

University of Vermont. The following courses are offered by Professors Bullard, Butterfield, Millington and Swift: Courses in algebra, plane geometry for teachers, plane trigonometry, solid geometry, differential and integral calculus, differential equations, astronomy and history of mathematics.

Professor W. H. Bristol, inventor, and formerly professor of mathematics at Stevens Institute of Technology, has died at the age of seventy.

Dean Howard L. Hodgkins, for forty-eight years a member of the faculty of the George Washington University, died on Friday, February 13, 1931.

Mr. E. W. Hyde, for twenty-five years professor of mathematics at the University of Cincinnati, and formerly treasurer and actuary of the Columbia Life Insurance Company, has died at the age of eighty-seven.

CORRIGENDA

The following corrections should be made in the February issue of this MONTHLY:

On page 110 the signature at the end of the review of the Graham-John "Advanced Algebra" should be H. M. Hosford.

On page 69, the first line should read:

$$d=-2\alpha(y_2-y_1)(x_2-x_1)^{-1}+\tfrac{1}{2}(x_1+x_2).$$

The error was the omission of the $+$ sign before the number $\frac{1}{2}$.

On page 71, the first displayed equation should read:

$$\left\{y_1(x_2-x_1)^2[4y_1^2+(x_2-x_1)^2]^{-1}\right\}=\tfrac{1}{4}(x_2-x_1)^2(y_1+y_2)^{-1}.$$

The error was the omission of the -1 exponent which appears with the expression in brackets.

On page 78, in the 17th line the last term of the left hand member of the equation should be $81y_1^4$ instead of $81y_1^2$.

On page 80, in the first line the exponent $3/2$ should be $-3/2$; and in the denominator of the fraction in the next line the exponent $-3/2$ should be $3/2$.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, April 15.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 14.
TEXAS, Fort Worth, Tex., Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE CO-OPERATION OF

ELIZABETH CARLSON, N. A. COURT, OTTO DUNKEL, H. S. EVERETT, B. F. FINKEL, R. E. GILMAN, R. A. JOHNSON, H. W. KUHN, J. R. MUSSELMAN, H. L. OLSON, D. E. SMITH, F. M. WEIDA

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 4, APRIL

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

TWO THEOREMS ON THE PARTITIONS OF NUMBERS

By WALTER B. FORD, University of Michigan

Despite the very large existing literature upon the partitions of numbers the two following theorems do not appear to have been noted. In particular, they do not seem to occur in such extensive summary treatises as Dickson's History of the Theory of Numbers or MacMahon's Combinatory Analysis. At the same time they seem noteworthy inasmuch as they afford a means of determining rapidly the number of ways in which an integer n may be expressed as the sum of mth powers of other integers, either when repetitions among the latter integers are allowed or not allowed:

Theorem 1. *Let* $P_{(n)}^{(m)}$ *represent the number of ways in which the integer* n *may be expressed as the sum of* m *th powers of integers when repetitions among the latter integers are allowed. Then, if* $\sigma_{(n)}^{(m)}$ *represents the sum of those divisors of* n *which are* m *th powers of integers (including* 1 *and* n *itself if it is the* m *th power of an integer), the following recurrence relation exists:*

$$P_{(n)}^{(m)} = \frac{1}{n}\left[\sigma_{(1)}^{(m)} P_{(n-1)}^{(m)} + \sigma_{(2)}^{(m)} P_{(n-2)}^{(m)} + \cdots + \sigma_{(n-1)}^{(m)} P_{(1)}^{(m)} + \sigma_{(n)}^{(m)}\right].$$

Theorem 2. *Let* $p_{(n)}^{(m)}$ *represent the number of ways in which the integer* n *may be expressed as the sum of the* m *th powers of other integers when repetitions among the latter integers are not allowed. Then, if* $\tau_{(n)}^{(m)}$ *represents the sum of those divisors of* n *which are* m *th powers of integers (including* 1 *and* n *itself if it is the* m *th power of an integer) and which give odd quotients, minus the sum of the similar divisors which give even quotients, the following recurrence relation exists:*

$$p_{(n)}^{(m)} = \frac{1}{n}\left[\tau_{(1)}^{(m)} p_{(n-1)}^{(m)} - \tau_{(2)}^{(m)} p_{(n-2)}^{(m)} + \cdots + (-1)^{n-2} \tau_{(n-1)}^{(m)} p_{(1)}^{(m)} + (-1)^{n-1} \tau_{(n)}^{(m)}\right].$$

As to the proofs, let us consider first the simplest case of Theorem 1, namely, that in which $m=1$. The actual (though inaccessible) value of $P_{(n)}^{(1)}$ will then be, as is well-known,[1] the coefficient a_n in the Maclaurin development

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = 1 + \sum_{n=1}^{\infty} a_n x^n. \tag{1}$$

Now

$$\log F(x) = -\sum_{i=1}^{\infty} \log(1-x^i) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{x^{ij}}{j}\cdot$$

Upon collecting the coefficients of the various powers of x here entering on the right, this equation takes the form

[1] See for example the paper by A. J. Kempner on *Partitio Numerorum* in this Monthly, vol. 3 (1923), page 360.

$$\log F(x) = \sum_{n=1}^{\infty} b_n x^n, \tag{2}$$

where

$$b_1 = 1,\ b_2 = 1 + \tfrac{1}{2} = \tfrac{1}{2}(2+1),\ b_3 = 1 + \tfrac{1}{3} = \tfrac{1}{3}(3+1),\ b_4 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{1}{4}(4+2+1),$$

$$b_5 = 1 + \tfrac{1}{5} = \tfrac{1}{5}(5+1),\ b_6 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} = \tfrac{1}{6}(6+3+2+1), \text{ etc.}$$

Thus, in general, it appears that b_n is equal to the sum of the integral divisors of n divided by n. Moreover, the law of formation of the b_s being known, it is easy to formulate an expression for the coefficient a_n of (1); that is, an expression for $P_{(n)}^{(1)}$. In fact, upon differentiating (2), subsequently using for $F(x)$ its series form (1), we may write

$$\frac{F'(x)}{F(x)} = \frac{\sum_{n=1}^{\infty} n a_n x^{n-1}}{1 + \sum_{n=1}^{\infty} a_n x^n} = \sum_{n=1}^{\infty} n b_n x^{n-1};$$

and if we clear this equation of fractions, then multiply together the two series forming a product on the right, subsequently collecting coefficients and equating those pertaining to like powers of x, we arrive at the following system of linear equations connecting the unknown a_s with the known b_s:

$$\begin{aligned} a_1 &= b_1, \\ 2a_2 &= 2b_2 + a_1 b_1, \\ 3a_3 &= 3b_3 + 2a_1 b_2 + a_2 b_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ n a_n &= n b_n + (n-1) a_1 b_{n-1} + (n-2) a_2 b_{n-2} + \cdots + a_{n-1} b_1. \end{aligned}$$

We thus have the following recurrence relation pertaining to a_n:

$$a_n = \frac{1}{n}\left[b_1 a_{n-1} + 2b_2 a_{n-2} + 3b_3 a_{n-3} + \cdots + (n-1) b_1 a_1 + n b_n\right].$$

We have now but to recall the properties already indicated of the quantities $b_1, b_2, \cdots b_n$ and remember that $a_n = P_{(n)}^{(1)}$ to see that this relation is coextensive with that stated in Theorem 1 for the special case $m = 1$.

In case $m > 1$ it suffices for the proof of Theorem 1 to carry through the steps of the above reasoning when one starts from the infinite product obtained from (1) by replacing the exponent i in x^i by the exponent i^n , noting the corresponding generalized properties of the coefficients b_1, b_2, b_3, etc.

As to Theorem 2 the proof may be supplied in an analogous manner in which, instead of starting with an infinite product of the type (1), one uses the product

$$\prod_{i=1}^{\infty}(1 + x^{i^m}).$$

TWO APPLICATIONS OF TSCHIRNHAUS TRANSFORMATIONS IN THE ELEMENTARY THEORY OF EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

In a beginning course in the theory of equations the student is introduced to a number of very simple transformations of equations, such as $y=mx$, $y=x-a$, $y=1/x$, and perhaps a few others. In his later work, if he continues the study of algebraic equations, he meets transformations of much more complicated nature in the works of Brioschi, Cayley, Hermite, Gordan, Weber, and others, in connection with quintic normal forms, invariants, and various topics. It seems to me both possible and desirable to lead up to these more difficult transformations by making some use, in the elementary course, of Tschirnhaus transformations of higher than the first degree.[1] This step is also justified by the fact that such transformations can be employed advantageously in treating some of the usual matters. Further, they serve as an application of symmetric functions, especially those usually denoted by s_k, where s_k means the sum of the kth powers of the roots of a given equation.

In this paper I wish to point out two important and familiar topics which can be discussed effectively with the aid of quadratic transformations, namely, the evaluation of the discriminant of the general cubic equation in terms of the coefficients, and the development of criteria for the number of real roots of the general quartic equation. Each of these can be handled in a number of different ways, as the reader doubtless knows, but the present method may nevertheless be of some interest and value.

To obtain the value of the discriminant of the general cubic, taken for convenience in the reduced form[2]

$$x^3 + px + q = 0, \tag{1}$$

with roots x_1, x_2, x_3, we apply to (1) the transformation $y=x^2$. The transformed equation can be set up in a number of ways. If its roots are denoted by $y_1=x_1^2$, $y_2=x_2^2$, $y_3=x_3^2$, their sum, the sum of their products taken two at a time, and their product can be easily computed with the aid of symmetric functions of (1). [We shall assume throughout this presentation that the symmetric functions which are required have been studied in an earlier part of the course.] Or we may follow the procedure customary in Tschirnhaus transformations and find the coefficients of the transformed equation from $\sum y$, $\sum y^2$ and $\sum y^3$, which can also be determined with the aid of $y=x^2$ and (1). (The symbol $\sum y$ means $y_1+y_2+y_3$, $\sum y^2$ means $y_1^2+y_2^2+y_3^2$, and so on.) In the present case this method is a little longer than the other. Or, we may use a special device, which consists in writing (1) as $x(x^2+p)=-q$, replacing x^2 by y, squaring and again replacing x^2 by y.

[1] This suggestion is, of course, not new, being carried out in some texts and, undoubtedly, by many teachers. It might well be extended to cover linear fractional transformations as well.

[2] The coefficients in (1) and (3) are real.

The transformed equation is found to be

$$F(y) = y(y + p)^2 - q^2 = 0, \tag{2}$$

or

$$y^3 + 2py^2 + p^2y - q^2 = 0.$$

Its factored form is

$$(y - x_1^2)(y - x_2^2)(y - x_3^2) = 0.$$

Now by definition the discriminant of (1) is

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2,$$

and it will be denoted by D. Since

$$x_1 + x_2 + x_3 = 0, \quad x_1x_2 + x_1x_3 + x_2x_3 = p, \quad x_1^2 + x_2^2 + x_3^2 = -2p,$$

we find

$$x_1x_2 = x_3^2 + p, \quad (x_1 - x_2)^2 = -3x_3^2 - 4p = 3\left(-\frac{4p}{3} - x_3^2\right).$$

The other factors of D reduce similarly, so that

$$D = 27\left(-\frac{4p}{3} - x_1^2\right)\left(-\frac{4p}{3} - x_2^2\right)\left(-\frac{4p}{3} - x_3^2\right).$$

But this is 27 multiplied by $F(-4p/3)$, which gives at once $D = -4p^3 - 27q^2$. I know no more convenient method of securing this result.

Equation (2) can also be used in the investigation of the relation between the value of D and the number of real roots of (1), since, from $y = x^2$, there is a one-to-one correspondence between the real roots of (1) and the *positive* real roots of (2). To discuss the latter, we may consider the graph of $F(y) = y^3 + 2py^2 + p^2y - q^2$. The derivative $F'(y)$ vanishes at $y = -p/3$ and $y = -p$, and $F(0) = -q^2$, $F(-p/3) = D/27$, $F(-p) = -q^2$. If p is positive, and D therefore negative, $F(y)$ obviously increases continuously as y increases from zero; since $F(0)$ is negative[1] $F(y) = 0$ has one and only one positive root. If p is negative and D also negative, the maximum and minimum points on the graph are both to the right of $y = 0$ and below the y-axis; again there can be only one positive root. If p is negative and D is zero, $F(y) = 0$ has a double root, since the maximum point of the graph is at $y = -p/3$, $F(y) = 0$. The third root is also positive. If p is negative and D is positive, the maximum point of the graph is *above* the y-axis, while the minimum point is below; hence there must be three positive roots. The student is here supposed to be familiar with the essential features of the graph of a cubic polynomial.

We pass now to the consideration of the general reduced quartic,

---

[1] The trivial case $q = 0$ may be treated separately.

$$x^4 + qx^2 + rx + s = 0, \tag{3}$$

with roots x_1, x_2, x_3, x_4. Its discriminant is, by definition the product

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2,$$

which we shall again call D. The value of D in terms of q, r, s is easily obtainable, since it is equal, except possibly for a constant factor, to the discriminant of any of the resolvent cubics which are commonly introduced in the solution of (3). This is proved in almost any text on the subject, and gives a simple and satisfactory evaluation of D. It is, however, not quite so easy to determine conditions on q, r, s, D which will insure that (3) has a certain definite number of real roots. Some authors obtain these conditions with the aid of Sturm's functions, while others use the resolvent cubic; we shall employ a quadratic transformation and a few symmetric functions.

In the first place, it is well known that D will be positive if (3) has four real roots or two pairs of conjugate complex roots, while D will be negative if (3) has two real roots and two complex. These statements can be proved at once, by choosing the notation for the roots so as to exhibit their reality or complexity and then substituting in the above value of D. Thus, to treat the case of four complex roots, we may write

$$x_1 = a + bi,\quad x_2 = a - bi,\quad x_3 = -a + ci,\quad x_4 = -a - ci,$$

after which it is easy to see that the product $(x_1-x_3)(x_2-x_3)$ is positive, the product $(x_1-x_4)(x_2-x_4)$ is positive, while $(x_1-x_2)(x_3-x_4) = -4bc$. Therefore, D is surely positive. Conversely, we may say that if D is negative (3) will have two (distinct) real roots and two complex, while if D is positive (3) will have all its roots real or none of them real. That is, the character of D alone is no longer sufficient, as it is for the cubic, to determine completely the number of real roots. However, if D is positive, the following familiar conditions can be stated:

$$\begin{aligned} &\text{If } D>0,\ q\geqq 0, \text{ (3) has no real roots,}\\ &\text{If } D>0,\ L\leqq 0, \text{ (3) has no real roots, } (L\equiv 8qs-2q^3-9r^2),\\ &\text{If } D>0,\ q<0,\ L>0, \text{ (3) has four real roots.}\end{aligned} \tag{4}$$

The first of these follows at once, since, for (3), $s_2 = -2q$, and s_2 cannot be zero or negative if all the roots are real and distinct. To prove the third, we note that, if $q<0$ and $L>0$, q^2-4s must be positive. The conditions $q<0$, $q^2-4s>0$ are equivalent to $s_2>0$, $s_4-\frac{1}{4}s_2^2>0$. Now suppose that (3) has no real roots, so that

$$x_1 = a + bi,\quad x_2 = a - bi,\quad x_3 = -a + ci,\quad x_4 = -a - ci,$$

with b and c different from zero. Then, by simple algebra,

$$s_2 = 4a^2 - 2b^2 - 2c^2,\quad s_4 - \tfrac{1}{4}s_2^2 = (b^2 - c^2)^2 - 8a^2(b^2 + c^2).$$

Now if $s_2>0$, $8a^2>4b^2+4c^2$, and $s_4-\frac{1}{4}s_2^2>0$ implies

$$(b^2 - c^2)^2 > 4(b^2 + c^2)^2,$$

which is clearly false. Hence (3) can not have no real roots, and must have four.[1]

The second condition of (4) does not seem to lend itself to similar treatment; we now proceed to apply a transformation to (3), so chosen that the present case will reduce to the first case of (4). That is, we make use of

$$y = x^2 + k_1 x + k_2, \tag{5}$$

where the parameters k_1 and k_2 are to be determined so that $\sum y = \sum y^2 = 0$, that is, so that the transformed equation in y will have no term in y^3 or y^2. Now $\sum y = s_2 + k_1 s_1 + 4k_2$, which is zero if $k_2 = q/2$. Now square (5) and sum over the roots of (3); the equation $\sum y^2 = 0$ turns out to be a quadratic[2] in k_1 whose discriminant is exactly $-4L$, as defined above. Hence, if L is not positive, k_1 is real, and we are thus able, by a real transformation, to reduce (3) to an equation in y which has no terms in y^3 or y^2 and cannot, therefore, have all its roots real. But (5) shows that, when k_1 and k_2 are real, each real root of (3) will give a real root of the transformed equation. We conclude that (3) cannot have four real roots, and since $D > 0$, it must have no real roots.

Here then are two fundamental matters in the theory of equations which can be treated conveniently with the aid of simple transformations. In each case the desired results are obtained, I believe, as easily as is possible, and, in addition, the student is becoming acquainted with an algebraic tool of some importance.

H. von KOCH'S FIRST LEMMA AND ITS GENERALIZATION

By A. A. SHAW, University of Arizona

In a letter to Poincaré, Helge von Koch wrote the following in 1895:[3]

LEMMA 1. *$\alpha_1, \alpha_2, \cdots$, and $\beta_1, \beta_2, \cdots$, being any given quantities whatever, the necessary and sufficient condition for the absolute convergence of the infinite determinant*

$$\begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots \\ 0 & 0 & \beta_3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is that the series

---

[1] No claim of originality is made for this argument, though I have not seen it used elsewhere.

[2] $\sum y^2 = -2qk_1^2 - 6rk_1 + (q^2 - 4s)$. The reader may take $k_1 = 0$ and show that (4) can be written with $q^2 - 4s$ in place of L.

[3] See Comptes Rendus de l'Academie des Sciences, vol. 120 (1895), p. 144. See also Whittaker and Watson, *Modern Analysis*, 3d edition, p. 37.

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \cdots$$

shall be absolutely convergent.

It is the purpose of this paper to give a proof of the above lemma, and to show, in addition, that the convergence of the given determinant depends on the absolute convergence of the series

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \alpha_3 + \beta_3 + \cdots .$$

As a generalization of von Koch's lemma, we shall give also proofs of convergence of more general infinite determinants which arise in the solution of homogeneous linear difference equations by means of infinite determinants.[1]

1. *Proof*: Consider this determinant of order m:

$$D_m = \begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots & 0 \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots & 0 \\ 0 & 0 & \beta_3 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot\cdot\beta_m & 1 \end{vmatrix};$$

and, to compare with it, construct the auxiliary determinant, Δ_m, all the elements in which are zeros except those of the principal diagonal, which are $1+|\alpha_m\beta_m|$, $m=1, 2, \cdots, m$:

$$\Delta_m = \begin{vmatrix} 1+|\alpha_1\beta_1| & 0 & 0 & \cdots & 0 \\ 0 & 1+|\alpha_2\beta_2| & 0 & \cdots & 0 \\ 0 & 0 & 1+|\alpha_3\beta_3| & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1+|\alpha_m\beta_m| \end{vmatrix}.$$

Now in the expansion of Δ_m all the terms are positive and are formed by certain[2] combinations of $2, 4, 6, \cdots, 2m$ letters at a time, except the first term which is unity. Hence Δ_m contains the absolute values of all the terms of D_m plus additional terms all positive, since the upper limit of the suffixes in the development of Δ_m is m while that of D_m is $m-1$; and since some of the terms of D_m are negative, it follows at once that Δ_m is greater than $|D_m|$.

---

[1] See the author's thesis (the solution of homogeneous linear difference equations by means of infinite determinants) which is in the library of the University of California, Berkeley, California.

[2] But not every possible combination; for when $m=3$, $\Delta_3=1+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_1\alpha_2\beta_1\beta_2+\alpha_1\alpha_3\beta_1\beta_3+\alpha_2\alpha_3\beta_2\beta_3+\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$ which does *not* contain such terms as $\alpha_1\beta_2$, $\alpha_1\alpha_2\alpha_3\beta_1$, etc.

Similarly the difference $(\Delta_{m+p}-\Delta_m)$ contains the absolute values of all the terms of the difference $(D_{m+p}-D_m)$ and other terms all positive. Consequently

$$\left| D_{m+p} - D_m \right| < \Delta_{m+p} - \Delta_m.$$

Hence the sequence D_m converges if the sequence Δ_m converges.

Now Δ_m is convergent if the infinite product of the terms of the main diagonal is convergent, since Δ_m and that infinite product are identical.[1] But that infinite product is convergent if the series $\sum |\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, is convergent, by a well known theorem about infinite products. But we have shown that $\Delta_m > |D_m|$. Hence D_m is absolutely convergent if the series $\sum |\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, is convergent and that is the *sufficient* condition for convergence of D_m.

Evidently it is also a *necessary* condition for convergence of the given determinant, for, if $\sum |\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, be divergent, D_m will not converge absolutely, since this series, with negative signs, is found in the expansion of D_m as may easily be verified.

Thus according to von Koch the absolute convergence of D_m depends only on second degree terms, $\alpha_i\beta_i$, in the development of the given determinant.

As a corollary to the fundamental theorem for normal determinants[2] it may be noticed here that *D_m is convergent if the series*

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \cdots + \alpha_i + \beta_i + \cdots$$

is absolutely convergent.

But this theorem[3] does not give us the *necessary* condition for convergence of D_m as does von Koch's first lemma, for the series of *linear* terms

$$|\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + \cdots + |\alpha_i| + |\beta_i| + \cdots$$

does not occur in the development of D_m and hence we cannot assert that the divergence of the series

$$|\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + \cdots + |\alpha_i| + |\beta_i| + \cdots$$

makes D_m divergent as in the proof of von Koch's lemma. This can be shown by a simple example:

If

$$\alpha_1 = \beta_1 = 1, \quad \alpha_2 = \beta_2 = \tfrac{1}{2}, \quad \alpha_3 = \beta_3 = \tfrac{1}{3}, \text{ etc.},$$

the above series becomes

---

[1] See Whittaker and Watson, loc. cit., p. 36 (§2.81).

[2] G. Kowalewski, *Einführung in die Determinantentheorie*, first edition, p. 372: "Wenn die unendliche Reihe $c_{11}+c_{12}+c_{21}+c_{13}+c_{22}+c_{31}+\cdots$ absolut konvergent ist, so existiert

$$\begin{vmatrix} 1+c_{11} & c_{12} & c_{13} & \cdot\ \cdot \\ c_{21} & 1+c_{22} & c_{23} & \cdot\ \cdot \\ c_{31} & c_{32} & 1+c_{33} & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot \end{vmatrix} = \lim D_n."$$

[3] See the Bulletin de la Société Mathématique de France, vol. 14 (1886), p. 77.

$$\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\cdots$$

which is divergent; while Δ_m becomes

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots$$

which is convergent; hence D_m is convergent.

2. *Extensions.* Solution of homogeneous linear difference equations by means of infinite determinants[1] gives rise to the following determinant which is a generalization of the determinant of von Koch's first lemma:

$$\begin{vmatrix} 1 & b_0 & c_0 & 0 & 0 & 0 & \cdot & \cdot \\ a_1 & 1 & b_1 & c_1 & 0 & 0 & \cdot & \cdot \\ 0 & a_2 & 1 & b_2 & c_2 & 0 & \cdot & \cdot \\ 0 & 0 & a_3 & 1 & b_3 & c_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

This is absolutely convergent if the series $b_0a_1+b_1a_2+\cdots+b_{n-1}a_n+\cdots+c_0a_1a_2+c_1a_2a_3+\cdots+c_{n-1}a_na_{n+1}+\cdots$ *converges absolutely.*

Proof: Consider this determinant of order m:

$$D_m = \begin{vmatrix} 1 & b_0 & c_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ a_1 & 1 & b_1 & c_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_2 & 1 & b_2 & c_2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & a_3 & 1 & b_3 & c_3 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{m-1} & 1 \end{vmatrix};$$

and, to compare with it, construct the auxiliary determinant, Δ_m, in which all the elements are zeros except those of the principal diagonal; namely

$$\Delta_m = \begin{vmatrix} 1+b_0a_1 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 1+c_0a_1a_2 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 1+b_1a_2 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1+c_1a_2a_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

[1] See the author's thesis, previously mentioned.

where b_ia_{i+1} and $c_ia_{i+1}a_{i+2}$ are all positive ($i=0, 1, 2, 3, \cdots$) and are *alternate factors* in the product of the diagonal elements.

Now in the expansion of Δ_m all the terms are positive and are formed by certain combinations of 2, 3, 4, $\cdots$, n letters at a time, except the first term which is unity, and the suffixes of b_ia_{i+1} and $c_ia_{i+1}a_{i+2}$ may be *repeated* in *any* combination; thus we shall have $b_0a_1+b_1a_2+\cdots+b_{m-2}a_{m-1}+c_0a_1a_2+c_1a_2a_3+\cdots+b_0a_1c_0a_1a_2+$ etc. where suffix 0 is *repeated* in the last term. There is no such repetition of subscripts in any term of the expansion of D_m. Hence Δ_m contains absolute values of all the terms of D_m plus additional terms all positive, since the upper limit of the suffixes in Δ_m is m while that of D_m is $m-1$ and some of the terms of D_m are negative. Hence Δ_m is greater than $|D_m|$.

Similarly the difference $(\Delta_{m+p}-\Delta_m)$ contains the absolute values of all the terms of the difference $(D_{m+p}-D_m)$ and other terms all positive. Consequently

$$\left| D_{m+p} - D_m \right| < \Delta_{m+p} - \Delta_m.$$

Hence the sequence D_m converges if the sequence Δ_m converges.

Now Δ_m is convergent if the infinite product of the main diagonal

$$\prod_{i=0}^{\infty}(1 + b_ia_{i+1})(1 + c_ia_{i+1}a_{i+2})$$

is convergent, since Δ_m and that infinite product are identical. But the product of the main diagonal is convergent if the series

$$\left\{ \sum_{i=0}^{\infty} \left| b_ia_{i+1} \right| + \sum_{i=0}^{\infty} \left| c_ia_{i+1}a_{i+2} \right| \right\}$$

is convergent, by a well known theorem of infinite products. But we have shown that $\Delta_m > |D_m|$. Hence D_m is absolutely convergent if the series

$$\left\{ \sum_{i=0}^{\infty} \left| b_ia_{i+1} \right| + \sum_{i=0}^{\infty} \left| c_ia_{i+1}a_{i+2} \right| \right\}$$

is convergent, and that is the *sufficient* condition for convergence of D_m. Obviously it is also the *necessary* condition for convergence of the given determinant; for, if

$$\left\{ \sum_{i=0}^{\infty} \left| b_ia_{i+1} \right| + \sum_{i=0}^{\infty} \left| c_ia_{i+1}a_{i+2} \right| \right\}$$

be divergent, D_m will not converge absolutely, since this series, with signs changed, is found in the expansion of D_m as may readily be verified.

Thus the convergence of the determinant in question depends on the *quadratic* and *cubic* terms only in the development of the given determinant.

As a corollary to the fundamental theorem for normal determinants it may be noticed again that D_m *is convergent if the series*

$$b_0 + a_1 + c_0 + b_1 + a_2 + c_1 + b_2 + a_3 + c_2 + b_3 + a_4 + c_3 + b_4 + a_5 + c_4 + \cdots$$

is absolutely convergent. Here again, as on page 190, the converse is not true, i.e. the absolute convergence of Δ_m (and hence of D_m) does not imply the convergence of the above series of linear terms, as we may easily show by an example: Put

$$b_0 = c_0 = a_1 = 1, \quad b_1 = c_1 = a_2 = \tfrac{1}{2}, \quad b_2 = c_2 = a_3 = \tfrac{1}{3}, \quad b_3 = c_3 = a_4 = \tfrac{1}{4},$$

generally,

$$b_i = c_i = a_{i+1} = \frac{1}{i}. \quad (i = 0, 1, 2, \cdots).$$

Then the above series becomes

$$1 + 1 + 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} + \cdots,$$

which is divergent; while Δ_m becomes

$$\left\{\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right) \cdots + \left(1 + \frac{1}{1^2.2}\right)\left(1 + \frac{1}{2^2.3}\right)\left(1 + \frac{1}{3^2.4}\right) \cdots\right\}$$

which is obviously convergent.

Thus the convergence of the above series of linear terms implies the absolute convergence of D_m, while the convergence of D_m does not imply anything about the series.

Generally, the solution of homogeneous linear difference equations of order n by the said method gives rise to the following determinant which is the most general extension to the determinant of von Koch's first lemma:

$$D_m = \begin{vmatrix} 1 & a_0^{(2)} & a_0^{(3)} & a_0^{(4)} & \cdot & \cdot & \cdots & a_0^{(m)} & 0 & \cdot & \cdots & \cdot & 0 \\ a_1^{(1)} & 1 & a_1^{(2)} & a_1^{(3)} & a_1^{(4)} & \cdot & \cdots & \cdot & a_1^{(m)} & 0 & \cdots & \cdot & 0 \\ 0 & a_2^{(1)} & 1 & a_2^{(2)} & a_2^{(3)} & a_2^{(4)} & \cdots & \cdot & \cdot & a_2^{(m)} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & a_{m-1}^{(1)} & 1 & \cdots & \cdot & \cdot & \cdot & \cdot & a_{m-1}^{(m)} & 0 \end{vmatrix}$$

As before, we can construct an auxiliary determinant Δ_m in which the elements are zeros except those of the leading diagonal and show that (since Δ_m contains all the moduli of the terms of D_m plus additional terms all positive) $\Delta_m > |D_m|$ and $\Delta_{m+p} - \Delta > |D_{m+p} - D_m|$. Hence if Δ_m be convergent D_m will be also; and the convergence of the latter will depend on the quadratic, cubic,

$\cdots$, $(n-1)$-ic terms in the development of the given determinant; while if we apply Poincaré's fundamental theorem for normal determinants we can show that a *sufficient* condition of convergence of the above determinant is that the series

$$a_0^{(2)} + a_1^{(1)} + a_0^{(3)} + a_0^{(4)} + a_1^{(2)} + a_2^{(1)} + \text{etc.}$$

be absolutely convergent. But the converse is not true as before, and this can be shown by constructing special examples as was done in the first and second parts of this paper.

It is to be noted that not only such determinants as those considered in this paper are convergent, but the same is true of their minors and of all determinants obtained from them by replacing a finite number of their rows or columns by a set of quantities which are numerically less than some constant.

A FEW THEOREMS RELATING TO THE RHIND MATHEMATICAL PAPYRUS

By G. A. MILLER, University of Illinois

One of the most striking features of the *Rhind Mathematical Papyrus* is the table at the beginning thereof in which 2 divided by the various odd numbers from 5 to 101 is expressed in each case as the sum of the reciprocals of *different* natural numbers. Hence this table illustrates a very special case of the following theorem: *An arbitrary rational number can be expressed as the sum of the reciprocals of a finite set of distinct integers having the same sign and such that the smallest absolute value of the integers of the set is arbitrarily large and that the number of these integers exceeds any assigned natural number.* Hence it may be said that the integers of such a set can be selected in a multiply infinite number of ways.

To prove this theorem we may consider the infinite series formed by the reciprocals of the natural numbers in order. This series was proved to be divergent by Mengoli in 1650 and is commonly known as the harmonic series. It is obvious that we may confine our attention to the case when the arbitrary rational number n under consideration is positive since the proof of the case when it is negative can be derived from the proof when it is positive by merely changing the signs of all the numbers involved therein. When n represents an arbitrary positive rational number we may begin with any term of the harmonic series which satisfies the single condition that it does not exceed n, and add to it a sufficient number of terms which follow it to obtain a sum s which does not exceed n but that $n-s$ is less than any of the terms of the harmonic series thus added. It is well known that the latter can always be done in an infinite number of ways when $s \neq n$.

When $s=n$ the theorem under consideration has thus been proved for this particular value of n. When s is less than n we may find the common fraction

which is equal to $n-s$ and reduce it to lowest terms a/b. Since a and b are relatively prime it is always possible to find two natural numbers, x and y, $y<b$ so that $ay-bx=1$, and hence

$$\frac{a}{b}-\frac{x}{y}=\frac{1}{by}.$$

If $x=1$ our problem is solved since the unit fractions $1/y$ and $1/(by)$ are then distinct from each other and also from each one of the unit fractions which were added to obtain s. If $x>1$ we can find in a similar way another fraction x_1/y_1 such that $y_1<y$ and that

$$\frac{x}{y}-\frac{x_1}{y_1}=\frac{1}{yy_1},$$

where $yy_1<by$. When $x_1=1$ then $1/y_1$ is distinct from all of the unit fractions which have thus far been added and it is larger than $1/(yy_1)$. Since the absolute values of the denominators $y, y_1, \cdots$ are decreasing integers and all the fractions involved in these equations are proper fractions it is obvious that this process can be continued until $x_k=1$, and hence the theorem under consideration has been proved.

The history of elementary mathematics can frequently be enriched by entering more deeply into the mathematical questions involved than could have been done at the period to which this history relates since mathematical developments are frequently inspired by ideas whose bearings are only partially understood at the time. In particular, while the ancient Egyptians could not have proved the theorem noted above a knowledge of its existence shows us now that their efforts to express numbers as the sum of different unit fractions were likely to lead to an unexpected richness of results. It should also be noted that only a finite number of different unit fractions is needed to represent as such a sum any rational number while this is not always the case when such a number is expressed as the sum of different powers of a single number, such as powers of 10.

This fact may help to explain the late adoption of the positional decimal system, especially as regards fractions. The theorem noted above becomes especially significant when it is observed that no pre-Grecian civilization is known to have extended the number concept beyond that of the positive rational numbers. Hence this theorem seems to apply to all of their numbers and it may help to explain the popularity of unit fractions among them even if only special cases of this theorem could be understood at that time. The fact that all the unit fractions which appear in the table noted above are distinct indicates a theoretic interest on the part of the ancient Egyptians which deserves emphasis here since mathematical historians have frequently directed attention to the practical trend of the ancient Egyptian mathematics. It should also be noted here that J. J. Sylvester called attention to a different line of mathematical developments which he associated with the table under consideration.[1]

[1] American Journal of Mathematics, volume 3 (1880), pages 332 and 388.

The given theorem establishes the fact that a necessary and sufficient condition that a number is rational is that it is the sum of a finite number of the distinct terms of the harmonic series, or of the series obtained by changing all the signs thereof. Every rational integer is also the sum of a finite number of distinct powers of 2, or of such powers of 2 with all of their signs changed. This may explain why multiplication was so commonly performed by successive doubling in the *Rhind Mathematical Papyrus.* It is not likely that the ancient Egyptians actually proved this theorem but their confidence therein was probably established by numerous examples. Just as multiplication was commonly performed in this work by successive doubling so division was usually accomplished by successive halving. Their most important measures were divided into halves, quarters, eighths, etc., and it has been suggested[1] that there may have been a stage when instead of the general unit fractions the ancient Egyptians used only powers of $\frac{1}{2}$.

From what was stated above it results that it is not possible to express every positive fraction in terms of a finite number of different powers of $\frac{1}{2}$. Even the fraction $\frac{2}{3}$ which was extensively used by the ancient Egyptians and others can not be thus expressed. The mathematics of conscious approximation is naturally a later development than the mathematics of accuracy, or supposed accuracy since the former requires a consideration of the desirable degrees of approximation for various needs. It does not seem likely therefore that when the ancient Egyptians assumed the area of a circle to be equal to that of a square whose side is equal to 8/9 of the diameter of the circle they were conscious of the fact that this result is only approximately true. At any rate, it is very interesting to note that by the use of the sum of a finite number of unit fractions they could express any of their given fraction exactly, and that the table noted above contains some very small fractions of this type, for instance 1/890, which could probably have been omitted if their interest had been confined to practical applications. It is much smaller than the relative error resulting from their rule for finding the area of the circle.

Another theorem which is suggested by work in the *Rhind Mathematical Papyrus* but was probably not fully understood at the time is that the square represents an upper limit of the area of a rhombus with a given side. In the selection of units of area and units of volume squares and cubes were commonly chosen and thus certain numerical relations were established between geometrical quantities having different dimensions. We may now see in this decided steps towards the later subject of analytic geometry and towards a closer union between numbers and geometric figures, which are clearly the most fundamental concepts of all mathematics. The difficulty of determining whether the area of an isosceles triangle with a relatively short base was actually supposed to be the product of half the base and the altitude or half the base and a side may have been partly due to a lack of noting the difference between the area

[1] Peet, *Rhind Mathematical Papyrus* (1923), page 15.

of a square and that of a rhombus having an equal side and an angle almost equal to a right angle.

While the operation of addition is simpler than that of multiplication when dealing with integers the reverse is true when dealing with common fractions. In view of the very early use of such fractions, as is evidenced by special symbols for some of the most common ones, such as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, it is easy to see that the addition of fractions may have greatly influenced the selection of certain numbers which were especially favored. For instance, in adding $\frac{1}{2}$ and $\frac{1}{3}$ it is desirable to reduce them to the common denominator 6, and this may explain, at least in part, why the number 6 played such a prominent rôle in ancient mathematics. Similarly, in adding $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ it is convenient to reduce them to the common denominator 12, and in adding $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ the common denominator 60 is most convenient. This common denominator will suffice to add also $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$, and hence it is especially significant, being the first which does not require any change when our list of reciprocals is extended by unity.

It is always interesting to find unifying principles in the history of elementary mathematics. The addition of certain unit fractions as found in the *Rhind Mathematical Papyrus* seems to furnish such a principle. Doubtless other principles were potent in securing for the numbers 6, 12, and 60 the favor which they enjoyed in ancient times, as is evidenced by the sexagesimal system of numeration and the emphasis on fractions whose denominators are powers of 12 among the Romans. The fact that we have not yet found among the pre-Grecian mathematicians, nor even among the ancient Greeks, a special word for common denominator, or lowest common denominator, would seem to indicate that their mathematical ideas along this line were not as definite as one might have expected from their advancement along other lines. It must, however, be remembered that even indefinite ideas are sometimes potent, and these constitute often the most difficult problems of mathematical history.

While the rule which the ancient Egyptians employed to find the area of a circle is not accurate it implies the fundamental theorem that the areas of two circles are to each other as the squares of their diameters. This theorem is said to have been proved by Hippocrates and was extended by the ancient Greeks to various similar plane figures as well as to similar solids. It should not be assumed that the ancient Egyptians formulated this theorem but it is interesting to note how closely they approached to it in the *Rhind Mathematical Papyrus*. By the consideration of such approaches the history of elementary mathematics constitutes a powerful incentive for the study of certain subjects of mathematics itself and it enables us also to see more clearly the limitations of the attainments of ancient peoples. A study of these limitations is as important in securing clear historical insight as the study of the advances made in early times, since mountain tops alone fail to give a complete picture of a country.

A CERTAIN POLYNOMIAL EXPANSION

By FLORA STREETMAN and L. R. FORD, Rice Institute

1. *Introduction.* The present paper contains an investigation of the expansion of a function $f(z)$, analytic at the origin, in a series

$$f(z) = P_0(z) + P_1(z) + P_2(z) + \cdots, \tag{1}$$

where $P_n(z)$ is a polynomial of degree n at most whose explicit form is

$$\begin{aligned} P_n(z) = \frac{1}{(1+h)^{n+1}}\Big[& h^n f(0) + nh^{n-1}f'(0)z + \frac{n(n-1)}{(2!)^2}h^{n-2}f''(0)z^2 \\ & + \frac{n(n-1)(n-2)}{(3!)^2}h^{n-3}f'''(0)z^3 + \cdots + \frac{1}{n!}f^{(n)}(0)z^n\Big], \end{aligned} \tag{2}$$

h being a positive constant. If $f(z)$ has the series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

(2) may be written in the alternative form

$$P_n(z) = \frac{h^n}{(1+h)^{n+1}}\left[a_0 + na_1\frac{z}{h} + \frac{n(n-1)}{2!}a_2\left(\frac{z}{h}\right)^2 + \cdots + a_n\left(\frac{z}{h}\right)^n\right].$$

A region S about the origin in which the series converges and represents the function is constructed. This region is larger than the circle of convergence of Maclaurin's series for the function unless the circle of convergence is the natural boundary; hence the series (1) provides a formula for the analytic continuation of the function beyond the limits of the circle of convergence.

2. *Derivation of the Series.* Let Γ be a regular curve enclosing the origin, within which $f(z)$ is analytic and on which $f(z)$ is continuous. We have from Cauchy's formula for z in Γ

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}. \tag{3}$$

Writing $t-z=(1+h)t-(z+ht)$, we find

$$\frac{1}{t-z} = \frac{1}{(1+h)t} + \frac{z+ht}{(1+h)^2t^2} + \cdots + \frac{(z+ht)^m}{(1+h)^{m+1}t^{m+1}} + \frac{(z+ht)^{m+1}}{(1+h)^{m+1}t^{m+1}(t-z)};$$

whence, substituting in (3),

$$f(z) = P_0(z) + P_1(z) + \cdots + P_m(z) + R_m(z),$$

where

$$P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+ht)^n}{(1+h)^{n+1}t^{n+1}}f(t)dt,$$

$$R_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+ht)^{m+1}}{(1+h)^{m+1}t^{m+1}(t-z)}f(t)dt.$$

It is clear that $P_n(z)$ is a polynomial of degree n at most. Its value is

$$P_n(z) = \frac{1}{n!} \cdot \frac{1}{(1+h)^{n+1}} \frac{d^n}{dt^n}[(z+ht)^n f(t)]_{t=0}.$$

On differentiating n times and setting $t=0$, we have the polynomial appearing in (2).

3. *The Major Circle.* The major circle for any point z is the circle with center $-hz$ and radius $(1+h)|z|$. (See M, Figure 1, constructed for $h=1$.)

The major circles for all points t on Γ are now constructed. Let σ be a region whose points lie within all such major circles. A sufficiently small neighborhood of the origin, for example, has this property.

THEOREM I. *In any region σ' wholly within σ the series* (1) *converges uniformly to the function $f(z)$.*

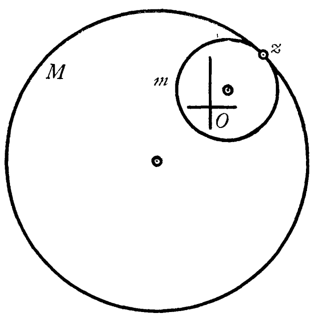


FIG. 1. Major and Minor Circles.

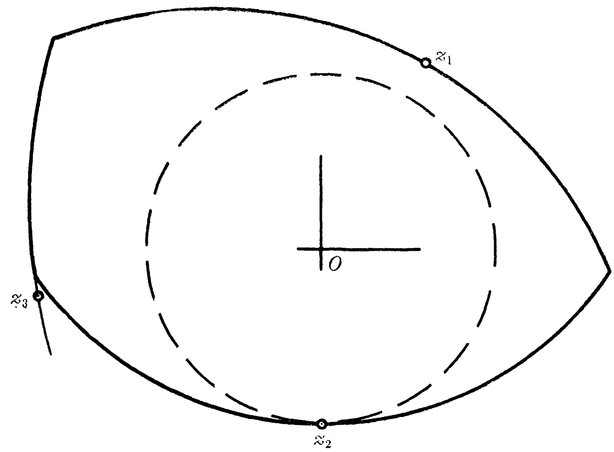


FIG. 2. The Region S.

If z lies in M_t, the major circle for a point t on Γ, its distance from the center $-ht$ of M_t is less than the distance from t to $-ht$; and

$$|(z+ht)/(t+ht)| < 1.$$

It can be shown that for any z in σ' and any t on Γ

$$|(z+ht)/(t+ht)| < r < 1.$$

Then since

$$\left|\frac{1}{2\pi i}\frac{f(t)}{t-z}\right| < K, \text{ a constant,}$$

we have

$$|R_m(z)| < r^{m+1}K\gamma/2\pi,$$

where γ is the length of Γ, and the remainder $R_m(z)$ approaches zero uniformly as m approaches infinity for all values of z in σ'.

4. *The Region S.* Let a ray, that is, a half-line, issue from the origin. We proceed from the origin along this ray until a singularity of the function is encountered, if any. We then construct the major circle for this singularity. We do this for each ray. The region within all the major circles so constructed will be called the region S.

In Figure 2 we show S for a function with three singularities, taking $h=1$. The broken circle is the circle of convergence of Maclaurin's series. It is clear that all points of the circle of convergence at which the function is analytic are interior points of S.

5. *The Minor Circle.* The minor circle for any point z is the circle with center $h(1+2h)^{-1}z$ and radius $(1+h)(1+2h)^{-1}|z|$. (See m, Figure 1.)

THEOREM II. *If z_1 lies inside, on, or outside the minor circle of z_2, then z_2 lies respectively outside, on, or inside the major circle of z_1.*

If z_1 lies inside, on, or outside the minor circle m_2 of z_2, its distance from the center of m_2 is respectively less than, equal to, or greater than the radius of m_2. We have then

$$\left| z_1 - \frac{h}{1+2h} z_2 \right| \lesseqqgtr \frac{1+h}{1+2h} | z_2 | ,$$

using the upper, middle, or lower sign when z_1 lies respectively inside, on, or outside m_2. Letting

$$z_1 = x + iy, \quad z_2 = m + in,$$

and substituting in the above, we have

$$\left| x + iy - \frac{h}{1+2h}(m + in) \right| \lesseqqgtr \frac{1+h}{1+2h} | m + in | ,$$

which gives

$$[(1 + 2h)x - hm]^2 + [(1 + 2h)y - hn]^2 \lesseqqgtr (1 + h)^2(m^2 + n^2).$$

On expanding and rearranging terms, we have

$$(m + hx)^2 + (n + hy)^2 \gtreqqless (1 + h)^2(x^2 + y^2).$$

Thus z_2 lies outside, on, or inside the circle with center

$$m = - hx, \quad n = - hy, \quad m + in = - hz,$$

and radius

$$(1 + h)(x^2 + y^2)^{1/2} = (1 + h) | z_1 | ;$$

that is, the major circle of z_1.

The region σ of Section 3 can now be given a new interpretation. It is a region such that the minor circle of any of its interior points lies within Γ.

Theorem III. *A necessary and sufficient condition that $f(z)$ be analytic within and on the minor circle of z_2 is that z_2 lie in S.*

(1). If z_2 lies in S, then $f(z)$ is analytic in and on the minor circle of z_2.

From Theorem II we know that if z_2 lies inside the major circle of z_1, z_1 lies outside the minor circle of z_2. Thus since z_2 lies within S, it lies within the major circles of all the singularities, and hence all the singularities lie outside of the minor circle of z_2.

(2). If $f(z)$ is analytic in and on m_2, the minor circle of z_2, then z_2 lies in S.

Since $f(z)$ is analytic in and on m_2, all singular points lie outside m_2. Hence by Theorem II, z_2 lies inside the major circles of all singularities, that is, in the region S.

It will be observed that the minor circle of a point S is not necessarily wholly within S.

6. *Convergence of the Series in S.* The proof of the convergence of the series rests on the following proposition.

Theorem IV. *For any point z_2 in S, there exists a circle, enclosing z_2 and the origin, within and on which $f(z)$ is analytic, and such that the major circles of all points on the circle enclose a region in S about the point z_2.*

A circle Γ concentric with the minor circle for z_2, m_2, and of slightly larger radius can be constructed such that $f(z)$ is analytic in and on Γ, since we know $f(z)$, by Theorem III, to be analytic in and on m_2.

Since the center and radius of the minor circle are continuous functions of z, it follows that for z in a sufficiently small circle σ about z_2, the minor circle of z lies in Γ. Any point t on Γ is outside the minor circle of any point of σ; hence, from Theorem II, any point of σ is within the major circle of t.

We know by Theorem I that the series of polynomials converges uniformly to $f(z)$ in σ. Since we can find such a circle σ for any point within S, we can state the following theorem:

Theorem V. *The series of polynomials*

$$P_0(z) + P_1(z) + \cdots + P_n(z) + \cdots$$

converges to $f(z)$ in the region S.

The following theorem is also readily established.

Theorem VI. *The convergence of the series of polynomials is uniform in any region S' lying wholly within S.*

About each interior or boundary point of S' there exists a circle in which the series converges uniformly. Since by the Heine-Borel theorem S' can be covered by a finite number of these circles, the convergence is seen to be uniform for the region S'.

INTEGRAL SOLUTIONS OF $ax^3+by^3=az^3+bt^3$

By PERRY A. CARIS, University of Pennsylvania

1. *Introductory Remarks.* The Diophantine equation

$$ax^3 + by^3 = az^3 + bt^3, \tag{1}$$

where a and b are any non-zero integers seems never to have been treated. The special case $a=b=1$ has been solved by Euler,[1] Binet,[2] Carmichael[3] and others. The case $a=2$, $b=1$, has been treated by Amsler[4] and Gérardin.[4]

It is the purpose of this paper to find non-trivial solutions of (1) no matter what permissible values a and b may have. Some trivial solutions are obvious. For example, $x=z$, $y=t$. But it is perhaps not so obvious that $xy=tz$ would imply a trivial solution. To make this clear let $t=xy/z$ in (1). Then (1) reduces to

$$ax^3z^3 + by^3z^3 = az^6 + bx^3y^3.$$

That is

$$az^3(x^3 - z^3) = by^3(x^3 - z^3)$$

which, since $x \neq z$, implies $az^3=by^3$ and therefore also $ax^3=bt^3$ which is trivial. Hence these assumptions are made: $x\neq z$ and $xy\neq tz$.

2. *Solution of the given equation.* Write (1) in the form

$$a(x^3 - z^3) = b(t^3 - y^3).$$

This equation will be satisfied if

$$\begin{aligned} ap(x-z) &= bq(t-y), \\ q(x + \tfrac{1}{2}z - \tfrac{1}{2}z\sqrt{(-3)}) &= (r + s\sqrt{(-3)})(t + \tfrac{1}{2}y - \tfrac{1}{2}y\sqrt{(-3)}), \\ (r + s\sqrt{(-3)})(x + \tfrac{1}{2}z + \tfrac{1}{2}z\sqrt{(-3)}) &= p(t + \tfrac{1}{2}y + \tfrac{1}{2}y\sqrt{(-3)}). \end{aligned}$$

These three equations imply the following in real numbers:

$$\begin{aligned} apx + bqy - apz - bqt &= 0, \\ 2qx - (r+3s)y + qz - 2rt &= 0, \\ (s-r)y + qz + 2st &= 0, \\ 2rx - py + (r-3s)z - 2pt &= 0, \\ 2sx - py + (r+s)z &= 0. \end{aligned} \tag{2}$$

From the first three of these equations $x:y:z:t=$

---

[1] Dickson's *History of the Theory of Numbers*, vol. 2, pp. 552–554.

[2] *Ibid.*, pp. 554–555.

[3] Carmichael's *Diophantine Analysis*, pp. 62–66.

[4] Dickson's *History of the Theory of Numbers*, vol. 2, p. 562.

$$\begin{vmatrix} bq & -ap & -bq \\ -r-3s & q & -2r \\ s-r & q & 2s \end{vmatrix} : \begin{vmatrix} ap & -ap & bq \\ 2q & q & 2r \\ 0 & q & -2s \end{vmatrix} : \begin{vmatrix} ap & bq & -bq \\ 2q & -r-3s & -2r \\ 0 & s-r & 2s \end{vmatrix}$$

$$: \begin{vmatrix} ap & -bq & -ap \\ 2q & r+3s & q \\ 0 & r-s & q \end{vmatrix},$$

whence

$$\begin{aligned} x &= \lambda_1(bq^2r + 3bq^2s - apr^2 - 3aps^2), \\ y &= \lambda_1(bq^3 - 3apqs - apqr), \\ z &= \lambda_1(bq^2r - 3bq^2s - apr^2 - 3aps^2), \qquad \lambda_1 \neq 0. \\ t &= \lambda_1(bq^3 + 3apqs - apqr). \end{aligned} \tag{3}$$

The substitution of these values in the fourth and the fifth equations of (2) gives

$$(bq^2 - apr + aps)(r^2 + 3s^2 - pq) = 0$$

and

$$(bq^2 - apr - 3aps)(r^2 + 3s^2 - pq) = 0,$$

respectively. These will both be satisfied if

$$pq = r^2 + 3s^2. \tag{4}$$

Now the x of (3) equals

$$\lambda_1[bq^2r + 3bq^2s - ap(r^2 + 3s^2)] = \lambda_1(bq^2r + 3bq^2s - ap^2q). \tag{5}$$

Similarly, the z of (3) equals

$$\lambda_1(bq^2r - 3bq^2s - ap^2q). \tag{6}$$

If $q=0$, $x=z=0$ and the result is trivial. Hence $q\neq 0$. Let $\lambda_1=\lambda/q$ and combine (5) and (6) with y and t of (3) and obtain the solution

$$\begin{aligned} x &= \lambda(bqr + 3bqs - ap^2), \\ y &= \lambda(bq^2 - 3aps - apr), \\ z &= \lambda(bqr - 3bqs - ap^2), \\ t &= \lambda(bq^2 + 3aps - apr), \end{aligned} \tag{7}$$

with the condition $pq=r^2+3s^2$.

Consider now the converse problem. Let x, y, z, t be non-trivial values satisfying (1). It remains to show that λ, p, q, r, s can be found to give precisely these values of x, y, z, t. From the second and the third equations of (2),

$$q:r:s = [(2y + t)^2 + 3t^2]:(2xy + 4xt + 4yz + 2tz):(2xy - 2tz).$$

If q, r, s be taken equal to these values, then (4) will be satisfied if

$$p = (r^2 + 3s^2)/q = (2x + z)^2 + 3z^2.$$

Now from (7), $x-z=6bq\lambda s$, whence

$$\lambda = \frac{x - z}{48b(y^2 + yt + t^2)(xy - tz)}\,.$$

Since the supposed solution is non-trivial, λ is neither 0 nor ∞. Moreover these values of λ, p, q, r, s will satisfy (7). Consider, for example, $\lambda(bqr+3bqs-ap^2)$. This equals

$$\text{(8)} \qquad \frac{x - z}{48b(y^2 + yt + t^2)(xy - tz)}[8b(y^2 + yt + t^2)(xy + 2xt + 2yz + tz)$$

$$+ 24b(y^2 + yt + t^2)(xy - tz) - 16a(x^2 + xz + z^2)^2]$$

$$= \frac{x - z}{3(xy - tz)}\left[2xy + xt + yz - tz - \frac{a(x^2 + xz + z^2)}{b}\left(\frac{x^2 + xz + z^2}{y^2 + yt + t^2}\right)\right].$$

But, by hypothesis,

$$a(x^3 - z^3) = b(t^3 - y^3).$$

That is

$$a(x - z)(x^2 + xz + z^2) = b(t - y)(y^2 + yt + t^2),$$

whence

$$\frac{x^2 + xz + z^2}{y^2 + yt + t^2} = \frac{b(t - y)}{a(x - z)}\,.$$

Then (8) can be continued thus:

$$\frac{x - z}{3(xy - tz)}\left[2xy + xt + yz - tz - \frac{(t - y)(x^2 + xz + z^2)}{x - z}\right]$$

$$= \frac{1}{3(xy - tz)}(3x^2y - 3xtz) = x.$$

The work for y, z, t is similar.

The solution (7) may be expressed in terms of independent parameters by giving explicit values to p, q, r, s. Let $p=c^2+3d^2$; $q=f^2+3g^2$. Then by the method of composition of binary quadratic forms,

$$r = cf \pm 3dg; \; s = cg \mp df.$$

A SIMPLIFIED INTEGRAL TEST FOR THE CONVERGENCE OF INFINITE SERIES

By RAYMOND W. BRINK, University of Minnesota

In other papers[1] the author has presented certain integral tests for the convergence and divergence of infinite series. Such tests are interesting not only because they can be used for testing types of series which are very difficult to examine by other methods, but also because, through the natural connection between integration and summation, they offer a simple and attractive means of unifying and establishing many tests of other kinds.

DuBois–Reymond[2] gave the name "tests of the first kind" to series tests which make direct use of the general term of the series itself. And "tests of the second kind" are those which use the ratio of the general term to the preceding term. In this category are the d'Alembert test, which every student of Calculus knows as "the" ratio test, the first test of Raabe, and the sequence of tests of de Morgan and Bertrand.

In a similar way, the familiar Maclaurin-Cauchy integral test,[3] which requires the use of a function $u(x)$ where $u(n)$ is the general term of the series, may be called an "integral test of the first kind." And an "integral test of the second kind" is one in which an analogous rôle is played by a function $r(x)$, where $r(n)$ is the ratio of the nth term to the preceding term.

Hitherto the most generally useful integral test of the second kind was given by the author in one of the papers mentioned.[4] Its statement and proof are simple for convergence, but rather awkward for the divergence test. The purpose of the present paper is to give a modified form of the test—a form that is as widely applicable as the earlier one and that is simpler to state and to establish. Two proofs will be given, first a very short proof based on a theorem of the earlier paper and second a direct proof so simple as to be available to good students of intermediate grade. The condition of limited variation which is imposed on the ratio-function $r(x)$ is sufficiently general for all cases met in practice, and, of course, includes such cases as that in which $r(x)$ increases monotonically toward the limit 1.

Theorem: *Given a series*

$$u_0 + u_1 + u_2 + \cdots, \quad (u_n > 0,\ n \geqq \mu).$$

Let $r(x)$ be a function with a derivative $r'(x)$ satisfying the following conditions for values of x greater than or equal to a certain integer μ:

---

[1] Transactions of the American Mathematical Society, vol. 19 (1918), p. 186. Annals of Mathematics, Ser. 2, vol. 21 (1919), p. 39.

[2] Journal für Mathematik, vol. 76 (1891), p. 61.

[3] Maclaurin, *Treatise on Fluxions* (1742), vol. 1, p. 289. Cauchy, *Exercises de Mathématique* (1827), vol. 2, p. 221.

[4] Transactions of the American Mathematical Society, vol. 19 (1918), p. 195.

$$r(n) = r_n = u_{n+1}/u_n, \tag{1}$$

$$0 < A \leqq r(x) \leqq B, \tag{2}$$

$$\int_\mu^x | r'(x) | \, dx < D, \; x > \mu. \tag{3}$$

Then the convergence of the integral

$$\int_\mu^\infty e^{R(\mu, x)} dx, \text{ where } R(\mu, x) = \int_\mu^x [r(x) - 1] dx, \tag{A}$$

is sufficient for the convergence of the given series; and the divergence of the integral

$$\int_\mu^\infty e^{\rho(\mu, x)} dx, \quad \text{where } \rho(\mu, x) = \int_\mu^x [1 - 1/r(x)] dx, \tag{B}$$

implies the divergence of the series.

Under the conditions (1), (2) and (3) the convergence of the integral

$$\int_\mu^\infty e^{\sigma(\mu, x)} dx, \text{ where } \sigma(\mu, x) = \int_\mu^x \log r(x) dx, \tag{C}$$

is known[1] to be necessary and sufficient for the convergence of the series. Then the theorem follows from the familiar inequalities[2]

$$\left.\begin{array}{l} \log r(x) \leqq r(x) - 1 \\ \log r(x) \geqq 1 - 1/r(x) \end{array}\right\}, \quad 0 < r(x).$$

For the convergence of (A) is sufficient for the convergence of (C) and therefore for that of the series. And if (B) diverges, so do (C) and the series.

We will now establish the theorem without the use of the integral (C). Set

$$d_n = \int_n^{n+1} [r(x) - 1] dx - (r_{n+1} - 1) = \int_n^{n+1} [r(x) - r_{n+1}] dx. \tag{4}$$

Integrating by parts, we get

$$d_n = - \int_n^{n+1} (x - n) r'(x) dx \tag{5}$$

so that

$$| d_n | \leqq \int_n^{n+1} | r'(x) | \, dx. \tag{6}$$

Therefore, by (3),

$$\sum_{i=\mu} | d_i | < D, n > \mu, \text{ and } \sum_{i=\mu}^{n} d_i > -D.$$

[1] Transactions of the American Mathematical Society, vol. 19 (1918), p. 188, Theorem II.
[2] See, for instance, Pringsheim, Mathematische Annalen, vol. 35, p. 317.

If we use a bar to indicate that the integrals under it have the same integrand, for any value of x for which $n \leqq x \leqq n+1$,

$$R(\mu, x) = \int_{\mu}^{x} [r(x) - 1]dx = \overline{\int_{\mu}^{\mu+1} + \int_{\mu+1}^{\mu+2} + \cdots + \int_{n-1}^{n} + \int_{n}^{x}}[r(x) - 1]dx$$

$$(7) \qquad = \sum_{i=\mu}^{n-1} d_i + \sum_{i=\mu}^{n-1}(r_{i+1} - 1) + \int_{n}^{x} [r(x) - 1]dx$$

$$> -D - 1 + (r_{\mu+1} - 1) + (r_{\mu+2} - 1) + \cdots + (r_n - 1).$$

Therefore

$$(8) \qquad \int_{n}^{n+1} e^{R(\mu,x)}dx > e^{-D-1} \cdot e^{(r_{\mu+1}-1)} \cdot e^{(r_{\mu+2}-1)} \cdots e^{(r_n-1)}.$$

Since $e^{z-1} \geqq z$,

$$(9) \qquad \int_{n}^{n+1} e^{R(\mu,x)}dx > e^{-D-1} \cdot r_{\mu+1} \cdot r_{\mu+2} \cdots r_n = \frac{e^{-D-1}}{u_{n+1}} u_{\mu+1}, \ (\mu \leqq n).$$

Therefore if the integral (A) converges, the series

$$\sum_{n=\mu}^{n} u_n$$

also converges.

To establish the test for divergence we proceed in a similar way, this time setting

$$d_n = \int_{n}^{n+1} [1 - 1/r(x)]dx - (1 - 1/r_{n+1}),$$

and using the inequality

$$e^{1-(1/z)} \leqq z, \ (0 < z).$$

In its modified form, this integral test can be applied to many special series of the normal type

$$c + cr_0 + cr_0r_1 + cr_0r_1r_2 + \cdots$$

where $r_n = r(n)$, $r(x)$ being a known simple function. For example, consider the series

$$(10) \quad 1 + (1 - k) + (1 - k)\left(1 - \frac{k}{2}\right) + (1 - k)\left(1 - \frac{k}{2}\right)\left(1 - \frac{k}{3}\right) + \cdots.$$

Since the function $r(x)=1-(k/x)$ is finite and monotonic, the conditions for the test are satisfied. Then, if $k>1$, the series converges, for

$$R(\mu, x) = -\int_{\mu}^{x} (k/x)dx = k \log (c/x)$$

and the integral

$$\int_{\mu}^{\infty} e^{R(\mu,x)}dx = \int_{\mu}^{\infty} (c/x)^k dx$$

converges. Similarly, if $k \leqq 1$, the series diverges, for the integral

$$\int_{\mu}^{\infty} e^{\rho(\mu,x)}dx = \int_{\mu}^{\infty} \frac{c\,dx}{(x-k)^k}$$

diverges.

Moreover, as has already been suggested, it is possible to give a discussion, based on this integral test, of the tests of the logarithmic scale and of most of the other tests of the second kind. For example, to establish Raabe's first test,[1] which depends on the value of $l=\lim_{n=\infty}\omega_n$, where $r_n=1-(\omega_n/n)$, we need only to compare the given series with the series (10) of the preceding example, taking k between l and 1.

The extraordinary power of the present integral test and the simplicity of its conditions as compared with the divergence test of the earlier article is shown by the ease with which it can be used to establish the sequence of Bertrand's delicate tests:[2]

If

$$\frac{1}{r(x)} = 1 + \frac{1}{x} + \frac{1}{x \cdot l_1 x} + \cdots + \frac{1}{x \cdot l_1 x \cdots l_{k-1} x} + \frac{\omega(x)}{x \cdot l_1 x \cdots l_k x},$$

and if $\lim_{x=\infty}\omega(x)=l$, *the corresponding series converges when* $l>1$ *and diverges when* $l<1$. Thus suppose that $l<1$. Take λ between l and 1, and compare the given series with a series for which

$$\frac{1}{r(x)} = 1 + \frac{1}{x} + \cdots + \frac{1}{x \cdot l_1 x \cdots l_{k-1} x} + \frac{\lambda}{x \cdot l_1 x \cdots l_k x}.$$

For this series,

$$\int_{\mu}^{\infty} e^{\rho(x)}dx = c\int_{\mu}^{\infty} \frac{dx}{x \cdot l_1 x \cdot l_2 x \cdots l_{k-1} x (l_k x)^{\lambda}},$$

a divergent integral. The given series therefore diverges.

The present integral test can be extended to multiple series and also to series of functions. In the latter case, uniform convergence of the integral implies uniform convergence of the corresponding series.

---

[1] Journal für Mathematik, vol. 11, p. 309.

[2] Journal de Mathématiques, vol. 7, p. 42.

ON A CURVE ASSOCIATED WITH A TRIANGLE

By JAMES H. WEAVER, Ohio State University

Let there be a triangle $A_1A_2A_3$ and a point P in the plane of the triangle (see Fig. 1) determined as follows:

$$\angle A_iPA_j = \angle (m\pi + kA_n) \quad (i, j, n, = 1, 2, 3, i \neq j \neq n). \tag{1}$$

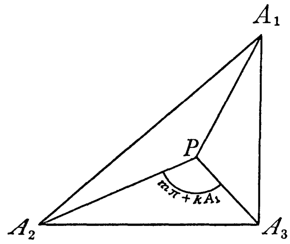


FIG. 1

From (1) we have, by addition, $3m+k=2$. If we transform (1) by means of

$$3m + k = 2, \ A_n = \tfrac{1}{3}\pi - A_n', \tag{2}$$

we obtain

$$\angle A_iPA_j = \angle (\tfrac{2}{3}\pi - kA_n'). \tag{3}$$

If now k is allowed to vary the point P will describe a curve C.

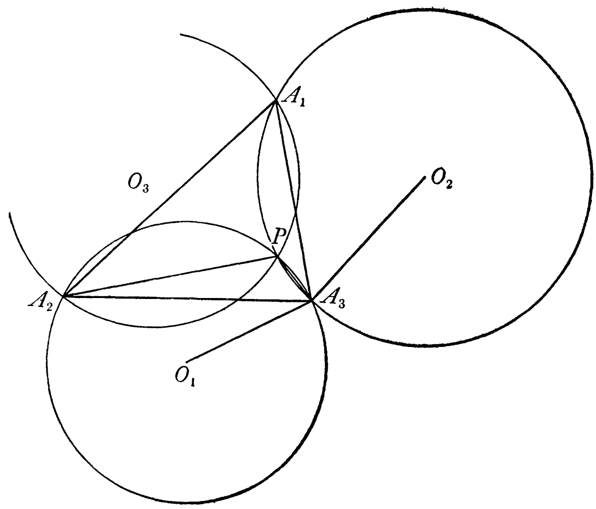


FIG. 2

Points on the curve C may be constructed by drawing on A_iA_j segments of circles which will contain the angles $\frac{2}{3}\pi - kA_n'$. (See Fig. 2). Equations (3) show that if $A_n'=0$, the circle through A_iPA_j will be fixed. We therefore have this theorem:

If one of the angles of the triangle is $\frac{1}{3}\pi$ the locus of P is a circle passing through the other two vertices.

It is obvious that the curve will pass through a vertex only when one of the circles is a circumcircle and the other two are tangent to each other at a vertex. To determine which circle must be a circumcircle we proceed as follows. Consider the point A_3. The circle on A_1PA_2 will be a circumcircle if

$$\angle\ (\tfrac{2}{3}\pi - kA_3') = \angle\ (A_3 + x\pi) \quad (x \text{ an integer}).$$

Draw from O_1 the center of the circle through A_2PA_3 a line to A_3. Then

$$\angle\ O_1A_3A_2 = \angle\ (\tfrac{1}{6}\pi - kA_1').$$

Similarly for the circle on A_3PA_1 we have

$$\angle\ O_2A_3A_1 = \angle\ (\tfrac{1}{6}\pi - kA_2').$$

Also

$$\begin{aligned}\angle O_1A_3A_2 + \angle O_2A_3A_1 + \angle A_3 &= \angle\ (\tfrac{1}{3}\pi - k(A_1' + A_2')) + \angle\ A_3.\\ &= \angle\ (\tfrac{1}{3}\pi + kA_3' + A_3)(\text{since } A_1' + A_2' + A_3' = 0)\\ &= \angle\ [\pi - (\tfrac{2}{3}\pi - kA_3') + A_3]\\ &= \angle\ (\pi - A_3 - x\pi + A_3\\ &= \angle\ (1 - x)\pi.\end{aligned}$$

Therefore angle $O_1A_3O_2$ is a straight angle. It is evident that the point A_3 will not be a point of the curve C unless these conditions hold. These facts give the following theorem:

A vertex of the triangle will be a point of the curve C, when and only when the circle on the opposite side as segment, becomes a circumcircle.

Let the differences A_n' be commensurable. We then have $|A_1'|:|A_2'|:|A_3'| = p_1:p_2:p_3$, where the p_i are relatively prime integers. Consider again the point A_3. It will be a point of the curve for more than one value of k. Let k_1 and k_2 be two consecutive values of k which make P_3 a point of C. Then $(k_2-k_1)A_3' = \pi$. But while this change takes place, the angle $O_2A_2A_1$ changes $(k_2-k_1)A_2' = (p_2/p_3)\pi$. And since p_2 and p_3 are relatively prime it will take p_3 rotations to bring the line O_2A_3 to its original position. Moreover all such rotations will be equal, and the construction shows that the tangents to C at A_3 will be perpendicular to O_2A_3. Hence the theorem:

If the angles of a triangle differ from $\pi/3$ in the ratio $p_1:p_2:p_3$ and the p's are relatively prime integers, the curve C has a multiple point of order p_i at A_i and the tangents at these multiple points divide the angular magnitude into $2p_i$ equal parts.

If there are other multiple points on C, the centers of the circles determining them will be identical for each branch of C through the point and will approach their respective positions always from the same direction. Hence there can be no other multiple points on C.

The coordinates of the point P may be written:

$$\alpha_i = M\frac{\sin\,(\tfrac{2}{3}\pi - kA_i')}{\sin\,(\tfrac{1}{3}\pi - (k-1)A_i')} \tag{4}$$

where $M = f(k)$.

If the A_i' are commensurable the equation of the curve will be algebraic. Its equation may be obtained as follows.

Equation (4) may be written:

$$\tan kA_i' = \frac{\alpha_i \sin(\frac{1}{3}\pi + A_i') - M \sin \frac{1}{3}\pi}{\alpha_i \cos(\frac{1}{3}\pi + A_i') - M \cos \frac{1}{3}\pi}, \tag{5}$$

Let $A_1' = p_1\theta$, $A_2' = p_2\theta$, $A_3' = -p_3\theta$. Then since

$$\tan(p_2 kA_1') = \tan(p_1 kA_2'), \tag{6}$$

a substitution from (5) in (6) gives

$$\begin{aligned} &\frac{\alpha_1^{p_2}\sin\left(\frac{p_2\pi}{3}+p_1p_2\theta\right)+p_2\alpha_1^{p_2-1}M\sin\left[(p_2-2)\frac{\pi}{3}+(p_1p_2-p_1)\theta\right]+\cdots+M^{p_2}\sin-\frac{p_2\pi}{3}}{\alpha_1^{p_2}\cos\left(\frac{p_2\pi}{3}+p_1p_2\theta\right)+p_2\alpha_1^{p_2-1}M\cos\left[(p_2-2)\frac{\pi}{3}+(p_1p_2-p_1)\theta\right]+\cdots+M^{p_2}\cos-\frac{p_2\pi}{3}},\\ &=\frac{\alpha_2^{p_1}\sin\left(\frac{p_1\pi}{3}+p_1p_2\theta\right)+p_1\alpha_2^{p_1-1}M\sin\left[(p_1-2)\frac{\pi}{3}+(p_1p_2-p_2)\theta\right]+\cdots+M^{p_1}\sin-\frac{p_1\pi}{3}}{\alpha_2^{p_1}\cos\left(\frac{p_1\pi}{3}+p_1p_1\theta\right)+p_1\alpha_2^{p_1-1}M\cos\left[(p_1-2)\frac{\pi}{3}+(p_1p_2-p_2)\theta\right]+\cdots+M^{p_1}\cos-\frac{p_1\pi}{3}}, \end{aligned} \tag{7}$$

an equation of degree p_1+p_2 in M.

Moreover $\tan(kA_1'+kA_2') = -\tan kA_3'$, and this equation by virtue of (5) simplifies to

$$M = \frac{\alpha_2\alpha_3 \sin A_1' + \alpha_1\alpha_3 \sin A_2' + \alpha_1\alpha_2 \sin A_3'}{\alpha_1 \sin A_1' + \alpha_2 \sin A_2' + \alpha_3 \sin A_3'}. \tag{8}$$

The value of M in (8) when substituted in (7) gives a curve of degree $p_1+p_2+p_3$ in the α's. We therefore have the following theorem.

If p_1, p_2 and p_3 are relatively prime integers the curve C is algebraic and of degree $p_1+p_2+p_3$.

Fig. 3 shows the curve for $p_1=1$, $p_2=2$ and $p_3=3$.

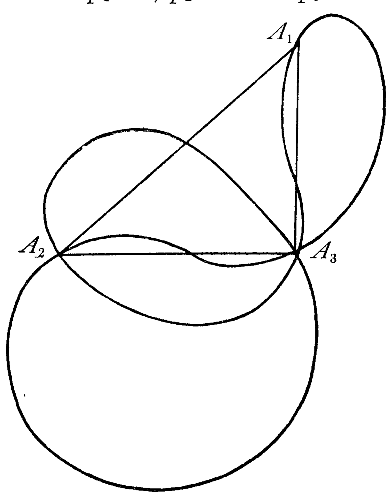


FIG. 3

MEAN VALUE OF THE ORDINATE OF THE LOCUS OF THE RATIONAL INTEGRAL ALGEBRAIC FUNCTION OF DEGREE n EXPRESSED AS A WEIGHTED MEAN OF $n+1$ ORDINATES AND THE RESULTING RULES OF QUADRATURE

By BENJAMIN F. GROAT, Brookline, Mass.

From time to time there have appeared summaries of such rules as Newton's, Simpson's (so called, but due to Cotes), and others, for "approximating" the areas of inclosures, and the subject of interpolation has received attention by Newton, Lagrange, and Gauss. But the author is not aware of any general formulas for finding the averaging weights of arbitrarily chosen ordinates, such as will give the true area under the general curve of rational integral algebraic form. Nor does he know of an instance in which a rule has been given for finding the mean ordinate of such a curve as the arithmetical mean of a determinate number of properly situated ordinates.

The following treatise produces the mean ordinate under a given segment of the general curve as a weighted mean, along with associated general formulas for the weights of the ordinates. All the usual special rules mentioned, and doubtless others, may be derived immediately from these formulas as special cases, while any new arbitrary disposition of ordinates may be analyzed. It is urged, in particular, that the rules herein formulated for the arithmetical mean are generally the most direct and most easily applied, especially those for which the ordinates are equally spaced. There is an infinite number of ordinate dispositions each furnishing the arithmetical mean.

Let the general integral algebraic curve be

$$y = a\xi^{n-1} + b\xi^{n-2} + \cdots + \epsilon, \tag{1}$$

n being the number of ordinates to be employed with it. The base above which the area is to be found is represented by D. As the results are independent of the coordinate axes, the origin may be at one extremity of the base. The coordinates of the heads of the ordinates are, respectively, $(\xi_1 y_1), \cdots, (\xi_n y_n)$. The mean value of the ordinate over the base, D, is represented by V, such that the area is $A = VD$.

The essence of the problem consists in finding V as an arbitrarily weighted mean of n ordinates, or, in terms of parameters, the n parameters, $a, \cdots, \epsilon$, are to be eliminated from the expression for the mean value of the ordinate, while the n ordinates, $y_1, \cdots, y_n$, erected above the base, are to be introduced in their stead.

The mean value of the ordinate of A is

$$V = \frac{A}{D} = \frac{aD^{n-1}}{n} + \frac{bD^{n-2}}{n-1} + \cdots + \epsilon. \tag{2}$$

The difference between the mean value, (2), of the ordinate, and the value, (1), of any particular ordinate may be found by subtraction. Then, since there

may be calculated (or measured) n ordinates at arbitrarily chosen distances from the origin, there may be formed n equations of the type

$$(3) \qquad (V - y_z) + aD^{n-1}\left(x_z{}^{n-1} - \frac{1}{n}\right) + bD^{n-2}\left(x_z{}^{n-2} - \frac{1}{n-1}\right) + \cdots = 0,$$

in which the relative abscissa, x, is written for $\xi \div D$, while z may be any integer from 1 to n.

From these n equations the remaining $n-1$ parameters, a, b, $\cdots$, may be eliminated, ϵ having been eliminated by the subtractions. The eliminating determinant is

$$(4) \qquad \left[(V - y_1), \left(x_2^{n-1} - \frac{1}{n}\right), \left(x_3^{n-2} - \frac{1}{n-1}\right), \cdots, \left(x_n - \frac{1}{2}\right)\right]$$

which, placed equal to zero and solved for V, yields

$$(5) \qquad V = \frac{\sum y_z C_z}{\sum C_z},$$

wherein C_z is the cofactor of y_z in the determinant (4).

Thus the mean ordinate is a weighted mean of the n calculated (measured in practical mensuration) ordinates. The weights depend entirely upon the relative distances of the ordinates from the origin, or conversely. In particular, the weight of y_m, in determinant form, is

$$(6)\ W_m = C_m = (-1)^{m+1}\left[\left(x_1^{n-1} - \frac{1}{n}\right), \left(x_2^{n-2} - \frac{1}{n-1}\right), \cdots, \left(x_n - \frac{1}{2}\right)\right]$$

from which the constituents containing the symbols $x_m{}^z$ must be omitted.

This determinant clearly breaks up into 2^{n-1} determinants,all but n of which vanish, thus leaving

$$(7) \quad W_m = (-1)^{m+n}\left\{\frac{1}{n}[x_1^{n-2}, x_2^{n-3}, \cdots, x_n, 1] - \frac{1}{n-1}[x_1^{n-1}, x_2^{n-3}, \cdots, x_n, 1] \right.$$
$$\left. + \cdots + (-1)^{n-1}[x_1^{n-1}, x_2^{n-2}, \cdots, x_n]\right\},$$

in which terms, excepting the last, the successive determinants, each containing $n-2$ exponents, are written out by omitting, progressively, one at a time, every exponent, beginning with the first, but including all in the last term. This is the general formula for the weight of an ordinate.

The series (7) of determinants is divisible by the determinant

$$(8) \qquad [x_1^{n-2}, x_2^{n-3}, \cdots, x_n, 1],$$

which is merely the product of all the binomial differences, $(n-1)(n-2)\div 2$ in number, (subscript m not appearing) which can be formed by taking all combinations of the constituent ordinates two at a time. If the mode of forming factors be: x_1-x_2, x_1-x_3, $\cdots$, x_2-x_3, x_2-x_4, $\cdots$, etc., following the order of the principal diagonal, the product is treated with the positive sign.

Therefore the relative weight C_m may be written

$$(9)\ W_m = (-1)^{m+n}P_m\left\{\frac{1}{n} - \frac{\sum x}{n-1} + \frac{\sum xx}{n-2} - + \cdots + (-1)^{n-1}(xxx\cdots x)\right\}$$

in which only $n-1$ relative abscissas appear, the abscissa subscript, m, being omitted on the right, and wherein

$$(10)\qquad P_m = (x_1 - x_2)(x_1 - x_3)\cdots(x_2 - x_3)(x_2 - x_4)\cdots(x_{n-1} - x_n),$$

the factors containing the subscript m being omitted in accord with (9).

Equation (9) is a convenient expression of the general formula (7) for the weight of any ordinate corresponding with the particular choice of the numerical value of m. By it, or any one of the formulas for W_m, one may deduce many of the rules of mensuration as special cases:

$n=2$. The curve is $y=a\xi+b$, a straight line. If $x_1=0$, $x_2=1$, i.e., if $\xi_1=0$, and $\xi_2=D$, in (1), then, by (9) the relative weights of y_1 and y_2 become equal, which is the well-known trapezoidal rule of mensuration, the mean ordinate being arithmetical mean of the two ordinates.

$n=3$. $x_1=0$, $x_2=\frac{1}{2}$, $x_3=1$. Then, by (9), the relative weights are as 1:4:1, "Simpson's rule," frequently employed in engineering to calculate approximate areas of irregular inclosures.

$n=4$. $x_1=0$, $x_2=\frac{1}{3}$, $x_3=\frac{2}{3}$, $x_4=1$. Then, by (9), $W_1:W_2:W_3:W_4=1:3:3:1$, Newton's rule, also employed to calculate areas.

$n=5$. $x_1=0$, $x_2=\frac{1}{4}$, $x_3=\frac{1}{2}$, $x_4=\frac{3}{4}$, $x_5=1$. By (9) the relative weights are, respectively, 7, 32, 12, 32, 7, which is a rule given in the 14th edition of the Encyclopaedia Britannica in an article on mensuration by W. F. Sheppard.

$n=6$. $x_1=0$, $x_6=1$, the base being divided into equal parts by equally-spaced ordinates as before. By (9) the weights are, respectively, 19, 75, 50, 50, 75, 19. This is probably a new rule altogether.

This method may be continued indefinitely.

Let attention now revert to $n=3$, but, instead of locating an ordinate at each end of the base, as has been customary, let the abscissas x and y (y may here represent a particular one of the abscissas for simplicity of symbols. y_z would be an ordinate.) be made arbitrary, but equally distant from the ordinate z in the central position. Then, by (9), since $z=\frac{1}{2}$ and $x+y=1$, the relative weights become

$$(11)\quad W_x = \frac{1}{24}(2x-1),\quad W_z = (2x-1)\left(x^2 - x + \frac{1}{6}\right),\quad W_y = \frac{1}{24}(2x-1)$$

when expressed in terms of x.

Equations (11) illustrate the general theorem that, in any system of ordinates symmetrically disposed as to the center of the base, the weights of any two ordinates equally distant from that center are equal, while the weight of the central ordinate, if there is one, depends upon the displacements of the others. Hence, if W_z be made to vanish, the mean ordinate of the area sought will be the arithmetical mean of the ordinates y_x and y_y. Thus, rejecting $x=\frac{1}{2}$,

$$x = \tfrac{1}{2} \pm \sqrt{(1/12)}, \tag{12}$$

while y is the complementary surd. This special case of the general formulas is well known to hydraulic engineers as the "two-tenths eight-tenths depth method" for making observations by current meters of velocities of flowing water at 0.2 and 0.8 the full depth of the stream, although the present analysis shows the terminology to be inexact.

This result shows, not only that the areas of all the parabolas passing through the two fixed points are equal, but, also, that this fixed area is equal to the area under the straight line through the two points as limited by the common base. Passing on to a cubic with four ordinates symmetrically placed as to center of base, it will be found, by making the weight of each of one pair of ordinates equal to zero, that the abscissas of the other pair must equal x in (12). Thus, the area under the cubic is equal to the area under the straight line, and also to a fixed area under a family of parabolas. Todhunter has called attention to the fixity of such an area under a variable curve of odd degree, but it does not seem that he called attention to the resulting equality of such areas for curves of lower degree, nor to the fact that the curve of highest degree may be of even degree (Todhunter's *Treatise on Integral Calculus*, p. 398, Art. 400).

Instead of making $W_z=0$, suppose it be demanded that all three weights shall be the same, z still being the central abscissa. It is merely necessary to equate W_z with either of the other weights, since, by the theorem just given, the latter are equal. Thus one easily finds

$$x = \tfrac{1}{2} - \sqrt{\tfrac{1}{8}}, \quad z = \tfrac{1}{2}, \quad y = \tfrac{1}{2} + \sqrt{\tfrac{1}{8}}. \tag{13}$$

This result gives the theorem: *When three velocity readings are taken respectively at* $\frac{1}{2}-\sqrt{\frac{1}{8}}$, $\frac{1}{2}$, *and* $\frac{1}{2}+\sqrt{\frac{1}{8}}$, *relative depth in the vertical velocity plane, the mean velocity in that plane is the arithmetical mean of the three observed velocities, or,*

$$V = (y_x + y_z + y_y) \div 3, \tag{14}$$

assuming that the velocity curve may be closely approximated by a parabola with horizontal axis, which is known to be the case. Of course the method applies to finding areas (also volumes) just as with Simpson's and Newton's rules. The theorem is exact for any parabolic curve or for a straight line. It has probably not been given before.

But the sole restriction, $W_x = W_z = W_y$, is only partial, for it amounts to two equations involving three unknowns, thus leaving two variables dependent upon

the third. Hence, with x, y, z, undetermined, and by eliminating z, it may be shown that

$$(15)\quad (y - x)^2(y + x - 2xy - \tfrac{2}{3})^2\{4(x^2 + y^2 - 1) + [3 - 2(x + y)^2]\} = 0,$$

when all three weights are the same. Consequently there is an infinite number of arrangements of three ordinates the arithmetical mean of which is the mean ordinate under a parabolic segment, as is otherwise very apparent.

After rejecting inapplicable solutions the values of y and z in terms of x may be shown to be

$$(16)\quad \begin{aligned} y &= \frac{3}{4} - \frac{x}{2} \pm \left[\frac{3}{4}(x - x^2) - \frac{1}{16}\right]^{1/2} \\ z &= \frac{-(x^2 - \frac{1}{3}) \pm [(x^2 - \frac{1}{3})^2 + (1 - 2x)G]^{1/2}}{1 - 2x}, \end{aligned}$$

where

$$G = x^2 - \frac{2}{3}x \pm \left\{\frac{5}{24} - (x - x^2)\right\}[12(x - x^2) - 1]^{1/2}.$$

This completely evaluates the problem for three ordinates. It is now merely necessary to ascertain what values may be assigned to the arbitrary variable so as to determine useful sets of values of x, y, z, the relative abscissas. Some of the results will be found in Table 1. They may be checked by means of the preceding formulas. The weights for other arrangements of ordinates may be calculated and appended as desired; or, ordinate dispositions may be determined for any assigned weights.

$n=4$. *New formulas.* The usual mensurational arrangement of ordinates for this case has previously been shown to be a special case easily soluble by (9). Further development is not only possible, but leads to that most facile form, the arithmetical mean. The conditions for this latter case are: $W_w = W_x = W_y = W_z$, although only one equation is available on account of the theorem of symmetry previously stated, which is here to be applied:

$$w + z = x + y = 1 \text{ and } w - x = x - y = y - z.$$

An appeal to (9) brings forth

$$(17)\quad w = \tfrac{1}{2},\ w = \tfrac{1}{2} - \sqrt{(0.15)} \text{ and } V = \tfrac{1}{4}[y_w + y_x + y_y + y_z],$$

in which z takes the complementary value of w, while the value $\frac{1}{2}$ for w is rejected. The values of the other abscissas and the results of other solutions appear in Table 1.

$n=5$. *New formulas.* The usual case of mensuration has been treated previously. Two other cases, however, appear in Table 1. They are: equally spaced ordinates symmetrically situated as to the center, which does not admit

the arithmetical mean; weights equal so as to admit the arithmetical mean, which case does not admit equal spacings when one ordinate is central and the others symmetrically situated as to it.

$n=6$. *New formulas.* Both formulas herein developed are new. The case corresponding to those of mensuration has already been treated.

$n=7$. *New formulas.* The base is divided into 6 equal parts. There are seven ordinates; one at each point of division, including the two ends of base. Weights are, respectively, 41, 216, 27, 272, 27, 216, 41, total, 840.

Weddle's rule gives weights in the order: 1, 5, 1, 6, 1, 5, 1, but it is not exact for seven ordinates to a curve of the sixth degree.

Ordinates beyond the limits of the base may have negative weights, as in the following cases of five ordinates: weights: 29, 124, 24, 4, -1; abscissas: 0, $\frac{1}{2}$,

TABLE 1

A few special quadrature rules derived from the general rule (9).

Three Ordinates

Locations R_x	R_y	R_z	*Weights* W_x	W_y	W_z	*Ordinate Spacing*	*Remarks*
0	$\frac{1}{2}$	1	1	4	1	$\frac{1}{2}$	Simpson's rule (Cotes)
.1	.5	.9	100	184	100	.4	
1/8	1/2	7/8	24	33	24	3/8	
1/7	1/2	6/7	49	52	49	5/14	
1/6	1/2	5/6	3	2	3	1/3	
1/6	1/3	5/6	3	4	5	(1/6−(3/6)	
1/5	2/5	4/5	16	3	17	(1/5)−(2/5)	
$\frac{1}{2}-r$	$\frac{1}{2}$	$\frac{1}{2}+r$	1	0	1	r	$r=\sqrt{(1/12)}$*
.1	$.7-r$	$.7+r$	1	1	1	$(.6-r):2r$	$r=\sqrt{(1/200)}$ Arith. mean
$\frac{1}{2}-r$	$\frac{1}{2}$	$\frac{1}{2}+r$	1	1	1	r	$r=\sqrt{(1/8)}$ " "
1/6	$\frac{2}{3}-r$	$\frac{2}{3}+r$	1	1	1	$(\frac{1}{2}-r):2r$	$r=\sqrt{(1/24)}$ " "

Four ordinates—Symmetry as to center

R_w, R_z	R_x, R_y	W_w, W_z	W_x, W_y		
$\frac{1}{2}\mp\frac{1}{2}$	$\frac{1}{2}\mp\frac{1}{6}$	1	3	1/3	Newton's rule 1:3:3:1
$\frac{1}{2}\mp r$	$\frac{1}{2}\mp\frac{1}{3}r$	1	1	$2r/3$	$r=\sqrt{.15}$ Arith. mean (New)

Five ordinates—Symmetry as to center

R_v, R_z	R_w, R_y	R_x	W_v, W_z	W_w, W_y	W_x		
$\frac{1}{2}\mp\frac{1}{2}$	$\frac{1}{2}\mp\frac{1}{4}$	$\frac{1}{2}$	7	32	12	1/4	†
$\frac{1}{2}\mp r$	$\frac{1}{2}\mp\frac{1}{2}r$	$\frac{1}{2}$	17	17	22	$\frac{1}{2}r$	$\sqrt{(3/17)}$ Probably new.
$\frac{1}{2}\mp r_1$	$\frac{1}{2}\mp r_2$	$\frac{1}{2}$	1	1	1	$(r_1-r_2):r_2$	‡

* The "two-tenths eight-tenths depth method" applied to velocity measurements by the current meter.

† Given by Sheppard in the article on *Mensuration* in the Encyclopedia Brittaninica, 14th edition

‡ $r_1=\left[\frac{1}{4}-\frac{7-\sqrt{11}}{48}\right]^{1/2}$; $r_2=\left[\frac{1}{4}-\frac{7+\sqrt{11}}{48}\right]^{1/2}$. Arithmetic mean. New.

1, 3/2, 2, and weights: $-1, 34, 114, 34, -1$; abscissas: $-\frac{1}{2}, 0, \frac{1}{2}, 1, 3/2$. All the ordinates may be beyond the limits of the base.

Forms (9) and (10) are general, and convenient for a small number of ordinates. In cases where the ordinates are symmetrically arranged as to a point on the base, or its extension, the product sums of (9) and the product (10)may be simplified when there are too many ordinates for convenient use of (9). Thus, spacings being symmetrically placed about a given point of abscissa, y, on the line of base, the following are examples of forms for total summations, including all the abscissas:

$$
\begin{aligned}
&\sum x = ny; \quad \sum xx = \sum_{1}^{\frac{1}{2}(n-k)} a_z + \frac{(n+k-2)(n-k)}{2}y^2;\\
(18)\qquad &\sum xxx = (n-2)y\left[\sum_{1}^{\frac{1}{2}(n-k)} a_z + \frac{n(n-4)+3k}{6}y^2\right];
\end{aligned}
$$

where $k = \frac{1}{2}[1-(-1)^n]$, $a_z = x_z(2y - x_z)$, and y is the abscissa of the point of symmetry on the base, or its extension, and the unlimited summations include the n abscissas.

The forms (9) and (10) do not include x_m, but they may be expressed in terms of (18) and x_m as follows:

$$
\begin{aligned}
(19)\qquad \sum\nolimits_m x = ny - x_m;\ \sum\nolimits_m xx &= \sum_{1}^{\frac{1}{2}(n-k)} a_z - a_m - (n-2)yx_m\\
&\quad + \frac{(n-k)(n+k-2)}{2}y^2;\\
\sum\nolimits_m xxx &= [(n-2)y - x_m]\left[\sum_{1}^{\frac{1}{2}(n-k)} a_z - a_m\right] + (n-2)\frac{n(n-4)+3k}{6}y^3\\
&\quad - \frac{(n-2)(n-k-4)+k(n-k)}{2}y^2x_m.
\end{aligned}
$$

These forms result from successive applications of

$$
\begin{aligned}
&\sum(x\cdots x_r) = \sum\nolimits_m x \sum\nolimits_m (x\cdots x_{r-1}) \div r,\\
(20)\qquad &\sum\nolimits_m (x\cdots x_r) = \sum(x\cdots x_r) - x_m \sum\nolimits_m (x\cdots x_{r-1})
\end{aligned}
$$

the last term of the first of which is exactly divisible by r. The subscript, m, of $\sum_m$, indicates omission of x_m. The number of factors in a product is shown by the subscript r. When n is odd, the odd ordinate has the abscissa, y, but the formulas may be arranged otherwise, if needed. If P_m is the product of all binomial differences of the n abscissas taken as shown in (10), omitting factors containing x_m, then

$$P_m = \frac{(n-1)_{m-1}(n-1)!!}{(m-1)!(n-1)!}\left(\frac{x_n - x_1}{n-1}\right)^{\frac{1}{2}(n-1)(n-2)} \tag{21}$$

wherein

$$(n-1)_{m-1} = (n-1)(n-2)\cdots(n-m+1)$$

and $(n-1)!! = (n-1)!\,(n-2)!\cdots 2!\,1$. Thus the relative values of P_1, P_2, etc., follow Newton's rule for binomial coefficients, when ordinate spacings are equal. (For five ordinates, $P_1 : P_2 : P_3 = 1:4:6$ etc.).

In solutions of complicated cases factors of the forms $2u-1$, $2v-1$, $u+z-1$, may be cancelled from equations expressing equal weights, and the degree of such an equation may be depressed by placing $a = u - u^2$, $b = v - v^2$, etc. For example, with $n=6$, the quantities to be equated for equal weights reduce to $(-1)^{m+n}\,F[1-5(b+c)+30bc]$; forms deducible from one another by cyclic changes, F being a factor dependent upon m. $F = 1, 3, 2$, respectively, when spacings are equal and $m = 1, 2, 3$.

A SIMPLE DERIVATION OF WARING'S FORMULAE

By FRANCIS D. MURNAGHAN, Johns Hopkins University

The formulae giving the expressions for the elementary symmetric functions of the roots of an algebraic equation in terms of the various sums of like powers of the same roots were published by Waring in his *Miscellanea analytica* in 1762. It is hoped that the following derivation of these expressions may prove of interest.

A square matrix A with elements a_s^r, may be regarded as a linear vector function [= mixed tensor of rank 2] which transforms a vector $x = (x^1, x^2, \cdots, x^n)$ into a vector y by means of the equality $\boldsymbol{A}\,x = y$ which may be written out explicitly as

$$a_\alpha^r x^\alpha = y^r; \qquad (r = 1, 2, \cdots n;\ \alpha \text{ a summation symbol}).$$

The proper or characteristic numbers of the matrix are those numbers λ for which $y = \lambda x$ so that $Ax = \lambda x$. On transforming this vector equality by means of the matrix A we obtain

$$AAx = A\lambda x = \lambda Ax = \lambda^2 x,$$

so that[1] the proper numbers of $AA (\equiv A^2)$ are the squares of the proper numbers of A. A continuation of this procedure shows at once, that the proper numbers of A^p, where p is any positive integer, are the p^{th} powers of the proper numbers of A. Here A^p is the matrix of which the element in the r^{th} row and s^{th} column is

$$a_{\alpha_1}^{r}a_{\alpha_2}^{\alpha_1}\cdots a_s^{\alpha_{p-1}};\ \alpha_1, \cdots, \alpha_{p-1} \text{ being summation symbols.}$$

[1] See Aurel Wintner, *Spektraltheorie der unendlichen Matrizen* (1929), p. 15.

Now the proper numbers of a matrix are the roots of the *proper equation* which is obtained by equating to zero the determinant of the matrix $A-\lambda E$ whose r, s element is $a_s^r-\lambda\delta_s^r$; $\delta_s^r=0$ if $r\neq s$ and $=1$ if $r=s$. Hence the sum of the proper numbers of the matrix A is a_α^α. More generally.

$$\sum\lambda_1 = a_\alpha^\alpha = \delta_\beta^\alpha a_\alpha^\beta,$$

$$\sum\lambda_1\lambda_2 = \frac{1}{2!}\delta_{\beta_1\beta_2}^{\alpha_1\alpha_2} a_{\alpha_1}^{\beta_1} a_{\alpha_2}^{\beta_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum\lambda_1\lambda_2\cdots\lambda_r = \frac{1}{r!}\delta_{\beta_1\beta_2\cdots\beta_r}^{\alpha_1\alpha_2\cdots\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r},$$

where

$$\delta_{n_1\cdots n_r}^{m_1\cdots m_r}$$

is the generalized Kronecker delta[2] and r runs from 1 to n.

We shall denote by S_r the r^{th} elementary symmetric function of the proper numbers λ of A and by P_r the sum of the r^{th} powers of these numbers so that (since P_r is the first elementary symmetric function of the proper numbers of A^r)

$$P_r = a_{\alpha_1}^{\alpha_r} a_{\alpha_2}^{\alpha_1}\cdots a_{\alpha_r}^{\alpha_{r-1}}.$$

The numerical coefficient

$$\delta_{\beta_1\beta_2\cdots\beta_r}^{\alpha_1\alpha_2\cdots\alpha_r}$$

in the expression

$$r!S_r = \delta_{\beta_1\cdots\beta_r}^{\alpha_1\cdots\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r},$$

being a determinant of which the element in the r^{th} row and s^{th} column is

$$\delta_{\beta_s}^{\alpha_r},$$

may be expanded as a sum of terms of the type

$$\pm\,\delta_{\gamma_1}^{\alpha_1}\cdots\delta_{\gamma_r}^{\alpha_r}$$

where $(\gamma_1,\cdots,\gamma_r)$ is a permutation of the numbers $(\beta_1,\cdots\beta_r)$. If this permutation is the identity (or, in other words, consists of r cycles of one letter each) the corresponding term in S_r is

$$\delta_{\beta_1}^{\alpha_1}\delta_{\beta_2}^{\alpha_2}\cdots\delta_{\beta_r}^{\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r}$$

[2] See *The generalized Kronecker symbol and its application to the theory of determinants*, This Monthly, vol. 32 (1925), pp. 233–241; O. Veblen, *Invariants of quadratic differential forms*, p. 3.

On summing with respect to the summation labels $(\beta_1, \cdots \beta_r)$ this becomes

$$a_{\alpha_1}^{\alpha_1} \cdots a_{\alpha_r}^{\alpha_r},$$

and on further summation with respect to the labels $(\alpha_1, \cdots \alpha_r)$ we get P_1^r. If, again, the permutation $(\gamma_1, \cdots \gamma_r)$ of the labels $(\beta_1, \cdots, \beta_r)$ has one cycle of two letters and $r-2$ cycles of one letter [e.g., $(\gamma_1, \gamma_2, \gamma_3 \cdots, \gamma_r) = (\beta_2, \beta_1, \beta_3, \cdots, \beta_r)$] the corresponding term in S_r is

$$-\delta_{\beta_2}^{\alpha_1}\delta_{\beta_1}^{\alpha_2}\delta_{\beta_2}^{\alpha_3} \cdots \delta_{\beta_r}^{\alpha_r} a_{\alpha_1}^{\beta_1} \cdots a_{\alpha_r}^{\beta_r}$$

and summation with respect to the labels β gives

$$-a_{\alpha_1}^{\alpha_2}a_{\alpha_2}^{\alpha_1}a_{\alpha_3}^{\alpha_3} \cdots a_{\alpha_r}^{\alpha_r}.$$

Further summation with respect to the labels α yields $-P_2P_1^{r-2}$. Now the number of permutations of r letters which are mere transpositions is

$$\binom{r}{2} = \frac{r(r-1)}{1\cdot 2}.$$

Hence there will be $(r/2)$ terms of the type just discussed and these will yield, on summation,

$$-\binom{r}{2}P_2P_1^{r-2}.$$

The general permutation of the r letters $(\beta_1, \cdots \beta_r)$ may be analysed into m_1 cycles of one letter, m_2 cycles of 2 letters, m_3 cycles of 3 letters, $\cdots m_r$ cycles of r letters where

$$m_1 + 2m_2 + 3m_3 + \cdots + rm_r = r$$

and a term in S_r corresponding to such a permutation $(\gamma_1, \gamma_2, \cdots, \gamma_r)$ of the r letters $(\beta_1, \cdots, \beta_r)$ will yield on summation with respect to the labels β and α the result

$$(-1)^{m_2+m_4+m_6+\cdots} P_1^{m_1} P_2^{m_2} \cdots P_r^{m_r}.$$

The number N of permutations on r letters which consist of exactly m_1 cycles of one letter, m_2 cycles of 2 letters, $\cdots m_r$ cycles of r letters, was calculated by Cauchy[1] and the derivation is extremely simple although the result does not, as far as we are aware, find its place in the text books on algebra. Since the order in which the cycles are given does not affect the permutation and since the particular letter of a given cycle with which we choose to start the cycle does not affect the permutation we obtain all the permutations on the r letters by multiplying N by $m_1!\, m_2! \cdots m_r!\, 2^{m_2}3^{m_3} \cdots r^{m_r}$. Hence

[1] Cauchy, Oeuvres, Ser. 1, Tome 9, p. 289.

$$N = \frac{r!}{m_1!m_2! \cdots m_r! 2^{m_2} \cdot 3^{m_3} \cdots r^{m_r}}$$

so that, finally,

$$(W) \qquad S_r = \sum (-1)^{m_2+m_4+m_6 \cdots} \frac{P_1^{m_1} P_2^{m_2} \cdots P_r^{m_r}}{m_1!m_2! \cdots m_r! 2^{m_2} 3^{m_3} \cdots r^{m_r}}$$

the summation being over all positive integral solutions of the equation

$$m_1 + 2m_2 + \cdots + rm_r = r.$$

This is the statement of Waring's formula. In our derivation we started with the matrix A and the $(\lambda_1, \cdots, \lambda_n)$, its proper numbers, were secondary. However it is quite evident that if the $(\lambda_1, \cdots \lambda_n)$ are given arbitrarily we need merely choose as our matrix A the diagonal matrix $a_s^r = 0$, if $r \neq s$, $a_r^r = \lambda_r$ in order to have, as its proper numbers, the arbitrary numbers $(\lambda_1, \cdots \lambda_n)$. Hence the proof given is valid for any algebraic equation. The main interest in the present derivation is the identification of the coefficients in the formula (W) with the number of permutations on r letters whose cycle structure is given.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE GEOMETRICAL PARADOX

By J. L. COOLIDGE, Harvard University

Question: A point P in three-dimensional Euclidean space is required to be collinear with two given points A and B; how many independent algebraic conditions are thereby imposed upon its coordinates?

First answer: In order to be collinear with A and B it is necessary and sufficient that P should be coplanar with A and B and each of two arbitrary points not in the same plane through A and B; *two conditions.*

Second answer: If P be collinear with A and B it is necessary and sufficient that the sum of two of the distances determined by two pairs from these three points should be equal to the distance determined by the third pair. That may be written

$$(-AB + BP + PA)(AB - BP + PA)(AB + BP - PA) = 0.$$

We rationalize the left hand side by multiplying through by a non-vanishing factor:

$$-(AB+BP+PA)(-AB+BP+PA)(AB-BP+PA)(AB+BP-PA)=0.$$

This expands into

$$AB^4+BP^4+PA^4-2BP^2\cdot PA^2-2PA^2\cdot AB^2-2AB^2\cdot BP^2=0.$$

The left hand side is an unfactorable rational expression in the coordinates of the point P, hence *but one condition is imposed.*

Which answer is right? How many conditions are imposed in a Euclidean space of n dimensions?

Prize Problems

By Tomlinson Fort, Lehigh University

Professor Radò in his lecture on "Mathematics in Hungary," delivered before the Mathematical Association of America at its meeting in Cleveland, Ohio (Jan. 1, 1931), described certain prizes given in Hungary for the solution of problems in elementary mathematics.

Professor Radò's remarks greatly interested the persons present. It consequently seems to me that the readers of The Monthly will be interested in learning of a series of prize problems which has been in existence at Lehigh University for the past three years. Each semester five problems are published in the college paper, one each two weeks, over a ten-week period. Problems are published on Fridays. Solutions must be handed in not later than the next subsequent Wednesday. A prize is offered for the best solution of the problem by a Lehigh undergraduate, and, in addition, the name of the winner is posted on our bulletin board and published in the college paper. At first we made each prize five dollars, but subsequently changed it, making each prize a mathematical book to be properly inscribed and not to exceed five dollars in value.

We consider the prize problems a success. There has been so much favorable comment that the President of the University, who was at first very dubious, now no longer questions the inclusion of this item in our budget. We have a departmental committee to choose problems, read papers, and handle other details. It has proved harder than we anticipated to originate problems of interest and of proper difficulty. However, I think the set of problems finally evolved a good one. Decision between papers has at times necessarily been made on minor points, and in a few instances prizes have been divided.

The prize problem series at Lehigh was suggested to me some years ago by a conversation with Professor Lester S. Hill, who told me of something of the kind that had been tried at one time at Yale University.

On Division with a Calculating Machine

By E. C. Kennedy, University of Texas, College of Mines

An eight-column calculating machine may be used to divide a number of sixteen (or less) digits by a number of sixteen digits in the following manner:

Let the quotient be N/n and let n_1 be the number consisting of the first eight digits of n.

Set
$$(N/n) = (N/n_1) + e,$$
where N/n_1 is taken to eight places and e is to be determined.

Then
$$n[(N/n_1) + e] = N$$
and
$$e = \frac{N - n(N/n_1)}{n_1}, \text{ approximately.}$$

Then the quotient $= (N/n_1) + e$.

This does not require division by a number of more than eight digits and requires only one such multiplication.

For example, evaluate .423456/.6823499215826795 to sixteen places.

Here
$$N/n_1 = .62058481 \text{ (to eight places only)};$$
and
$$N - n(N/n_1) = .0000000035610979$$
and
$$e = .0000000035610979/.68234992 = .0000000052188735.$$

Therefore the quotient $= .6205848152188735$ within an error of about one or less in the sixteenth place.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Lustiges und Merkwürdiges von Zahlen und Formen. By Walther Lietzmann. Breslau, F. Hirt, 1930. vi+307 pages, price 9.50 RM.

Mathematik und Bildende Kunst. By Walther Lietzmann. Breslau, F. Hirt, 1931. 150 pages.

It is safe to say that any book written by Dr. Lietzmann is worth reading for three good and sufficient reasons: (1) He knows mathematics from having sat at the feet of Klein as a student and having been closely associated with him in the International Commission on the Teaching of Mathematics from 1908

until the war began; (2) he knows the teaching problem, having tried out the Klein ideas and being at present an Oberstudiendirektor in Göttingen; (3) he writes in a style that appeals to teachers and pupils alike. Such a combination is rare, and hence such books as he writes are unusual.

The first of the works under review is well known to those who follow mathematical courses of the secondary schools of Germany. It first appeared in 1922 and was revised in 1928 and 1929, the present edition being the fourth and containing much material not found in the first two. The work is divided into three parts, which are also bound separately for those who prefer this form. The first covers the general field, with amusing problems, verses, dramatizations (with interesting excerpts from American sources), aphorisms, illustrations, games and fallacies. The second is devoted to numbers, with a collection of some of the best of the number puzzles of the past and some very good ones of recent years. Teachers wishing a supply of amusing problems in this field could not do better than consult this part of the book. Among the problems is this curious one (p. 173): "The American mathematician Augustus de Morgan said, when asked his age, 'I was x years old in the year x^2.' Knowing that he lived in the 19th century, when was he born?" Following out the lighter vein of the book, it might be well to add that the place in which he was born in America can be found by adding 1 to the sum of the two complex cube roots of this number to be added, and to the result adding the five fifth roots of unity. In the same lighter vein it may be stated that he always wrote his name with a capital letter used for that one which, in the Latin alphabet, corresponds to the one in the acrophonic Greek numerals, which represents the number ten.

The second of the books under review is a more recent study by Dr. Lietzmann, and an interesting one. It is largely a collection of pictures in which geometry plays an important part. It runs the course from the pyramids and the rope-stretchers of Egypt, through the perspective of the Renaissance, to the chamber of horrors of the modern jazz of interior decoration. On the way, however, some of the world's best art is shown, chiefly for the purpose of illustrating symmetry and proportion and of illustrating the familiar "golden section" in architecture. Even for those teachers of geometry who do not read German these pictures will be of service, illustrating as they do a quotation of a celebrated architect quoted in the preface, "Ars sine scientia nihil est,"—art without science is nothing. Personally, this reviewer does not believe it, but he is thoroughly in sympathy with this revision,—"Scientia sine arte nihil est." Dr. Lietzmann's notable success illustrates this very point. He teaches with the soul of an artist. Personally, too, this reviewer would have made from the world's architecture a somewhat different selection, as also from the world's gems of the potter's art, and the goldsmith's, and the etcher's, and that of the craftsmen of the Orient. But such criticism is egotistical,—chacun à son gout,—and Dr. Lietzmann's taste is one to be praised as helpful to the teachers of our gild.

DAVID EUGENE SMITH

Moderne Algebra, Teil I. B. L. v. d. Waerden: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 33. Julius Springer, Berlin, 1930. 243+8 pages.

The domain of abstract algebra has at present a flourishing period in Germany, and a number of recent textbooks on the subject gives evidence of this development. The present book by v. d. Waerden gives a very clear and satisfactory treatment of the main ideas of the theory of groups, fields and some of the fundamentals of the theory of rings and ideals. It shows many points of connection with the recent book by O. Haupt. The second volume promises to be of still higher interest; it will, according to the plans of the author, deal with general ideal theory and elimination, linear algebras and hypercomplex systems, and the so-called theory of representation. A more complete review of this book will be printed in the *Bulletin*.

OYSTEIN ORE

Darstellende Geometrie, I and II. By Dr. Robert Haussner. Sämmlung Göschen, Nos. 142 and 143.

These numbers of the Göschen collection are revisions and extensions of earlier editions by the same author. The introductory number, which is the fourth edition of the text, deals with the general principles of parallel and orthogonal projection with applications to plane and space figures. The second part, a third edition, treats of harmonic properties and cross-ratios with applications to conic sections. Pascal's and Brianchon's theorems, the pole and polar theory, and many metric properties, including those of the foci, are deduced. The treatment is not essentially different from that found in texts on the geometry of position. Many diagrams appear in each of the texts. These are well drawn and are remarkably clear when the restricted size of the page is considered. In each number also there is a helpful table of the literature of descriptive geometry.

THOMAS F. HOLGATE

Addition-Subtraction Logarithms, to Five Decimal Places. By L. M. Berkeley. White Book and Supply Company, 36 West 91st Street, New York, 1930. xii+136 pages, paper bound.

Addition-subtraction logarithms are based on the following simple relations:

$$\log (a + b) = \log \frac{a+b}{a} + \log a = \log \left(1 + \frac{1}{n}\right) + \log a$$

$$\log (a - b) = \log a - \log \frac{a}{a-b} = \log a - \log\left(\frac{n}{n-1}\right),$$

where $n = a/b$.

Further, if $m = n+1$, then $1+(1/n) = m/(m-1)$. The present table consists essentially of three parallel columns, of $\log n$, $\log [1+(1/n)]$, $\log (n+1)$, respectively. To find $\log (a+b)$, we find $\log n = (a/b) = \log a - \log b$, then enter the table with $\log n$, take out $\log [1+(1/n)]$, then add this to $\log a$, in accordance

with the first of the above equations. A similar procedure yields log $(a-b)$. Since in these tables no interpolation is necessary, the result can no doubt be obtained more easily than by the usual process of finding and adding the numbers themselves.

The tables are excellently arranged and clearly printed. A more substantial binding would be necessary if they were to be used extensively.

R. A. J.

Non-Interpolating Logarithms, Cologarithms, and Antilogarithms. By Frederick W. Johnson. The Simplified Series Publishing Company, 1381 Third Avenue San Francisco.

By a simple and ingenious device of arrangement, this table makes it possible to find at once and without interpolation the five-figure mantissa of the logarithm of any five-digit number. The table of logarithms occupies eighteen double pages, and there is also a table of antilogarithms of the same size. The page has an objectionably unfamiliar look, but the reviewer finds the table very workable in practice. One would like to see it tried out in competition with the "Graphic Table" issued a few years ago.

Following the five-place tables is a similarly arranged set of four-place tables. The antilogarithmic tables of the latter are also printed separately in a pamphlet offered by the publishers for free distribution.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3487. *Proposed by Vladimir F. Ivanoff, San Francisco, Calif.*

Prove the identities:

$$\frac{d^2y}{dx^2}+\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3\equiv 0, \tag{1}$$

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2+\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2+3\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2}\equiv 0, \tag{2}$$

the derivatives d^ny/dx^n and d^nx/dy^n being taken of the same function, $f(x, y)=0$.

3488. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Find numbers m such that the square of each of the $\phi(m)$ numbers less than m and prime to it will be congruent to 1, modulo m.

UNSOLVED PROBLEMS

Solutions are desired for the following problems, which are the remaining unsolved problems proposed in 1915.

309 [1915, 162]. *Proposed by Jos. B. Reynolds.*

The tangent at one cusp of a vertical, three-arched hypocycloid is horizontal, and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

232 [1915, 202]. *Proposed by E. T. Bell.*

If $F(x)$ is any function of x which vanishes with x, and which, for $0<|x| \lesseqgtr |\xi|$, can be expanded in an absolutely convergent series of positive powers of x, show that a function $f(n)$ may be found, essentially in one way only, such that

$$\int_0^\xi \frac{1}{x}F(x)dx = -\log \prod_{n=1}^{\infty}(1-\xi^n)^{(1/n)f(n)},$$

and find the form of $f(n)$ explicitly in terms of the coefficients in the expansion of $F(x)$. Hence, as particular examples, expand (when possible by the method) e^x as an infinite product, and show that

$$\frac{1}{e} = \prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)^{(1/n)\phi(n)},$$

where $\phi(n)$ is the totient of n.

313 [1915, 202]. *Proposed by Clifford N. Mills.*

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1+ekgw/2)$.

234 [1915, 268]. *Proposed by Frank Irwin.*

Start with any number, for instance 89, and add to it the number obtained by reversing the order of its digits: $89+98=187$. Now perform the same operation on the result: $187+781=968$. If we continue in this way we arrive, after a certain number of operations, at a number which reads the same forward and backwards (24 operations bring us to 8813200023188). Will this be the case no matter with what number we start?

NOTE: I am told that this is an old problem, but do not know whether it has ever been solved. (No other number under 100, except, of course, 98, requires so many operations to lead to the desired result as 89 does.)

308 [1915, 268]. *Proposed by H. S. Uhler.*

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on the principal section, the ray inside the prism generates a cone of elliptical right section, It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

315 [1915, 309]. *Proposed by H. S. Uhler.*

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

SOLUTIONS

3441 [1930, 380]. *Proposed by Solomon Kullback, Brooklyn, New York.*

In a given triangle, inscribe an equilateral triangle having a given point, P, on one of its sides. Suggested by problem No. 3405.

Solution by Ralph Deutsch, Brooklyn, N. Y.

Construction: Draw the arc of a circle AOB so that it contains the angle $C+60°$, and the arc BOC so that it contains $A+60°$. Let O be the intersection of these two arcs, and let the angles that the lines OA, OB, OC make with the sides of ABC be numbered as in the figure. Construct the angle OYP equal to angle 1 with Y lying upon CA, and then the angle OYX equal to angle 5, where YX cuts BC in X. Extend YP until it cuts AB in Z. Then XYZ is the required triangle.

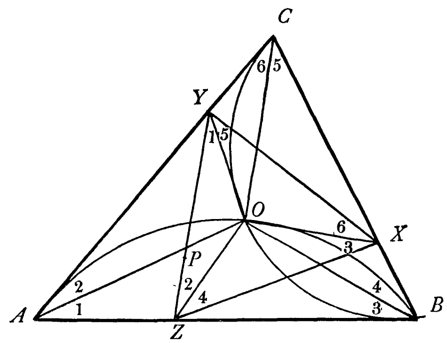


FIG. A.

Proof: Since O, Y, A, Z and O, Y, C, X are concyclic, O, X, B, Z are also concyclic. Therefore, angles OZY, OXY, OXZ, and OZX are equal, respectively, to angles 2, 6, 3, and 4. Also $\angle AOB+\angle BOC+B+1+5=360°$, or $C+A+B+1+5=240°$. Hence $1+5=60°$. Since $\angle AOC=60°+B$ it can be proved in a like manner that $2+4=60°$. This shows that XYZ is equilateral.

A discussion by Otto Dunkel: In the construction above the points X, Y, Z

have the same sense of rotation as A, B, C, and this case will be considered first. In general, if P lies within a certain region no construction will be possible, and for certain other regions the construction above is impossible although there may exist solutions. The directions for the construction and also the number of solutions vary with the location of P. There are four types of regions,and in order to locate these regions the requirement that a side of the required triangle shall pass through a given point P will be dropped.

The three envelopes. If no angle of the triangle ABC is as great as 120° the point O lies within this triangle, and this case will be considered first. For convenience suppose that XYZ is an equilateral triangle with X lying within the segment BC, Y lying within CA, and Z lying within AB. Then the three circles through A, Y, Z:B, Z, X:C, X, Y meet in a point, which is easily shown to be O; and all the equalities between the angles of the figure given above easily follow. Also $\angle YOZ=180°-A$, $\angle ZOX=180°-B$, $\angle XOY=180°-C$. It will be shown later how to construct one triangle $X'Y'Z'$ which is always inscribed in the strict sense above. When the angle YOZ rotates about O remaining always $180°-A$, the points Y and Z trace projective ranges on CA and on AB, respectively. Hence the envelope of YZ is a conic tangent to these last two lines. When Z is at A, Y is at the point of tangency B_a of CA, where $\angle OB_aA = \angle OAB$. Similarly, the point of tangency of AB is C_a, where $\angle OC_aA = \angle OAC$. When OY is parallel to AC, OZ is parallel to AB; and hence YZ is now the line at infinity. The conic is therefore a parabola. It also follows that O is the focus, for it is known that the circle circumscribing the triangle formed by three tangents to a parabola always passes through its focus. Here such a triangle is AYZ, and its circumscribed circle always passes through O. Hence YZ, ZX, XY are tangent always to corresponding parabolas α, β, γ with the common focus O. Moreover, from the theory of the parabola we know that the angles of the triangles OYZ, OXY, OZX remain constant. Hence if the figure formed by OX, OY, OZ rotates about O as a rigid figure preserving the angles at O, the lines XY, YZ, ZX will be tangent to their respective parabolas and will always form an equilateral triangle, since they do so for an initial position. Hence as Y traces the entire line on which CA lies we obtain all the inscribed equilateral triangles of the same order of vertices as ABC. These triangles contain O, and therefore a part or the whole of the area of ABC. Let X', Y', Z' be the feet of the perpendiculars from O to BC, CA, AB; then on account of the values of the angles subtended by pairs of these points at O, $X'Y'Z'$ is an equilateral triangle. A simple consideration of the figure will show that it is inscribed in the above strict sense. Let the perpendicular from O to $Y'Z'$ cut it in V_a and CA in B_a'. Then, since $\angle OY'Z' = \angle OAB = \angle OB_aA = \angle OB_a'B_a$, the triangle OB_aB_a' is isosceles with vertical angle at O. Hence OV_a is the axis of α and V_a is its vertex. Thus the feet of the perpendiculars V_a, V_b, V_c from O to the sides of $X'Y'Z'$ are the vertices of α, β, γ, and the sides of this triangle are the tangents at the vertices.

The regions. The directrices of α, β, γ form an equilateral triangle with sides

parallel to those of $X'Y'Z'$, with O as center of similitude for the two triangles, and with the ratio of $2:1$ for the corresponding parts. The intersections of any pair of the parabolas must lie upon an altitude of the directrix triangle, and hence the regions have in general six angular points lying in pairs upon the three altitudes of this directrix triangle. The focus O may lie anywhere within $X'Y'Z'$ according to the shape of ABC, but not on a side if we have an actual triangle ABC or a triangle in this case in which no angle is as great as 120°. Let I' and r' be the center and radius of the circle (I') inscribed in $X'Y'Z'$, while I'' and r'' are the corresponding symbols for the directrix triangle. If O lies within (I'), then $OI''=2\,OI'<2r'=r''$. Since for this position of O, $OI''<r''$, the point I'' lies within each parabola and therefore in the same region as O, which will be denoted by I. In this case α, β, γ intersect in three pairs of points, the points of each pair lying upon an altitude of the directrix triangle and separated by its center I''. The figure shows this case. There are in addition to the region I three regions II_a, II_b, II_c such that from any point within II_a, for example, two tan-

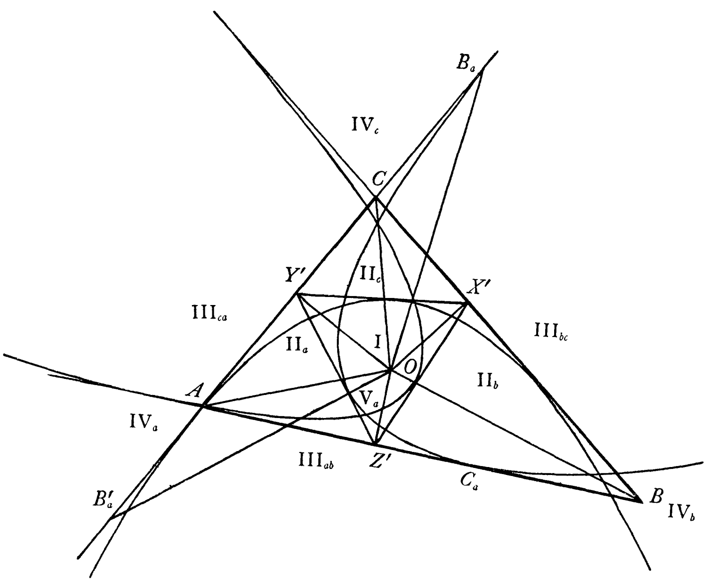


FIG. B.

gents may be drawn to α but none to β or γ; three regions III_{ab}, III_{bc}, III_{ca} such that from any point within III_{ab}, for example, two tangents may be drawn to α and two to β but none to γ; three regions IV_a, IV_b, IV_c such that from a point within any one of them two tangents may be drawn to each parabola.

If O lies outside the circle (I') but within $X'Y'Z'$, $OI''>r''$ and here I'' lies outside of each parabola. In this case the two points of each pair of intersections lie on an altitude on the same side of I'', and there are only two regions II but four regions III. If O lies on (I') but not at a point of tangency, $OI''=r''$

and all three parabolas pass through I''. In this case a region II has reduced to a point and there are only nine regions.

Constructions. If P lies within I there is no construction; if within a II there are two; if within a III, four; if within a IV, six. If P lies on the boundary of any of these regions there may be one, two, etc., up to five constructions depending upon the particular boundary. The cases are easily distinguished. If P lies for example within III_{bc}, an arc of a circle is described on OP as a chord so that it contains the angle OCB and lies on the side of OP towards C. This arc cuts AC in two points each of which serves for the Y vertex, and the balance of the construction easily follows. The lines PY are tangents to γ. Then an arc is drawn on OP on the side towards B so as to contain the angle OBC. This arc cuts AB in two points each of which serves for a Z. The lines PZ are tangent to β; again the balance of the construction follows in a simple manner. In this way four equilateral triangles are constructed for this position of P; and for each region definite directions may be given in a similar manner for the constructions.

The remaining cases. In the above no angle is as great as 120°. If an angle, say C, is 120°, then O lies at Z', and every triangle XYZ has a vertex at $O \equiv Z'$. Also CO bisects the angle C; the two parabolas α and β reduce to the point O. If P lies outside γ there are four constructions; if inside, but not at O, there are two; if at O, there are an infinite number.

If an angle, say C, is greater than 120°, O lies outside ABC and hence also outside $X'Y'Z'$, but within the vertical angle of 60° at Z'. This case may be treated in a manner similar to the above case for $C<120°$, but there is a slight difference in the expressions for the equalities of the angles. Here $\angle AOC = 60° + B$, $\angle COB = 60° + A$, $\angle AOB = 300° - C$. The parabolas α and β intersect on the internal bisector of the angle of the directrix triangle corresponding to Z', while the other two lines of intersections are the external bisectors of the angles corresponding to X' and Y'. The interiors of α and β lie outside ABC. The intersection of the three lines above is the center of an escribed circle of the directrix triangle, and it lies always outside the three parabolas. Thus the ten regions contain, as in the second case above, only two regions II, II_a and II_b; and of the four III regions there are two separate regions III_{ab}. The region I is necessarily outside ABC.

If we consider the triangle XYZ with an order of rotation opposite to that of ABC, there will be still other constructions possible. This gives rise to a second point O'. If $C>60°$, $B>60°$, then $\angle AO'B = C - 60°$, $\angle CO'A = B - 60°$, $\angle BO'C = 60° - A$. If $A<60°$, $B<60°$, then $\angle AO'B = C - 60°$, $\angle CO'A = 60° - B$, $\angle BO'C = 60° - A$. These constructions include the cases in which XYZ lies outside ABC, and they include, of course, no cases of a triangle inscribed in the strict sense. Let J' be the center of the escribed circle (J') in the angle of $X'Y'Z'$ in which O' lies, then as before there will be three cases according as O' lies inside, on, or outside (J'). The treatment of this case is similar to the above, and there may be from one to six, or no constructions according to the region

in which P lies. This concludes the cases arising from the various locations of the focus with respect to $X'Y'Z'$.

The problem of inscribed triangles XYZ of any given form may be treated in a similar manner, but there would be a greater variety of cases.

The minimum triangle. It is worthy of note that $X'Y'Z'$ is the inscribed equilateral triangle of minimum perimeter and minimum area. For in all cases

$$YZ = \frac{OY \sin A}{\sin OAC};$$

and, since YZ varies directly as OY, the minimum of YZ occurs when OY is perpendicular to CA.

3442 [1930, 380]. *Proposed by the late F. P. Matz.*

Solve the equation

$$\frac{d}{dw}\left(\frac{dw}{dr} + \frac{2w}{r}\right) = 0.$$

Solution by Eugene A. Rasor, Ohio State University

A first integration gives

$$\frac{dw}{dr} + \frac{2w}{r} = -\frac{3}{a}.$$

After multiplying by the integrating factor r^2, a second integration gives the final result,

$$r^3 + ar^2w + b = 0.$$

Also solved by A. W. Bailey, E. C. Kennedy, J. D. Leith, and A. Pelletier.

3445 [1930, 381]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If p is odd and greater than 1, prove that

(a) $$1^2\cdot 3^2\cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \bmod p;$$

(b) $$2^2\cdot 4^2\cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p.$$

Solution by W. Randolph Church, Hightstown, N. J.

The theorem of Wilson states that

(1) $$1\cdot 2\cdot 3\cdot 4 \cdots (p-1) \equiv -1 \pmod p,$$

where p is any odd prime. We have the following set of congruences:

(2) $$i \equiv -(p-i) \pmod p,\ p = 0, \pm 1, \pm 2, \cdots.$$

Replace the $(p-1)/2$ even numbers in the left hand side of (1) by the numbers

congruent to them as obtained by letting $i=2, 4, 6, \cdots, (p-1)$ in (2). After combining we have

$$1^2\cdot 3^2\cdot 5^2 \cdots (p-2)^2 \, (-1)^{(p-1)/2} \equiv -1 \pmod{p},$$

or

$$1^2\cdot 3^2\cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}. \tag{3}$$

Similarly, replace the $(p-1)/2$ odd numbers in (1) by the numbers congruent to them as obtained by letting $i=1, 3, 5, \cdots, (p-2)$ in (2). This gives

$$2^2\cdot 4^2\cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Also solved by L. S. Kennison, L. C. Mathewson, and the Proposer.

3448 [1930, 446]. *Proposed by Oliver D. Kellogg, Harvard University.*

Prove that whenever the infinite series, with positive terms,

$$u_1 + u_2 + u_3 + \cdots$$

converges, the series

$$\frac{u_1}{r_1} + \frac{u_2}{r_2} + \frac{u_3}{r_3} + \cdots$$

diverges,

$$r_n = u_n + u_{n+1} + u_{n+2} + \cdots$$

being the remainder of the first series after $n-1$ terms.

Solution by the Proposer.

By definition, $r_n = u_n + r_{n+1}$. Hence

$$\frac{r_{n+1}}{r_n} = 1 - \frac{u_n}{r_n},$$

$$\frac{r_{n+2}}{r_n} = \left(1 - \frac{u_n}{r_n}\right)\left(1 - \frac{u_{n+1}}{r_{n+1}}\right) > 1 - \frac{u_n}{r_n} - \frac{u_{n+1}}{r_{n+1}},$$

and, as is easily established by induction,

$$\frac{r_{n+p}}{r_n} > 1 - \frac{u_n}{r_n} - \frac{u_{n+1}}{r_{n+1}} - \cdots - \frac{u_{n+p}}{r_{n+p}}.$$

As the given series converges, r_{n+p} approaches 0 as p becomes infinite. Hence, to any n, there corresponds a p, such that

$$\frac{u_n}{r_n} + \frac{u_{n+1}}{r_{n+1}} + \cdots + \frac{u_{n+p}}{r_{n+p}} > 1 - \frac{r_{n+p}}{r_n}$$

exceeds any preassigned number less than 1. It follows that the series

$$\frac{u_1}{r_1} + \frac{u_2}{r_2} + \frac{u_3}{r_3} + \cdots$$

diverges, as was to be proved.

Also solved by Ralph Agnew, C. H. Dix, R. L. Jeffrey, J. D. Leith, J. F. Locke, G. E. Raynor, F. Underwood, and Morgan Ward.

NOTE: Professor Raynor, in his solution, calls attention to the fact that this theorem is a special case of Dini's theorem a proof of which is given on page 293 English translation of Knopp's *Theory and Application of Infinite Series.* Professor Norman Miller also cites this authority.

The Proposer notes that it seems amusing that exactly this result should be needed in the study of a problem in potential theory where apparently he came across it.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

THE SUMMER SESSION FOR TEACHERS OF MATHEMATICS TO BE HELD BY THE SOCIETY FOR PROMOTION OF ENGINEERING EDUCATION

One of the sessions of the Summer School for Engineering Teachers, held annually by the Society for the Promotion of Engineering Education, will be devoted this year to the study of methods of teaching mathematics.

This Summer School, a unique undertaking in the field of higher education, was established in 1927 as one of the outgrowths of a general investigation of Engineer Education begun by the S.P.E.E. in 1924. The school has as its purpose intensive study and discussion of methods of teaching the principal subjects or divisions of engineering education. Sessions are held annually in various institutions throughout the country and are devoted to particular portions of the engineering curriculum. The Session of 1927, with which the school was inaugurated, dealt with the teaching of Engineering Mechanics. Sessions were held at Cornell and the University of Wisconsin. Since that year other Sessions of the school have been devoted to the teaching of Physics, Electrical Engineering, Mechanical Engineering, Engineering Drawing and Descriptive Geometry, and Civil Engineering. These sessions have been held at the Massachusetts Institute of Technology, The University of Pittsburgh, Purdue University, the Carnegie Institute of Technology, and Yale University.

From modest beginnings the school has developed into one of the really important enterprises in the field of engineering education. Sessions are attended by teachers from all parts of the country, the enrolment ranging from 50 to 100 individuals. The teaching staffs are recruited from among the foremost teachers, scientists, and practising engineers.

It will be of interest to teachers of mathematics to learn that the Summer School Session of this year will be held at the University of Minnesota immediately preceding the annual meetings of the Mathematical Association of America and the American Mathematical Society. The S.P.E.E. session will open on August 24th and continue for two weeks, closing on September 5th. The program of the Session is arranged in five principal divisions:

1. General principles of teaching.
2. Teaching of mathematics to engineering students.
3. Advanced mathematics.
4. Application of mathematical principles in engineering practice.
5. Historical and miscellaneous.

These five divisions of the program are carried forward simultaneously by means of sequences of lectures and discussion periods, informal conferences and the like. As an illustration, lecture and conference periods on the following topics are scheduled in the second division of the program:

Teaching the Fundamentals of Trigonometry.

The Teaching of College Algebra (A) Arrangement of course content.
(B) Teaching problems.

Analytic Geometry (A) Arrangement of course problems.
(B) Teaching problems.

Differential Calculus (A) Arrangement of course content.
(B) Teaching the nature of the derivative.

Integral Calculus (A) Arrangement of course content.
(B) Teaching problems.

Differential Equations for Engineering Students.

Among the topics in other divisions of the program is a series of lectures and discussions on the general principles of teaching. These are to be conducted by a teacher of educational method. A part of the program will be devoted to specific educational problems, such as the question of the size of class sections, the coordination of mathematics with the related engineering subjects, the adjustment between college and secondary school mathematics, and combined versus unit courses in mathematics. Two lectures on the history of mathematics are scheduled, as well as a number of lectures on higher mathematics and on relationship between mathematical principles and current engineering practice.

The teaching staff of the Session is headed by Dean O. M. Leland, College of Engineering and Architecture, University of Minnesota, who will serve as the Director of the Session, and by Professor C. A. Herrick of the University of Minnesota who will act as the Secretary. Other members of the staff include—F. T. Spaulding, of the Graduate School of Education, Harvard University; E. R. Hedrick, University of California at Los Angeles; Louis O'Shaughnessy, Virginia Polytechnic Institute; S. P. Timoshenko, University of Michigan; Dunham Jackson, W. E. Brooke, and M. E. Haggerty, University of Minnesota; H. L. Rietz, State University of Iowa; W. J. Berry, Polytechnic Institute of Brooklyn; R. C. Archibald, Brown University; E. V. Huntington, Harvard University; J. W. Young, Dartmouth College; Warren Weaver and C. S. Slichter, University of Wisconsin; Thornton C. Fry, Bell Telephone Laboratories; Charles N. Moore, University of Cincinnati; Leigh Page, Yale University.

Attendance at the Session is open to teachers of mathematics in general and particularly to teachers of mathematics to engineering students. The cost of attendance, aside from traveling expenses, is limited to a registration fee of \$10 and to the charge for room and meals which will be approximately \$35 for the two week period.

H. P. Hammond of the Polytechnic Institute of Brooklyn is the general director of the S.P.E.E. Summer Schools. Further information concerning the session can be secured from Professor Hammond by addressing him at 99 Livingston Street, Brooklyn, New York.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. N. B. MacLean, formerly professor of mathematics at the University of Winnipeg, has been appointed professor of mathematics at McGill University, as successor to Professor D. A. Murray, retired.

The following appointments to instructorships in mathematics are announced:

Iowa State College, J. C. Hempstead.

University of Michigan, Dr. J. D. Elder.

Dr. R. A. Fisher, Chief Statistician of Rothamsted Experimental Station, Harpenden, England, will be on the staff of the Department of Mathematics at the University of Minnesota during the second half of the summer session, July 27 to August 29. He will lecture on "The Theory of Estimation" and on "The Statistical Theory of Experimental Design". Professor Griffith C. Evans of the Rice Institute will also be in residence at the University of Minnesota during the entire summer session. During the first term, from June 17 to July 25, Professor Evans will lecture on "The Mathematical Theory of Economics". In the second term he will present "Potential Theory". The courses of Professor Evans and Dr. Fisher are in addition to the usual summer offerings of the Department of Mathematics.

The following courses in mathematics are announced for the summer 1931:

University of Chicago, first term, June 22–July 24; second term, July 27–Aug. 28. In addition to the regular courses in differential and integral calculus, the following advanced courses will be offered: By Professor H. E. Slaught: Differential equations; Elliptic integrals. By Professor G. A. Bliss: Algebraic functions. By Professor A. C. Lunn: Thermodynamics; Vector analysis in Riemann-Einstein space. By Professor M. I. Logsdon: Projective geometry of Hyperspaces. By Professor L. M. Graves: Advanced calculus; Theory of functions of real variables. By Professor R. W. Barnard: Linear algebras. By Professor E. Hille: Theory of equations; Fourier series. By Professor O. Zariski: Synthetic projective geometry; Applications of linear series on an algebraic curve. By Professor W. Bartky: Celestial mechanics; Modern theories of differential equations I.

University of Colorado, first term, June 22 to July 25; second term, July 27 to August 28. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, and calculus, the following courses will be offered: First term —By Professor Light: Teachers' course in mathematics; Statistics; Differential equations of mathematical physics. By Professor Kempner: Differential equations; Functions of a complex variable. Second term—By Professor Light: His-

tory of mathematics; Theory of finance; Differential equations of mathematical physics (continued). By Professor Kempner: Differential equations (continued); Functions of a Complex variable (continued); Theory of equations.

University of Michigan, June 29 to August 21. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor J. W. Bradshaw: Projective geometry; Teaching of geometry. By Professor P. Field: Vector analysis; Engineering problems. By Professor W. B. Ford: Advanced calculus; Infinite series. By Professor T. H. Hildebrandt: Topics in calculus; Functions of a real variable. By Professor G. Y. Rainich: Mathematics of relativity. By Professor L. A. Hopkins: Analytic mechanics. By Professor H. C. Carver: Mathematical theory of statistics. By Professor R. L. Wilder: Foundations of mathematics; Theory of sets of points. By Professor Norman H. Anning: Differential equations; History of mathematics. By Professor W. L. Ayres: Foundations of Euclidean geometry. By Professor C. J. Coe: Integral equations. By Professor A. H. Copeland: Theory of probability. By Professor N. C. Fisk: Graphical methods. By Professor J. A. Nyswander: Finite differences.

Northwestern University, June 22 to August 15. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor E. J. Moulton: Theory of numbers. By Professor F. E. Wood: Teaching of mathematics in secondary schools. By Professor H. S. Wall: Advanced calculus.

Ohio State University, June 15 to August 28. In addition to the usual elementary courses in algebra, analytic geometry, and calculus, the following advanced courses will be offered: By Professor H. W. Kuhn: Advanced calculus; Theory of finite groups. By Professor H. Blumberg: Introduction to the theory of relativity; Methods and problems in the theory of real functions. By Professor F. R. Bamforth: Solid analytic geometry; Modern theories in ordinary differential equations. By Dr. P. M. Swingle: Introduction to higher algebra.

University of Pennsylvania, July 6 to August 15. In addition to elementary courses, the following advanced courses will be offered: By Professor H. H. Mitchell; Projective geometry. By Professor G. G. Chambers: Advanced calculus. By Professor S. P. Shugert: Galois theory of equations. By Professor P. A. Caris: Diophantine analysis. By Professor J. A. Shohat: Theory of approximation and differential equations.

University of Pittsburgh, June 15 to August 21. In addition to the usual undergraduate courses, the following advanced courses will be offered. By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics. By Professor F. A. Foraker: Modern synthetic geometry; Solid analytic geometry. By Associate Professor Taylor: Advanced calculus; Functions of a real variable. By Assistant Professor Culver: Differential equations; Theory of equations.

University of Texas, first term, June 9 to July 20; second term, July 18 to August 31. In addition to freshman courses given in both terms, the following courses will be offered. First term—By Professor R. L. Moore: Foundations of geometry; Functions of real variables. By Professor E. L. Dodd: Probability; Infinite processes. By Professor H. J. Ettlinger: Ruler and compass constructions; Research in differential equations. By Professor H. S. Vandiver: Number theory. By Professor P. M. Batchelder: Teaching problems in mathematics. By Professor Mary E. Decherd: Calculus. By Professors C. M. Cleveland and H. V. Craig: Advanced calculus. Second term—By Professor R. G. Lubben: Non-Euclidean geometry; Analytic functions. By Professor J. H. Roberts: Calculus; Functions of real variables. By Professor E. G. Keller: Advanced calculus; Potential theory.

University of Wisconsin, six weeks session, June 29 to August 7. By Professor H. P. Evans: Differential equations; Theory of equations; Definite integrals. By Professor W. W. Hart: Advanced course in the teaching of high school mathematics; Teaching of mathematics. By Professor R. E. Langer: Advanced calculus; Modern analytic geometry. By Professor M. Marden: Differential and integral calculus. By Professor Warren Weaver: Mechanics of a particle. Special nine weeks session for graduates, June 29 to August 28. By Professor M. H. Ingraham: Introduction to general analysis; Theory of numbers. By Professor Warren Weaver: Tensor analysis and its applications to the theory of relativity.

Information has just been received from Japan of the death of Norifumi Okamoto, who has recently been connected with the Imperial Academy of Tokyo, and has been in charge of the cataloguing of the old mathematical manuscripts contained in that library.

Mr. Okamoto was one of the last representatives of the old Japanese school of mathematics, but in addition to his knowledge of the ancient works, he was thoroughly conversant with the nature of the western mathematics introduced after the Restoration of 1868. In his younger years he translated some of the western literature in this subject. He had a long experience in teaching in the normal schools, and Peers' College, and the Military Officers' School.

He had not completed the work of the cataloguing of the large collection of ancient mathematical books and manuscripts, but it is to be hoped that this can be carried on by some one who has worked with him. It would seem to American scholars that Mr. Mikami would naturally take up such an important work.

THE CHAUVENET PRIZE

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period of 1929-1931.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, April 15.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 14.
TEXAS, Fort Worth, Tex., Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE CO-OPERATION OF

ELIZABETH CARLSON
N. A. COURT
OTTO DUNKEL
H. S. EVERETT
B. F. FINKEL
R. E. GILMAN
R. A. JOHNSON
H. W. KUHN
J. R. MUSSELMAN
H. L. OLSON
D. E. SMITH
F. M. WEIDA

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 5, MAY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

A REPORT OF THE COMMITTEE ON COLLEGE ENTRANCE REQUIREMENTS IN GEOMETRY

Early in 1929 a committee was appointed jointly by the Mathematical Association of America and the National Council of Teachers of Mathematics, to study the feasibility of a proposal that college entrance requirements in geometry should be modified so as to bring about the more extensive introduction of courses including the essentials of plane and solid geometry in a single year's work, in place of the traditional year of plane geometry. The committee begs leave to report as follows:

There is a widespread conviction held by a group of teachers impressive both as to numbers and as to scientific and educational standing, that the proposed rearrangement will be a fundamental improvement in the teaching of geometry. The leaders of this group are teachers who believe in the importance of mathematics as part of a liberal education. Their desire is not to diminish the amount of time given to the teaching of geometry, but to increase the effectiveness of that teaching. Their chief concern is to remedy the situation created by the progressive disappearance of solid geometry from the course of study of the average high school pupil, and to bring some important parts of it into the program of pupils who under present conditions would learn nothing of it at all.

The committee believes that they should be given all practicable support and encouragement in their endeavor to solve the problems involved in the rearrangement, and believes in particular that participation by the College Entrance Examination

[1] Presented to the Trustees of the Association at Cleveland, December 31, 1930, and to the Board of Directors of the National Council of Teachers of Mathematics at Detroit, February 20, 1931. A preliminary announcement of the Committee was published in an earlier number of the Monthly, vol. 36 (1929), pp. 347–348, as well as in the Mathematics Teacher for December, 1929.

Board will be a peculiarly effective and highly essential influence toward the accomplishment of the purpose.

At the same time it is clear that very many teachers, probably a great majority numerically, are unconvinced as to the desirability of the change, and that anything like a complete replacement of the existing treatment of plane geometry is not to be anticipated until the new courses have had ample time to establish and justify themselves.

There are conspicuous instances already of activity along the lines of the proposed revision. Carefully planned one-year courses in plane and solid geometry have been adopted as standard in the public high schools of Denver, and as alternative to the regular course in plane geometry in the schools of Oakland. (It may not be inappropriate to point out that although somewhat overshadowed by a larger neighbor, Oakland itself is a city of approximately the same size as Denver or Providence.) Definite steps toward the introduction of similar courses have been taken by the organization representing the teachers of mathematics in the State of Kansas, by the schools of Detroit and New York City, and by the Secondary School Board, an organization of influential private preparatory schools in the East.

As to the attitude of the colleges, letters of inquiry were addressed by the committee to representatives of twenty-four institutions which make extensive use of the examinations of the College Entrance Examination Board, and rather more than half of the replies indicated favorable interest in the project either on the part of the departments of mathematics concerned or on the part of some of their individual members. The present discussion, as far as it relates specifically to college entrance requirements, appears to be an outgrowth of communications published in the American Mathematical Monthly in 1928 by Professor Beatley of Harvard University and Professor H. W. Tyler of the Massachusetts Institute of Technology.

The launching of the committee itself was a result of the active personal interest of both President Young of the Association and President Barber of the Council.

The present standard course in geometry is still powerfully influenced by the tradition of Euclid, who wrote before the methods of algebra, trigonometry, analytic geometry, and the calculus were available. It is inevitable therefore that the treatment of the course should be subject to perpetual re-examination, in the light of modern methods for the analysis of its mathematical content, as well as of prevailing conditions in the body of students to be taught. It is noteworthy that the teaching of geometry was the dominant theme of the recent meeting of the Council in Atlantic City, as well as of its Fifth Yearbook. The fact that the consolidation of plane and solid geometry is only one of a great variety of changes that might be proposed, is all the more reason for concentrating effort toward the solution of this particular problem while the attention of teachers is focussed upon it.

On the other hand, if geometry in the schools has more than merely utili-

tarian value—and the sponsors of the present movement are among those who believe that it has—that value is bound up with some degree of continuity in its teaching. Much of its importance lies in the fact that it is a part of the common background and experience of educated men. The Pythagorean theorem is significant not merely because it is true, but because intelligent people everywhere know that it is true. Changes in method and content must not be so radical as to make one man's geometry unrecognizably different from another's. The skill acquired by experienced teachers in years of service is not to be redirected too abruptly. Those colleges whose avowed purpose is to attract the best pupils from small and remote schools must continue to recognize the traditional methods which will prevail in those schools for a long time to come.

Although it is not the function of this committee to formulate detailed recommendations, it may be appropriate to set down some further considerations which bear on the working out of the problem.

A most essential feature of the Board's examinations is the presence of really substantial "originals." The power exhibited by the better candidates in attacking such originals bears eloquent testimony to the ability of the candidates and to the effectiveness of the instruction that they have received. Unless the revision can be carried through without detriment to this feature of the requirement, its purpose will have been defeated at the outset. If it were demanded that the range of the course should be largely increased without diminution of its average depth, the attempt might be dismissed as obviously hopeless. But if it is granted that the maximum depth rather than the average depth is the important thing, the task may not be impossible. Specifically, it may be recalled that originals to try the powers of any candidate can be based on the content of the first three books of plane geometry.

If the pupil can once be brought to appreciate what consecutive deductive reasoning means, he will have a healthier attitude toward geometry if he then learns more facts than are completely proved, with frank and explicit recognition of logical omissions, than if he tries to believe that the last word has been said on the subject when the last page of the book is reached. To admit that geometry as we actually know it, personally and individually, is far from abstract finality, is not to surrender its essential character, but to bring it closer to the rest of human experience.

The detailed choice of material will naturally be a subject for extensive study and debate. There will be differences of opinion as to the amount of stress to be placed on the content of the first book of solid geometry, according as it is considered to be the most important or the most repellent part of the subject. A distinction may perhaps be made between the numerous and elusive proofs whose precise form is significant only in connection with a particular order of propositions, and the few characteristic and substantial demonstrations which erect the first three-dimensional structures from two-dimensional elements. The measurement of solid figures will undoubtedly be included, but is susceptible of treatment varying all the way from theoretical prominence to incidental

computation. The geometry of the surface of the sphere may be regarded as of prime importance in a science which still goes by the name of earth-measurement, or as comparatively inaccessible. A no less fundamental problem will be the selection of the parts of plane geometry to be omitted.

The amount of ground that can be covered is not to be estimated by a mere count of propositions. Much time and maturity of experience are required for the assimilation of fundamental concepts, which when once mastered appear so obvious that it is hard to make due allowance for the process of their introduction. Much may be accomplished by informal and repeated reference to the concepts of solid geometry early in the course, so that they are long since familiar when they first appear in the formal order of propositions. Or it may be that intuitive geometry in the junior high school is an essential foundation for the satisfactory carrying out of the whole program.

It is not to be overlooked, on the other hand, that an integral part of the project is the formulation of a second course for the senior high school, to be articulated with the first and to include the parts of plane and solid geometry not covered by the first year's work. While not a matter of concern to the numerical majority who take only one year of demonstrative geometry, adequate provision for the scientifically and professionally important minority who go further is a vitally essential part of the undertaking.

Various suggestions have been made for utilizing this opportunity of bringing about a closer coordination between geometry and other branches of mathematics. It has been urged, for example, that more extensive use should be made of algebraic methods. On the other hand, it may be worth while to point out that the "paradox" of extraneous roots in algebra is merely a matter of failing to distinguish between a proposition and its converse. It has been proposed that numerical trigonometry should be included in the geometry course as well as in the course in algebra. The work in geometry may both profit by the use of coordinates and clarify the pupil's understanding of them if he encounters them elsewhere. Without any attempt at a premature introduction of methods of the calculus, some hints as to the nature and efficacy of those methods may be left where the intelligent student will find them.

The working out of these suggestions is largely a matter for teachers and textbook writers rather than for the framers of formal entrance requirements. It may be possible nevertheless for the latter also to exert some influence toward putting them into effect, without impairing the distinctive character of the course in geometry as such.

A necessary auxiliary to the satisfactory establishment of the new courses will be the preparation of suitable textbooks. It is to be anticipated, however, that such textbooks will be forthcoming, if it becomes clear that the attitude of schools and colleges is such as to render them commercially feasible.

As a means of putting into effect the proposals under discussion, *the present committee suggests that a small committee, of three or five members, be appointed by the College Entrance Examination Board, to formulate appropriate requirements*

for tentative adoption as alternatives to the present Mathematics C and Mathematics D. A small committee is recommended because the speedy and vigorous adoption of a temporary working basis is regarded as more important than an attempt to reconcile all varieties of opinion in the present transitional stage. The committee would naturally be authorized and expected to avail itself of the advice of persons outside its membership. It is further suggested that the same committee be asked to construct examination papers under the new requirements for a period of years, subject to suitable revision and control, instead of calling upon the regular examiners to assume this additional responsibility, and that it be regarded accordingly as a standing committee to observe the progress of the experiment and to make subsequent recommendations as occasion demands.

Respectfully submitted,

Gertrude E. Allen	J. O. Hassler
C. M. Austin	Dunham Jackson (Chairman)
Ralph Beatley	C. N. Moore
Walter F. Downey	W. D. Reeve
Elizabeth L. Hall	Edwin W. Schreiber

RANDOM SAMPLING[1]

By W. A. SHEWHART, Bell Telephone Laboratories, Inc.

I. INTRODUCTION

Have you ever noticed a number of small boys playing in a vacant lot when one of the gang comes along with something new? Immediately there is an outburst of questions—"What you got?" "What's it good for?" This afternoon I am thinking of a somewhat similar scene. Instead of a group of small boys we have a host of research workers and industrial artisans worrying over the interpretation of chance variations in samples of data which they cannot explain. Into that group comes another from a strange land with a new tool—the theory of random sampling for use in interpreting chance phenomena. Immediately we hear the same two questions: "What you got?" "What's it good for?" What I have in mind to say this afternoon has to do with the answers to these two questions. To begin, however, let us look at the practical and almost universal need for such a tool, because what I have to say is tempered by such needs.

II. NEED FOR A THEORY OF SAMPLING

1. *Universal Need for Sampling Theory.*

The ultimate aim of research is to reduce everything to known laws, thereby doing away with chance. In the laboratory where all but one variable can often be quite thoroughly controlled, research progress has been very encouraging.

[1] A paper presented at the invitation of the program committee of the Mathematical Association of America at its meeting at Brown University, Providence, Rhode Island, September 8, 1930.

Even here, however, when the most refined measurements are made, such, for example, as the determination of the charge on an electron by Millikan, there remains a nucleus of effects of uncontrolled or chance causes. In other words, chance we have with us always.

FIG. 1

Bridgman in *The Logic of Modern Physics* admirably sums up the state of affairs thus:

> A situation like this merely means that those details which determine the future in terms of the past may be so deep in the structure that at present we have no immediate experimental knowledge of them and we may for the present be compelled to give a treatment from a statistical point of view based on considerations of probability.

Even the field of "exact" science, so-called, is being invaded by statistical concepts. No longer do we think of physical properties as constant quantities—instead, they are *frequency distribution functions*. In the same way, we think of relationships between physical quantities as frequency distributions in two or more dimensions. For these reasons it appears that all prediction must be based

upon samples and that *prediction within limits* is the only kind possible. The need for sampling theory appears to be universal.

To emphasize the statistical nature of properties of materials still often treated as constants, let us look through a microscope at a piece of ordinary steel, Fig. 1. What we see is anything but a homogeneous, isotropic body. Why this heterogeneous structure? The answer is—it is attributable to the effects of chance or unknown causes.

What is the effect of such irregularities upon the physical properties of steel when produced in some useful form as, for example, a supporting strand, a piece of which is shown in Fig. 2? The answer is that a physical property, say the

FIG. 2

breaking load, of such strand will be distributed as indicated in Fig. 3.

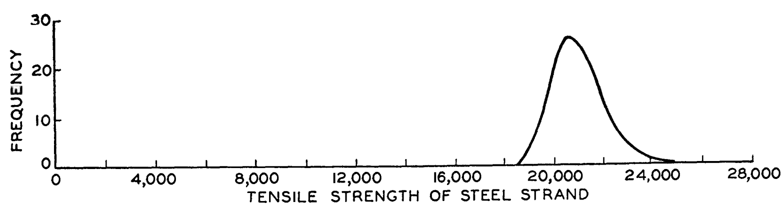


FIG. 3

Now, let us look at a cross section of another important structural material —wood, Fig. 4. This time we do not need a microscope to see the effects of chance causes upon the structure of the material.

Fig. 5 shows roughly what such irregularities do to the modulus of rupture of four kinds of telephone poles. Note the wide spreads of these distributions as compared with their means.

2. *Typical Specific Needs.*

We have said enough about the general need for a theory of sampling. Let us now consider some specific typical illustrations from the field of engineering and physical research, although similar ones could have been taken from any other field of human endeavor.

A. *Protection Against Loss of Life or Property.*

Two classes of conditions arise where an engineer must use sampling theory

in protecting against loss of life. One is where the thing upon which life depends cannot be measured otherwise than by sampling. The other is where the thing cannot be measured except through destructive tests.

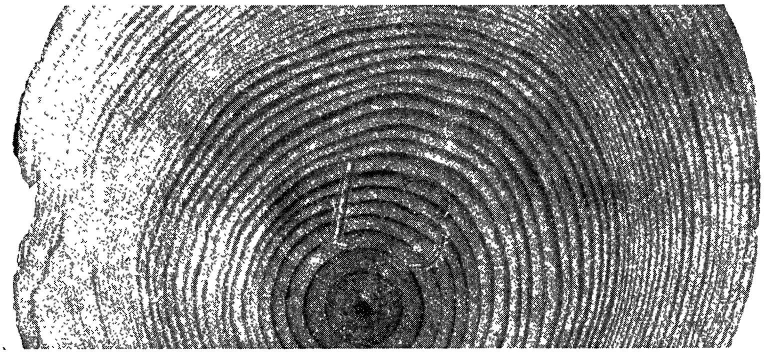


FIG. 4

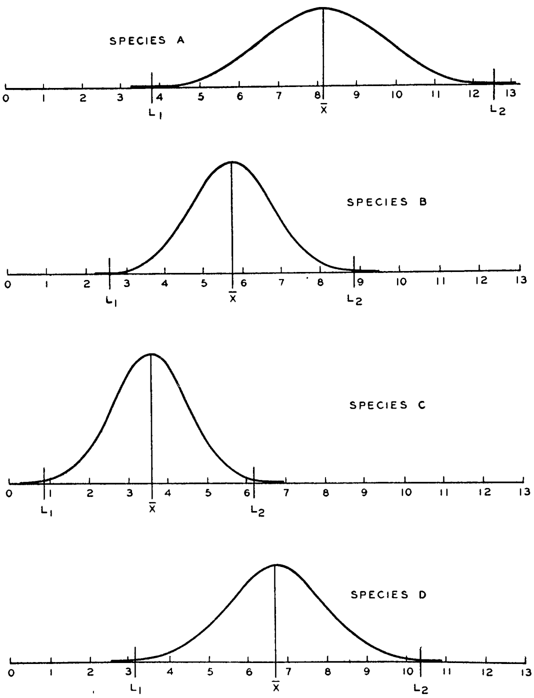


FIG. 5

As an example of the first class a civil engineer wants to build a dam or levee to hold back the flood waters of a given area, and hence wants to know the maximum run-off of that area to be expected in the future. If he does not build his

structure to take care of this maximum load, loss of life and property may ensue. From a sample of data such as that in Table 1 together with any other pertinent information, he must derive an estimate of the maximum run-off to be expected.

TABLE 1

Average Weekly Run-off—cu. ft. per second per sq. mile

Run-Off	Freq.	Run-Off	Freq.	Run-Off	Freq.	Run-Off	Freq.
29.30	1	6.82	1	4.27	8	1.85	57
15.50	1	6.77	2	4.16	7	1.74	60
13.60	1	6.52	2	4.05	5	1.64	50
12.65	1	6.45	2	3.95	10	1.53	45
12.50	1	6.36	2	3.85	9	1.43	50
11.55	1	6.25	2	3.75	10	1.32	50
11.30	1	6.06	2	3.64	10	1.21	57
10.52	1	5.95	4	3.54	9	1.10	43
10.35	1	5.85	2	3.43	15	1.03	15
9.90	1	5.74	1	3.32	17	.97	20
9.70	1	5.64	3	3.21	15	.92	13
9.65	1	5.53	1	3.10	17	.87	15
9.40	1	5.43	3	3.00	17	.82	15
9.33	1	5.32	2	2.90	20	.76	11
8.82	1	5.22	1	2.80	22	.71	8
8.60	1	5.11	4	2.69	26	.66	7
8.43	1	5.00	4	2.59	20	.61	2
8.05	1	4.90	3	2.48	30	.55	3
7.50	1	4.79	2	2.37	23	.50	2
7.45	1	4.69	5	2.26	30	.40	1
7.30	1	4.58	2	2.16	33	.34	3
7.23	1	4.47	9	2.06	35	.29	3
7.16	1	4.37	4	1.95	40		

The following typical questions are suggestive of examples of the second class. What is the breaking load of any structural member such as the steering-rod in your car? How long may you safely use any physical thing which deteriorates with age, such as the rubber in the tires of your car? Will your fire extinguisher work? At what pressure will the boiler in the basement of your home explode?

As a specific illustration, what can we say about the quality of the clay conduit piled up in the background of Fig. 6 from the results of the breaking tests conducted on the sample of four pieces shown in the foreground of this figure?

B. *How Shall we Take a Sample?*

How would you select a sample of 100 poles from those in the pole-yard of Fig. 7 to be inspected for modulus of rupture?

As another illustrative question, what plan would you propose for sampling the quality of the soldered terminals in a telephone office. To be more specific,

let us consider the panel in Fig. 8 on which there are 4500 such terminals. In a large office there are about 50 such panels. Of course we must keep in mind that the very process of inspecting the terminals may tend to loosen the soldered connections and that the taking of a second sample, if the first is found to be too small, is an expensive procedure.

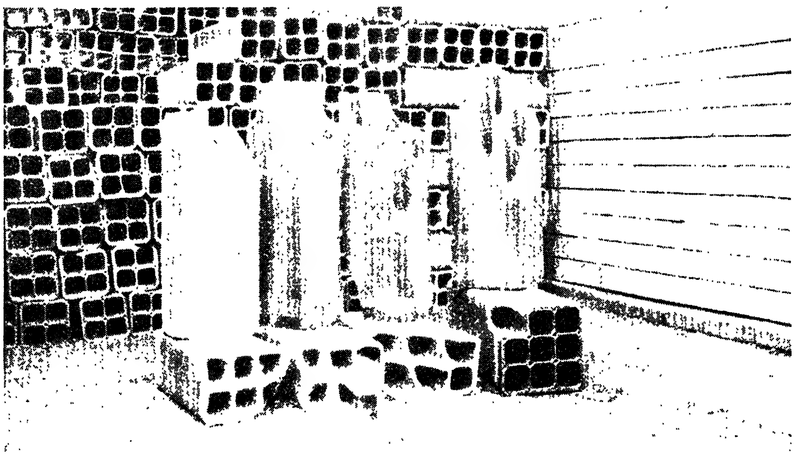


FIG. 6

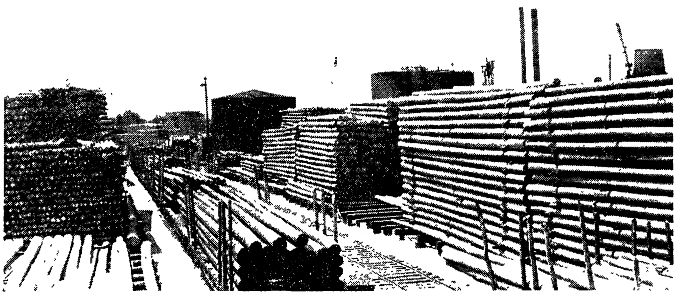


FIG. 7

In a similar way we may ask: How would you sample the bricks from a kiln, a sheet of brass, a keg of nails, a boat load of hides, a shipment of 50 barrels of oil, a barrel of flour, your drinking water, etc.? We might extend at great length the list of such problems.

Furthermore, the question as to how large a sample shall be taken enters into

every sampling problem. In fact, we need to know how to take a sample and how many to take in almost everything we do.

C. *How Shall We Specify the Standard of Quality?*

Typical questions are: How shall we estimate and tabulate the physical and chemical properties of materials in such a way that they may be used to greatest advantage in satisfying human wants, remembering that such qualities are al-

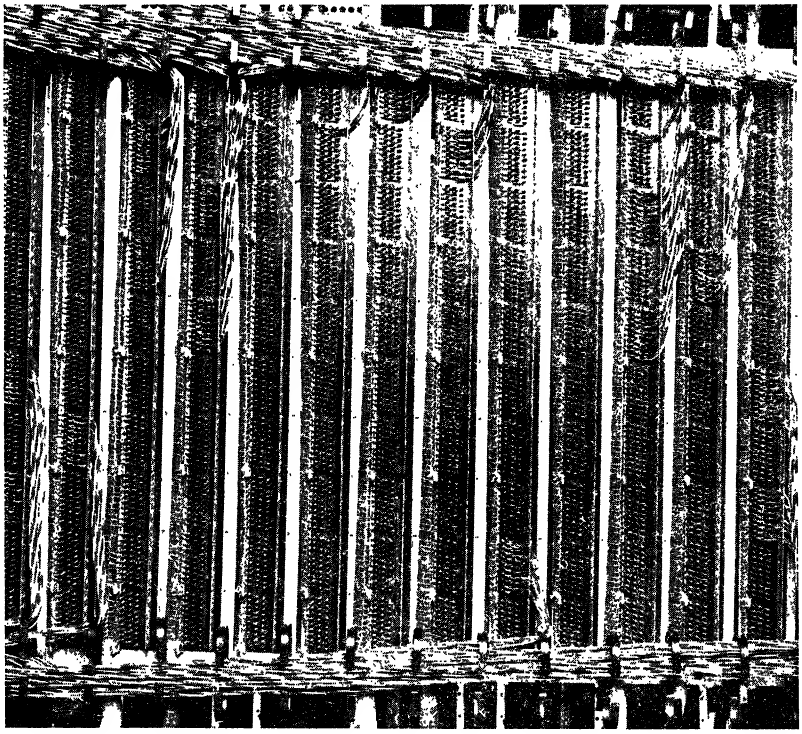


FIG. 8

ways frequency distribution functions? How shall we find such functions from the available sample of data? Shall we bother to try to specify the functional form or will it be sufficient merely to specify certain characteristics of a distribution function such as the average and standard deviation? How much information of this character is it necessary for a design engineer to have in order to secure minimum variability in the qualities of a designed structure at a given cost?

The standard qualities of the things we make and the things we eat to sustain life are frequency distributions. How shall we specify these to make it possible for inspection to give the greatest assurance of satisfactory quality?

D. *When Must Variability Be Left to Chance?*

When is the difference in quality of material from two or more sources greater than should be left to chance? For example, are the monthly variations

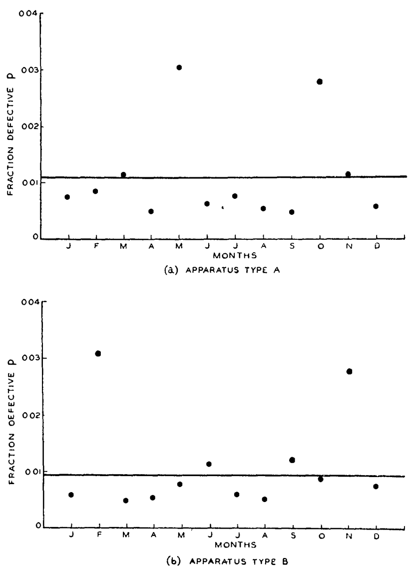


FIG. 9

TABLE 2

5045	4565	4895	4335	4550	4685	4850	4700	4500
4350	4410	4255	5000	4700	4430	4450	4500	4765
4350	4065	4170	4615	4310	4300	3635	4840	4500
3975	4565	3850	4215	4310	4690	3635	5075	4500
4290	5190	4445	4275	5000	4560	3635	5000	4850
4430	4725	4650	4275	4575	3975	3900	4770	4930
4485	4640	4170	5000	4700	2965	4340	4570	4700
4285	4640	4255	4615	4430	4080	4340	4925	4890
3980	4895	4170	4735	4850	4080	3665	4775	4625
3925	4790	4375	4215	4850	4425	3775	5075	4425
3645	4845	4175	4700	4570	4300	5000	4925	4135
3760	4700	4550	4700	4570	4430	4850	5075	4190
3300	4600	4450	4700	4855	4840	4775	4925	4080
3685	4110	2855	4700	4160	4840	4500	5250	3690
3463	4410	2920	4095	4325	4310	4770	4915	5050
5200	4180	4375	4095	4125	4185	4500	5600	4625
5100	4790	4375	3940	4100	4570	4770	5075	5150
4635	4790	4355	3700	4340	4700	5150	4450	5250
5100	4340	4090	3650	4575	4440	4850	4215	5000
5450	4895	5000	4445	3875	4850	4700	4325	5000
4635	5750	4335	4000	4050	4125	5000	4665	
4720	4740	5000	4845	4050	4450	5000	4615	
4810	5000	4640	5000	4685	4450	5000	4615	

in qualities of the two types of product shown in Fig. 9 indicative of the presence of assignable causes of variation or those which should not be left to chance?

Or again suppose an industrial research engineer interested in the production of a high resistance material of a given type finds that the first 204 successive pieces gave the results shown in Table 2 expressed in megohms. Suppose that this engineer asks if he has gone as far as he can reasonably expect to go in eliminating variability. What does sampling theory enable him to say?

E. *When Do We Know that Data are Good?*

If one gets to talking with any group of engineers about the application of sampling theory to their problem, he will likely run up against a criticism like the following:

"I agree that statistical or sampling theory has its place, perhaps, if we start with good data such as Millikan's measurements of the charge on an electron. In that case some of the high-brow theory may help one to get what he believes to be the most satisfactory estimate of a probable error. However, suppose that one starts with almost any ordinary set of engineering data as, for example, the observed values of modulus of rupture of 58 telephone poles, Table 3. Do you mean to tell me that a statistician claims to be justified in using all his high-brow formulas on this set of data to find the expected value and standard deviation of the modulus of rupture to the same degree that he would be in using the same formulas to find the best estimate of the charge on an electron and the probable error of his estimate?"

TABLE 3

Modulus of Rupture in lbs. per sq. in. of Type A Telephone Poles.

6930	6700	5910	7290	5090	6830
7100	5270	6370	5880	6870	7800
6650	6830	7330	5310	8120	6640
4690	7540	6140	6040	6650	9360
7550	7620	5510	8230	5550	6890
7620	6770	6560	4580	7720	9360
5990	7630	4880	5400	6770	5090
6970	4690	6010	5560	8220	5760
6490	5980	4580	5480	7600	
7630	8230	4610	6930	8190	

This kind of question is likely to give a statistician quite a jolt when he hears it for the first time. All of us know that when a man of research gets as good a set of data as that of Millikan he doesn't fuss much with the statistical theory of estimation. We also know that there are many sets of data not worth the paper they are written on—certainly not worth fussing over with high-brow instruments of estimation.

If sampling theory is in a position where those who have good data don't need it and those who do not have good data can't use it, *the industrial statistician might just as well shut up shop and find a new interest in life.* Discussions of sampling theory should show the way out of this apparent dilemma.

3. *Résumé of Situation to be Met by Theory.*

We have seen the kinds of questions that confront the theory of random sampling. In general, the theory must assist us in interpreting a sample in terms of the future under practical conditions where the variations in the sample are produced by unknown or chance causes. It is just this thing that the theory can be made to do for the practical man.

I fear, however, that great progress in application will not come until we have a comprehensive treatise of the theory in which emphasis is laid upon some of the essential elements of the theory often completely omitted from available discussions. The theory should be made to rest upon assumed physical laws and yet such laws are often not mentioned; the theory should be made broad enough to include practical problems and yet we shall soon see that certain current definitions of random sample exclude this possibility; some of the very important theorems of specification such as those of Tchebycheff and Camp-Meidell should be made available even though to do so pushes out some of the high-brow specifications of frequency curves of so much less practical importance. I fear that it is not sufficient merely to inform the practitioner that the theory is to be used as a tool. Instead, it is necessary to show him at what point the theory becomes of use and to show him what the theory will do *that he cannot hope to do as well without it.* The theory of random sampling *can be made* to meet this situation as we shall now see.

III. THE THEORY OF RANDOM SAMPLING—WHAT IS IT?

1. *What is a Random Sample?*

After I had decided what I wanted to say, it occurred to me that it might be well to see if the proposed remarks had to do with the subject assigned for discussion, namely, *Random Sampling.* Thereupon I gathered about me a goodly number of books on the theory of probability and the theory of statistics, and tried to find out the generally accepted meaning of random sampling. I thought it interesting to start by finding out what is meant by a sample. Two unabridged dictionaries gave the following definitions:

1. "A sample is a portion taken at random out of the quantity supposed to be homogeneous so that the qualities found in the sample may reasonably be expected to be found in the whole."[1]

2. "A sample is a part of anything taken at random out of a large quantity and presented for inspection or intended to be shown as evidence of the quality of the whole."[2]

If one accepts either of these definitions, it is obvious that the subject of my discussion is tautological. It should have been *Sampling.* To accept either of these definitions would play havoc with what I had in mind to say. If either of these definitions were accepted, much of what is written in the theory of sampling would have to be revised; because in practically every discussion we find the author talking about samples that are not random, a thing which obviously could not exist if, in accord with either of the definitions, a sample is

[1] Funk & Wagnall's Unabridged Dictionary.

[2] Century Dictionary and Encyclopedia.

something that is random. Is it any wonder that many of the possible applications of sampling theory go unmade?

The more I read, the more uncertain I became as to the prevalent conception of random sample. One thing certain, however, is that many of those found in standard places are not of much interest to a practical man.

For example, Yule, in that treasure house for statisticians, *An Introduction to the Theory of Statistics*, indicates that the usual concept of random sample is one drawn with replacement, although he explicitly rejects the term *random* because he takes it to mean simply *haphazard.* Jones also would have us believe that a random sample is one drawn with replacement. For example, he says in effect: To select 99 sheep from 999, number each sheep and place in a box 999 tickets numbered 1 to 999, one to correspond to each sheep, then pick out 99 tickets in succession being careful to replace each and shake up the box before picking out the next; if there were absolutely no difference between the tickets, such as would cause one to be picked more easily than another, the selection made in this way would be random.

Now, if a random sample were only that kind of a sample and if the theory of random sampling had to start with that kind of a sample, you can imagine how enthusiastic a purchaser of 999 sheep would be about the theory. To such a man that method of sampling would be foolish.

Not only is such a method of sampling often foolish from a practical viewpoint—very often indeed it is impossible to sample in this way. How would you make this kind of a random test of the tensile strength of a coil of wire—Fig. 10? I can't.

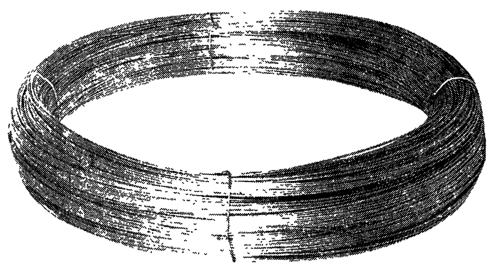


FIG. 10

Such a concept of random is certainly too narrow to get very far with in practice. The sample described by Yule is random, of course, but so are certain other kinds of samples as we shall soon see.

To start with, however, we must do as the physicist does in formulating a theory—adopt certain physical hypotheses or laws. This will give us a basis for a general definition.

2. *Law of Large Numbers.*

In the simplest form this law can be stated as follows: If an event which can

happen in two different ways be repeated a number n times under the same essential conditions, the ratio p of the number of times that it happens in one way to the total number n of trials will approach a definite statistical limit p' as the number n is increased indefinitely. Formally this law may be written

$$\lim_{s\, n \to \infty} p = p',$$

where $\lim_s$ stands for the statistical or stochastic limit.

A slightly more extended form of this law is as follows: If we make a series of n measurements

$$X_1, X_2, \cdots, X_i, \cdots, X_n$$

of some quality characteristic X in a way such that each measurement is made under the same essential conditions, the ratio p of the number of times that an observed value X will be found to lie within any specified range X_r to X_s to the total number n will approach a statistical limit p' as the number n is increased indefinitely.

An even more general statement of this law is: If we take a series of m samples of n measurements

$$\begin{array}{cccccc} X_{11}, & X_{12}, & \cdots, & X_{1i}, & \cdots, & X_{1n}, \\ X_{21}, & X_{22}, & \cdots, & X_{2i}, & \cdots, & X_{2n}, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{m1}, & X_{m2}, & \cdots, & X_{mi}, & \cdots, & X_{mn}, \end{array}$$

in such a way that each one of the m samples is drawn under the same essential conditions, and if we let θ be a symmetric function or statistic of the n values of X in a sample of size n, the ratio p of the number of times that the observed value of θ will be found to lie within the range θ_1 to θ_2 to the total number m of samples will approach a definite statistical limit p' as the number m of samples is increased indefinitely.

No doubt this law can be generalized even further, but in this form it serves the purposes of the present discussion.

We shall now assume without any consideration of the fine points involved—adequately discussed in so many places—that the statistical limit p' is an objective probability that can be substituted for the *a priori* probability in the mathematical formulas of the theory of specification, distribution, and estimation soon to be considered.

Of course, the Law of Large Numbers is perhaps the most universal law of nature. Even the parameters in the so-called exact laws are apparently constant only to the degree to be expected of the statistical limits which they really are.

3. *Definition of Random Sample.*

A sample drawn under conditions such that the Law of Large Numbers applies will be termed a random sample.

Obviously a random sample thus defined may be one drawn with replacement as characterized by Yule and Jones; one drawn without replacement, or one in which any part of the sample is drawn with replacement and the remainder without. So also is a Poisson sample a random sample, not to mention others that come under our general definition.

4. *Theory of Random Sampling.*

The statement of the methods of applying the Law of Large Numbers in the interpretation of random samples in terms of the future will be termed the theory of random sampling. At least five specific elements in this theory must be considered. They are:

1. The Law of Large Numbers and physical postulates making this law useful.
2. The mathematical theory of specification.
3. The mathematical theory of distribution.
4. The theory of estimation.
5. The rôle of human judgment.

We shall now see how these elements function in the practice of the theory.

IV. WHAT IS THE THEORY GOOD FOR?

1. *What do We Want the Theory to Do?*

In Part II we considered several types of problems. It will be helpful to generalize these problems still further by reducing them to two types. We may do this by developing a theory which will do two things.

a. Establish rules for prediction when observed data have been obtained under the same essential conditions.

b. Establish criteria to assist an experimentalist in determining when his data have been taken under the same essential conditions.

Suppose we note how the specific types of questions A, B, C, D, and E, of Part II can be reduced to one or the other of these two fundamental problems.

To predict such things as the maximum run-off of a flood area or the breaking load of any structural member, to set up standards of quality, and to determine how large a sample to take, involve either directly or indirectly the problem of going from a sample of n data to a set of m similar data not yet taken.

The Law of Large Numbers tells us that we can take this step *provided* the measurements have been made under the same essential conditions. So far as we know, however, there is no available basis for taking this step unless the samples are random in the sense of the generalized definition of Part III. Questions as to how we shall take a sample and what good data are, therefore, resolve themselves into the problem of taking data under the same essential conditions insofar as the effects of chance causes are concerned. So also, the very important problem of determining when variability in the characteristic of the phenomenon must be left to chance reduces to the second general problem stated at the beginning of this section.

In this same way, it appears that we can reduce any question involving the interpretation of a sample to one or the other of these two fundamental problems.

Hence, if we can show that the theory of random sampling helps in solving these two problems, we will have shown that it has in it something useful in the interpretation of samples.

2. *Contribution of the Theory of Random Sampling to the Solution of the First Fundamental Problem.*

It follows from what has been said that this problem can be considered as that of going from a sample to what we ordinarily term a universe, as schematically illustrated in Fig. 11. This is also the problem of estimation, so-called by R. A. Fisher.

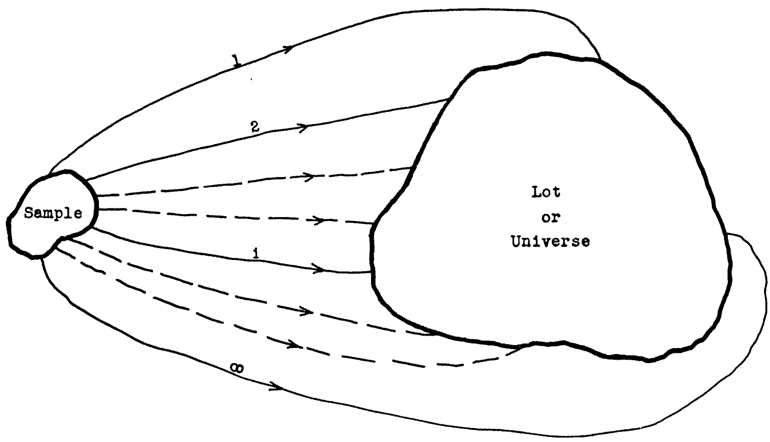


FIG. 11

There are in the literature the following three general different methods of going from a sample to its universe:

a. The *a posteriori* method.

b. The method of maximum likelihood.

c. The empirical method.

To mention these three in the same breath in the presence of a group of statisticians is almost certain to start an argument, for, as is well known, there is a wide divergence of opinion as to the comparative validities of these methods.

This afternoon, however, we have no desire to start such an argument. The only thing that we are interested in doing, is to call attention to the fact that we are apparently in a position to understand the nature of the problem better than ever before, and to state, as we shall soon do, certain specific things which seem to be true *independent* of the method we choose.

The *a posteriori* method is, of course, tied up with the name of Bayes and the discussions of probability of causes. The method of maximum likelihood is

largely the outgrowth of the work of R. A. Fisher, in particular, his paper[1] in the *Philosophical Transactions*, 1922, "On the Mathematical Foundations of Theoretical Statistics." The empirical method, so-called for the want of a better name, is that which involves the use of some constant c by which to multiply the observed value of a statistic θ in a sample of size n in order to get the best estimate of this same statistic of the universe.

One of the most interesting things from a practical viewpoint is that no one of these methods gives a unique answer which does not depend upon certain assumptions as to the nature of the universe. There are an indefinitely large number of such assumptions that may be used in connection with any one of the three generalized methods, and, depending on the assumption we wish to make, we get, through the use of available theory, a definite path of going from the sample to the universe as represented schematically by the lines connecting these two entities, Fig. 11.

Of course the practical man must choose *one* of the infinite set of possible ways of going from the sample to the universe. His choice is a matter of judgment. However, judgment can be exercized best after he becomes acquainted with available information contained in the detailed accounts of the significance of the choice of path.

Out of this theory there come four *fundamental contributions* of interest to the practical man, in spite of the fact that the formal theory does not give a unique answer to the fundamental problem.

1. The theory helps us to settle the right sort of difficulties and to raise the right sort of ulterior questions.

2. Through the use of judgment and the application of the Tchebycheff inequality, it follows that for most practical purposes it is sufficient to try to estimate only the average $\overline{X}'$ and the standard deviation σ' of the universe, there being little advantage to be gained for most practical purposes in attempting to go further in the way of specification.

3. No matter what one of the known ways of going from the sample to the universe we choose in a given case, it is desirable to start wherever possible with a knowledge of the average $\overline{X}$ and standard deviation σ of the observed set of n data.

4. It is evident that, in general, the best estimate of the standard deviation σ' of the universe derivable from the observed standard deviation σ of the sample is some multiple c times the observed standard deviation where, irrespective of the method used, c is greater than unity.

As a specific illustration, this part of the theory of random sampling indicates that the set of observed values of modulus of rupture of telephone poles given in Table 3 can best be summarized in terms of the average and standard deviation upon the assumption that these poles came from a constant system of chance causes or a condition wherein the probability that the modulus of rupture

[1] This paper also sets forth clearly the nature of the problems of specification, distribution, and estimation.

of a given pole lying within a given range is the same as the probability that any other pole will lie within this range. Furthermore, the best estimate of the standard deviation of the universe—this class of poles—is most likely something greater than the observed standard deviation, our particular choice of estimate depending upon which one of the infinite number of paths we choose to take. Under any conditions, however, provided the number in the sample is greater than, let us say, twenty-five as it is in this case, the magnitude of the correction factor is for all practical purposes unity.

The third contribution is of particular importance in the reduction of data in that it indicates that more extensive use than customary should be made of the two statistics, the average $\overline{X}$ and standard deviation σ of a sample, and furthermore that the sample size n should be recorded.

It is, of course, also of interest to know in any given case how the investigator himself would go from the sample to the universe, or, in other words, it is of interest to know the investigator's best estimate of the average $\overline{X}'$ and standard deviation σ' of the universe. Obviously, this estimate of the investigator contains in it not only the element of the original data but also the human element put in through his own personal judgment in choosing the requisite estimates. However, for some time to come, engineers and scientists in general will most likely want to be in a position where they can, if they please, choose some other set of estimates than those of the investigator. To make this possible, as already said, the investigator must tabulate the average $\overline{X}$ and the standard deviation σ in addition to his estimates.

It must be kept in mind, of course, that this infinite set of paths for going from the sample to the universe all rest upon a common assumption, namely, that each observation in the sample has been taken under the same essential conditions as any other observation.[1] Of course, if we do not start with such a set of data, it is of interest to know that there does not appear to be any available generalized method for going from the sample to the universe. This is of importance in that it indicates the necessity of eliminating causes of variability other than those which must be left to chance before attempting to apply the theory of random sampling as outlined above.

This situation seriously limits the application of this particular part of the theory. I believe that it is this state of affairs which so often prompts the engineer and scientific worker to criticize the applicability of the theory of random sampling on the score previously noted that his data are not sufficiently good to start with. In spite of this, however, it is the discussion of this first fundamental problem that receives the greatest attention in the literature which the man interested in the application of the theory of random sampling is most likely to read.

We shall now see that this situation calling for criticisms practically disap-

[1] We may, of course, think of the data as being statistics of samples of size n in which the samples have been taken under the same essential conditions even though the observations in a sample were not.

pears when we proceed with the discussion of the contribution of the theory to the solution of the second fundamental problem which in practice really arises before the first one does.

3. *Contribution of the Theory of Random Sampling to the Solution of the Second Fundamental Problem.*

By analyzing a set of data, can we get any indication as to whether or not they arose under the same essential conditions? As a specific example, did the set of 204 observations of resistance previously given in Table 2 arise under the same essential conditions? The theory of random sampling which includes the element of human judgment does provide a useful test as to whether or not the data have arisen under the same essential conditions.

Assuming the existence of the Law of Large Numbers, it follows that, if samples of size n are taken under the same essential conditions, there exists a frequency distribution function, Fig. 12, for any given statistic θ such that the

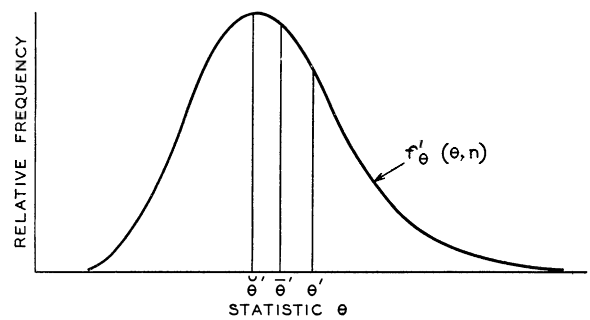


FIG. 12

integral of this function between any two limits θ_1 and θ_2 gives the probability that an observed value of θ will fall within these limits.[1] If, then, samples of size n do arise under the same essential conditions, and if this distribution function is known for these same essential conditions, we can say that P' of the observed values of the statistic θ for samples of size n should fall within the range fixed by the two limits $\bar{\theta}' \pm t\sigma'_\theta$ where $\bar{\theta}'$ is the expected or average value of θ and σ'_θ is the standard deviation of θ for samples of size n. The theory of specification and in particular Tchebycheff's theorem tells us that, if we take some range corresponding to the limits $\bar{\theta}' \pm t\sigma'_\theta$, then, no matter what the functional[2] form of the distribution, the probability of the value of the statistic θ falling within this range is greater than $1-t^{-2}$, where t is greater than unity. Thus Tchebycheff's theorem focuses our attention upon two characteristics of the frequency distribution function, namely, the average $\bar{\theta}'$ and the standard deviation σ'_θ. Knowing these, we can establish limits such as the dotted ones in Fig. 13 within which at least 89 per cent (if $t=3$) of the observed values of the statistic should lie provided the samples of size n arose under the same essential conditions.

[1] For our present purpose we shall assume that this distribution function is continuous.

[2] Subject to restrictions of no practical importance.

In a practical case such as that of the 204 measurements of resistance, we do not have the *a priori* distribution function, but if we are dealing with random samples, we can estimate the two desired characteristics of the distribution function for any given statistic. However, we do not usually know whether or not the data satisfy this condition. In fact, it is this very point that we want to settle. At first it looks like a hopeless tangle and it is, I believe, unless we invoke the element of judgement and the human faculty of postulation.

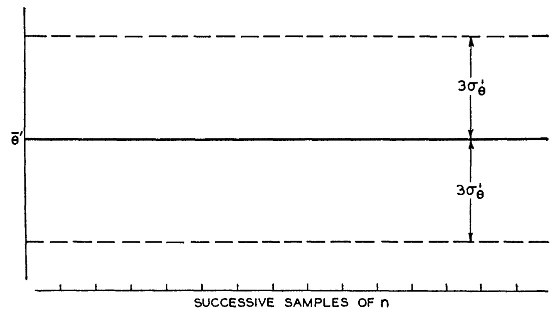


FIG. 13

Without more ado, we shall state two postulates, the reasonableness of which we cannot consider here.

Postulate 1: When we have done everything that we can do to find and to eliminate the chance or unknown causes of variation in a statistic θ of a sample of size n from sample to sample, the resultant variation in the statistic will be produced by a constant system of chance causes.

Postulate 2: It is humanly possible to find and eliminate causes of variability in the statistic until the remaining causes represent a constant system.

In what we shall say henceforth we shall assume that the constancy of a cause system applies to a single observation in the sample or, in other words, that we are dealing with the second class of random samples previously defined.

Suppose now that we have a set of N observed values that we wish to test to see whether or not it is likely that one observation arose under the same essential conditions as any other. It can be shown that it is reasonable to believe that if there is no *a priori* rational basis for subdividing these data into groups, then statistical theory is perhaps almost useless in helping us to decide the issue. If, however, there is a basis for dividing the total set of N observed values into m rational subgroups, the theory of random sampling does something very useful indeed. It tells us how to set up certain limits $\bar{\theta} \pm t\sigma_\theta$ for samples of size n from each of the m rational subgroups in such a way that, when an observed value θ falls outside these limits, it is unlikely that the individual observations arose under the same essential conditions.

Far more important, however, is the fact that this theory enables one to do economically what he wants to do by making possible the establishment of limits $\bar{\theta} \pm t\sigma_\theta$ within which observed values of a statistic θ for samples of size n

should fall with a probability P approximately equal to any previously desired probability P' *provided he has succeeded in eliminating causes of variability which can and should be found and weeded out.* Furthermore, these limits are such that

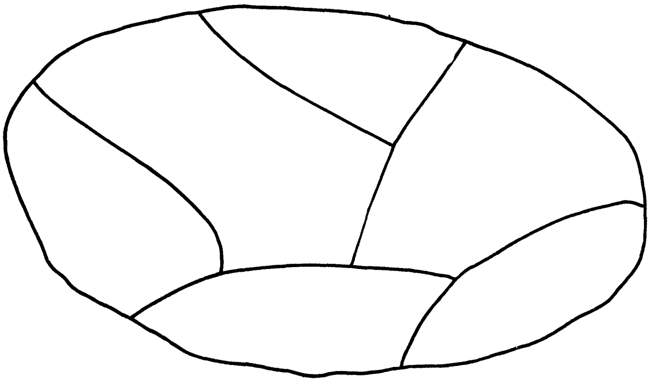


FIG. 14

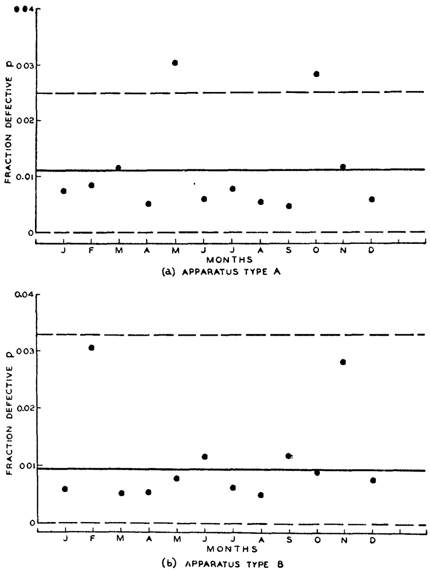


FIG. 15

if these causes do exist, it is likely that the probability $1-P$ of detecting them will be greater than $1-P'$ whereas in the long run, if the causes do not exist,

$$\lim_{\substack{s\\ n\to\infty}} \bar{\theta} = \bar{\theta}' \text{ and } \lim_{\substack{s\\ n\to\infty}} \sigma_\theta = \sigma_\theta',$$

so that P approaches the desired value P'.

Hence, if we are to take a sample of n observations about which we are to ask later whether or not each observation has been taken under the same essen-

tial conditions, we should first of all divide the objective set of observations into rational subgroups. Schematically we may think of the boundary of Fig. 14 as representing the universe of possible measurements and the subdivisions as representing the boundaries of the rational subgroups. We should then divide the sample of size n into subsamples, taking one subsample from each of the m rational subgroups where, of course, it is assumed that $n \geqq m$. We cannot here go into the details of the method of doing this under practical conditions.

It must suffice to give illustrations of how this method has been found to work in practice. For example, applying this test to the variations in Fig. 9, we get the control charts of Fig. 15. Although the variations in the qualities of these two types of apparatus appear to be about the same, we see that the test indicates trouble in one case and not in the other. The indication of trouble was found to be justified.

Similarly, applying the test to the data of Table 2, we find, Fig. 16, that the variations in resistance are such that they should not be left to chance as indicated by points falling outside the limits in the left side of this figure. In this case it is interesting to note that further research found at least some of the assignable causes of variability and removed them with the result that the variability then fell within the control limits as indicated in the right side of Fig. 16.

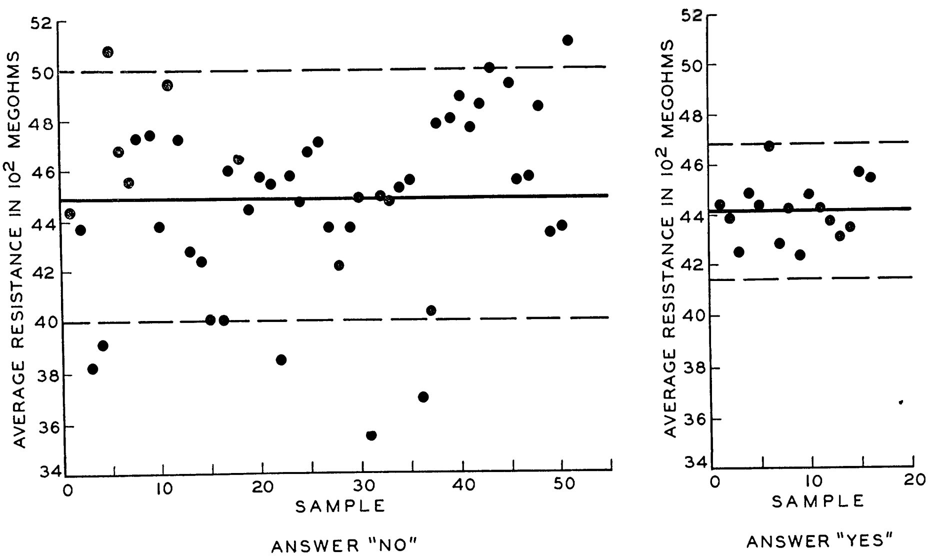


FIG. 16

If we apply this criterion to a set of data which *a priori* we have every reason to believe arose under a constant system of chance causes, such as Millikan's measurements of the charge on an electron, all of the points should lie within the dotted limits. They do, at least in this instance, Fig. 17.

4. *More About Applications.*

We may now consider further the contributions of the theory in answering some of the specific questions raised earlier in this paper. For example, how does the theory help the civil engineer to estimate the maximum run-off in terms of the data of Table 1 and the inspection engineer to establish plans for sampling the poles of Fig. 7 and the soldered terminals of Fig. 8? Let us consider the dam problem first.

A knowledge of the theory certainly makes an engineer very cautious about trying to infer anything about the maximum run-off through the use of the theory of estimation unless he feels sure that the causes of variation in run-off constitute a constant system of chance causes—and I fear that the causes do not usually behave in this way. On the other hand, the theory provides the engineer with a method for testing whether or not the data did come from a constant system of chance causes.

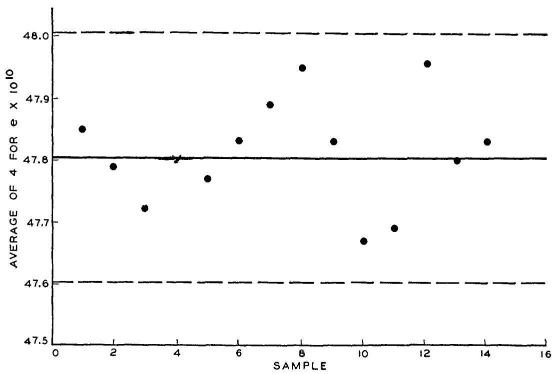


FIG. 17

Incidentally, it is of interest to note that, even assuming constant causes of variation in the run-off, such theory indicates that some of the attempts made in the literature to estimate maximum run-off through the use of high-brow frequency curves cannot hope to do much more than simply befog the issue by the welter of complicated formulas with their parameters almost as numerous as the available data themselves. On the positive side, however, the theory directs attention to the importance of estimating the average $\overline{X}'$ and standard deviation σ' of the universe corresponding to the assumed constant system of chance causes and of using these *intelligently* in the theorem of Tchebycheff.

In the inspection of poles and terminals, we wish to find out as much as possible about a finite set of N things by looking at only n of them. I think you will agree that in neither case is it feasible to tag all of the poles or all of the terminals shown in Figs. 7 and 8 and to set up a bowl experiment whereby we may get a random sample. What we must do under such conditions is to try to divide on *a priori* grounds the objective set of N things into rational subgroups. Then we must take some from each of the m rational subgroups so that the

sample of size n to be taken will be made up of component parts $n_1, n_2, \cdots, n_i, \cdots, n_m$. We can then use the theory of estimation for going from each subsample to its corresponding universe. By putting these predictions together we can go from the sample of size n to the lot N.

Thus, the theory shows us how to take a sample. It also tells us how many to take. More important, however, is the fact that it enables us to do what we want to do *economically*. For example, it helps a manufacturer to attain the following five economic advantages:[1]

1. Reduction in the cost of inspection.
2. Reduction in the cost of rejections.
3. Attainment of maximum benefits from quantity production.
4. Attainment of uniform quality even though inspection test is destructive.
5. Reduction in tolerance limits where quality measurement is indirect.

5. *Interplay of the Five Components in the Theory.*

The formal theory of specification, distribution, and estimation starts with numbers supposed to represent magnitudes of physical characteristics or entities. The operation of the formal part is independent of the nature of these entities and naturally cannot tell us whether or not we have established a set of numbers which represent the real entities in which we are interested. The choice of such a set of numbers is a function of the human mind.

To illustrate—what we sense through any one of our senses depends partly upon our previous use of these senses. For example, a child looking at a straight stick extending beneath the surface of a pool of water sees a bent stick. Similarly, the first time you saw what is shown in Fig. 18, you saw the length of the

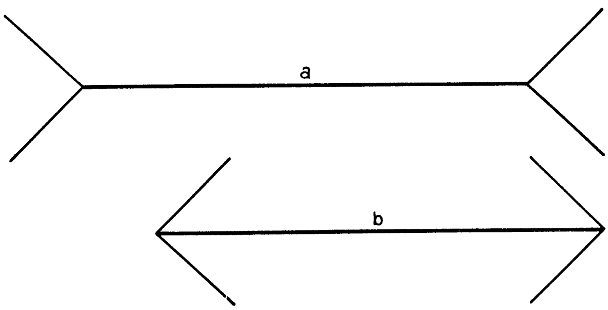


FIG. 18

line a to be different from the length of the line b. Without experience we see as one without experience.[2]

---

[1] For further discussion of this subject see "The Economic Quality Control of Manufactured Product," by W. A. Shewhart, Bell Telephone Laboratories Monograph 496, June, 1930.

[2] As William James so aptly puts it: "And the random irradiations and resettlements of our ideas which supervene upon our experience and constitute our free mental play are due entirely to the secondary internal processes which vary enormously from brain to brain even though brains be exposed to the same 'outer relations'." *Principles of Psychology*, Vol. II, p. 638.—Whitehead puts the same idea in slightly different words: "One main law which underlies modern progress is that,

Hence in the establishment of a set of numbers we must make sure that the mind exerts its proper influence. Whether or not we start with the proper set of numbers or, in other words, good data, in this sense depends almost exclusively upon the human element. It does not appear to depend in any way upon the formal mathematics or logic involved in the theory of specification, distribution, and estimation.

Not only in choosing the method of expressing our experience quantitatively in the form of numbers does judgment enter. We have seen that it must play an important rôle in dividing the objective set of conditions into rational subgroups. Here again the mind of man, his imagination, his intuition, his store of facts, his prehensions, and his knowledge of the world in which he lives function almost exclusively.

It is, however, not only in the taking of the data that human judgment functions. When we come to set up control charts for a given statistic, we must make further choices. Three important ones are:

1. The choice of statistics to be used.
2. The choice of the probability P' associated with the limits for a given statistic.
3. The choice of the way of using the statistics.

In making such choices, the mind alone is *not nearly so powerful* as mind plus the formal logical and mathematical theory of specification, distribution, and estimation.

I can do no more than briefly illustrate what I mean. In choosing a statistic from the indefinitely large number that might be chosen, we must consider the usefulness of such a statistic together with the cost of computing it. One of the factors which enters into such a choice is the variability of the statistic measured in terms of the variability of some statistic taken as standard. Such a measure has been called efficiency.

In this connection the comparatively recent theory of distribution becomes of great value in that it makes possible the comparison of the efficiencies of various estimates of a given statistic and in the end the choice of that one which it is most economical to use. For example, Fig. 19 shows the approximate efficiencies of the median and $\frac{1}{2}$(max.+min.) relative to the efficiency of the arithmetic mean taken as unity over the range of sample size from two to infinity.

Similarly, Fig. 20 shows the variation in efficiencies of the mean deviation and the range as estimates of the standard deviation in respect to the variation

---

except for the rarest accidents of chance, thought precedes observation. It may not decide the details, but it suggests the type. Nobody would count, whose mind was vacant of the idea of number. Nobody directs attention when there is nothing that he expects to see. The novel observation which comes by chance is a rare accident, and is usually wasted. For if there be no scheme to fit it into, its significance is lost. The way of thoughtless nature is by waste—a million seeds, and one tree; a million eggs, and one fish. In the same way, from a million observations of fact beyond the routine of human life it rarely happens that one useful development issues." *The Function of Reason,* p. 57.

of the root mean square deviation taken as unity.[1] Sometimes, of course, the economics of the situation, as for example the cost of computation, may counterbalance the efficiency of a statistic, particularly when the sample sizes are small. Hence it is essential that curves such as those presented in Figs. 19 and 20 should

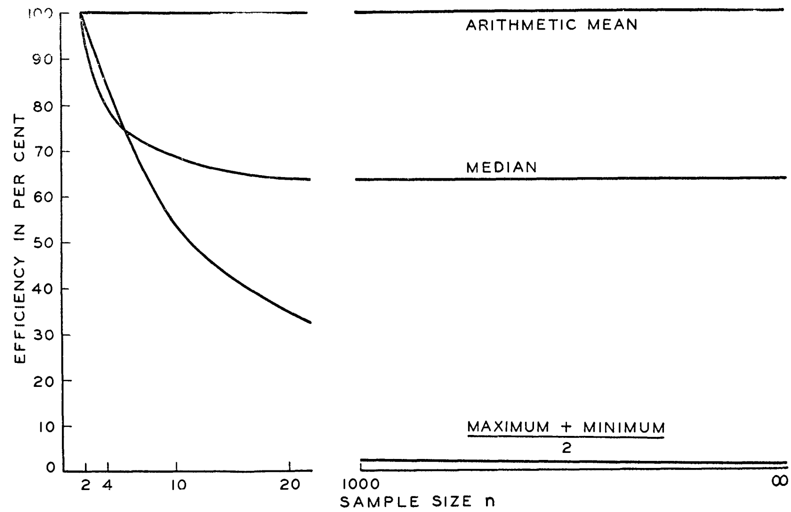


Fig. 19

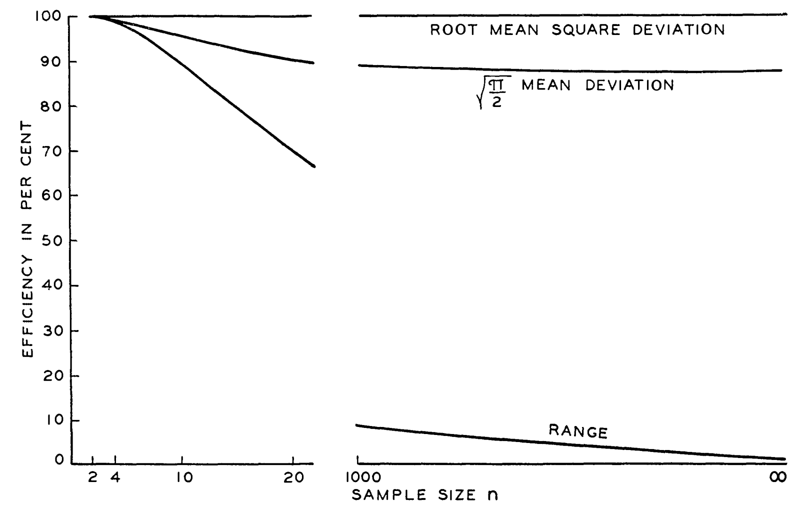


Fig. 20

be available to the applied statistician, if he is to make economical application of the theory. We see that, for comparatively small sample sizes (which for reasons we cannot go into here must often be used in control work because they

[1] Without the developments in the theory of distribution of the last decade these curves could not have been drawn.

catch trouble when larger samples would not), the efficiencies of the various estimates are not so widely different.

Moreover, recent theory enables us to compare the advantages of different ways of using statistics. For example, if we are to make use of the average and standard deviation, we may construct control charts on each statistic separately or we may construct an enclosed area in a plane based upon a consideration of the probability of the simultaneous occurrence of a given average and a given standard deviation. Or again, we may construct control charts upon the basis of differences in averages and differences in standard deviations. As an example, Fig. 21 illustrates the first two methods. On the basis of the first method, if an observed point such as either of the two shown in this figure, falls outside the

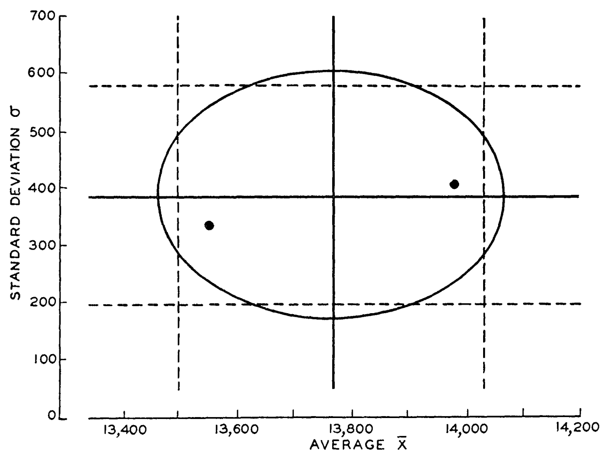


FIG. 21

dotted limits, this is taken as an indication of lack of constancy of the cause system. Similarly, if any point falls outside the ellipse, it is taken as a similar indication based upon the second method of using the statistics. It appears that there is always one of these many ways which is considerably better from a practical viewpoint than the others and it is the use of modern theories of distribution that enables one to choose the best way of using the statistics.

V. CONCLUSION

We started out to consider two questions often asked by the practical man about the theory of random sampling—"What is it?" and "What's it good for?" We have tried to throw a little light on the kind of answers that can be given and that seem to be of interest to the scientist and industrial artisan faced with practical problems that are not usually treated in books touching upon the theory of sampling.

Summary answers proposed are:

A. The theory of random sampling is that part of the scientific method which enables one to predict the future in terms of the past when the future is

related to the past through the Law of Large Numbers. The theory consists of a statement of the more or less accepted methods of using the human mind, and certain formal logical and mathematical elements classified usually under the subjects of specification, distribution, and estimation.

B. The useful purpose served by the theory is to define clearly those conditions under which prediction of the future can be made from a sample and to set forth definite procedures for predicting when possible.

Emphasis has been laid on the application of the theory in helping us to do what we want to do by telling us when variability need not be left to chance and in helping us to do this economically. Perhaps a little light has been thrown on the answer to another question: Why does the theory not enjoy broader application? Possibly there is need for a little shuffling of emphasis in discussions of the theory in the literature. Be that as it may, industry owes a debt of gratitude to the mathematicians who are doing so much to extend our knowledge of the subjects of specification, distribution, and estimation.

A NOTE ON THE PRINCIPLES OF MECHANICS[1]

By H. M. DADOURIAN, Trinity College

When the foundations of mechanics are cleared of special laws and theorems there remains a single fundamental principle from which the entire science of mechanics may be developed. The principles of *virtual work*, of *least action* and of *least constraint* are different forms of this principle. In some books on advanced mechanics this fact is recognized and made use of with great success in developing the subject in an orderly and logical manner, somewhat similar to the development of geometry from postulates. Lagrange's Mecanique Analytique and Hertz's Principles of Mechanics are conspicuous examples of such books.

Authors of elementary mechanics have ignored this important fact and have produced text-books which consist largely of collections of special theorems, laws and methods for solving problems. These authors usually give Newton's three laws of motion but they make little or no direct reference to them in the course of the presentation of the subject.

It is not to be wondered that these authors do not attempt to use Newton's three laws as the postulates of mechanics. Critics, such as Mach, Karl Pearson and Hertz, have severely criticised Newton's laws and have claimed that they are not satisfactory. They have pointed out that the first law is only a special case of the second, that the second law is nothing more than a definition of force, that the third law, while being a true principle, is not sufficiently broad to be used as a foundation of mechanics.

The fact that for sixty years after the publication of Newton's *Principia*

[1] This note will appear as an appendix in the third edition of the author's *Analytical Mechanics*.

students of mechanics had to appeal to all sorts of special theorems in order to solve problems of rigid bodies indicates that there is some basis to these criticisms. In commenting on D'Alembert's principle Routh (Elementary Rigid Dynamics, p. 47) says, ". . . It required the exercise of ingenuity and skill to detect the most suitable (theorem) in each case. Such problems were for some time a sort of trial of strength among mathematicians. The *Traité de Dynamique* published by D'Alembert in 1743 put an end to this kind of challenge by supplying a direct and general method of solving, or at least throwing into equations, any imaginable problem. The mechanical difficulties were in this way reduced to difficulties of pure mathematics."

These criticisms have produced little or no effect on the presentation of elementary mechanics, for two reasons. The first reason is obviously inertia and traditionalism. The second and more important reason is the fact that the well known forms of the principle are too difficult for the beginner to understand and demand too much mathematical knowledge. D'Alembert's principle is probably the easiest, but the six differential equations, which form this principle, are too forbidding to be handed to the beginner.

Before proceeding further I must confess that I have hesitated a very long time before deciding to write this note, because what follows may be construed as self-advertisement. But the subject seemed to me important enough to justify running this risk.

In the second edition of my Analytical Mechanics I expressed the underlying principle of mechanics in a form which is not forbidding to the beginner. It is broad enough to form a sound basis from which the entire subject may be developed in logical manner. This form of the principle, which I have called the *action principle*, states:

The vector sum of all the external actions to which a system of particles or any part of it is subject at any instant, vanishes.

$$\sum A = 0.$$

Two types of actions are postulated, namely *forces* and *kinetic reactions*. A force is defined as the action of one particle upon another particle. A kinetic reaction is defined as the action of the material universe as a whole upon a particle as a result of the acceleration of the latter. It is equal to the product of the mass of the particle and its acceleration, and has a direction opposite to the acceleration. This distinction between forces and kinetic reactions removes once for all the confusion which has prevailed in connection with uniform circular motion, as indicated by the term *centrifugal force*.[1] It makes very clear the fact that there is a complete agreement between the dynamic equilibrium of a particle in linear motion with acceleration and the dynamic equilibrium of a particle in uniform circular motion, and that in both cases the particle is in

[1] Hertz has a very interesting discussion of this subject in his *Principles of Mechanics*, p. 5,

dynamic equilibrium under the action of the resultant force and an equal and opposite kinetic reaction.

Introducing forces and kinetic reactions explicitly in the last equation, we obtain

$$\text{(A} \qquad \sum(\boldsymbol{F} - m\dot{\boldsymbol{v}}) = 0.$$

This vector equation may be split into the following three scalar equations.

$$\sum(X - m\ddot{x}) = 0, \quad \sum(Y - m\ddot{y}) = 0, \quad \sum(Z - m\ddot{z}) = 0.$$

The action principle is equivalent, therefore, to the first three equations of D'Alembert's principle. It has, however, a number of advantages over D'Alembert's principle. The action principle is stated in spoken language. It states explicitly that we are dealing with *vector* magnitudes; that we need consider only *external* actions and consequently may neglect internal forces; that we need consider only those actions to which the body, whose equilibrium or motion is being studied, is *subject*, and consequently may neglect the forces which that body exerts upon other bodies; and finally that the actions at different *instants* must be considered separately.

The pedagogical value of these advantages can hardly be exaggerated. The student, in his attempt to solve a problem in equilibrium, often represents not only the forces which are acting upon the body in equilibrium but also those which it exerts upon bodies it is in contact with. As a result he gets into difficulties. The action principle, which states explicitly that only those forces are to be considered *to which the body is subject*, helps avoid such difficulties when this principle is made the starting point of the problem.

Another advantage of the action principle lies in the fact that it emphasizes actions and not forces. The beginner is familiar with action in the form of push, pull and resistance, but he has only a vague conception of the idea of force. The concept of force is an abstraction which is difficult for the beginner to grasp fully. Consequently when he thinks in terms of forces he does strange things. He introduces forces where none existed and leaves out of consideration forces which actually exist. Ask a student to represent the forces acting upon a bullet in the projectile problem, for example, and you will find that he is just as likely as not to place a force back of the bullet pushing it forward in its path. If he were thinking in terms of action he could not make such an absurd blunder because his experience and intelligence would tell him that no body follows the bullet in order to push it forward. Again, in a problem, say, on the equilibrium of a body on an inclined plane he is likely to forget that the plane exerts a force upon the body. He is not likely to make such a blunder if he thinks in terms of actions and asks himself, "What bodies are acting upon the body under consideration?"

The action principle lends itself readily to a gradual development of the subject. It is easily understood by the beginner, at least to the extent of his needs at the different stages of his development as a student of mechanics. While

studying the equilibrium of a particle, for example, he need only be reminded that $\dot{v}=0$ in order to be able to obtain $\Sigma F=0$ as the condition of equilibrium. Newton's second law of motion, $F=m(dv/dt)$, is obtained by assuming that the system is a single particle acted upon by the resultant force F. Newton's first law is obtained by letting $F=0$. Other important laws and theorems, such as Newton's third law of motion, the principle of the conservation of dynamical energy, the theorem on the motion of the center of mass of a system of particles and the principle of the conservation of linear momentum are progressively derived from the action principle.

In order to put the action principle in a form adapted to motion of rotation, a new concept is introduced, namely, that of *angular action*. An angular action is defined as the moment of a linear action relative to a given exis. The angular action of a force is called a *torque*. The angular action of a (linear) kinetic reaction is called an *angular kinetic reaction*. Then the following equation is derived from equation (A).

$$\text{(Aa)} \qquad \sum(\boldsymbol{G} - m\boldsymbol{p}\dot{\boldsymbol{v}}_p) = 0,$$

where $\boldsymbol{G}$ represents the moments of external forces, $\dot{v}_p$ the projection of accelerations upon a plane perpendicular to the axis of reference, $-m\boldsymbol{p}\dot{\boldsymbol{v}}_p$ the angular kinetic reactions of the particles of the system, and $\boldsymbol{p}$ the lever arm of $-m\dot{v}_p$. Equation (Aa) states:

The vector sum, relative to any axis, of all the external angular actions to which a system of particles or any part of it is subject at any instant vanishes.

$$\sum A_a = 0.$$

This new principle is called the *angular action principle*.

When the vector equation (Aa) is resolved into three scalar equations the following are obtained.

$$\sum m(yZ - zY) = \frac{d}{dt}\sum m(y\dot{z} - z\dot{y}),$$

$$\sum m(zX - xZ) = \frac{d}{dt}\sum m(z\dot{x} - x\dot{z}),$$

$$\sum m(xY - yX) = \frac{d}{dt}\sum m(x\dot{y} - y\dot{x}).$$

The angular action principle is thus equivalent to the last three equations of D'Alembert's principle. The angular action principle, however, has a number of advantages over these equations. As in the linear action principle the important concepts represented by the terms *vector*, *external*, *subject to*, and *at any instant* are made explicit. The conditions of equilibrium of a rigid body, the

equation of motion of rotation of a rigid body, the theorem on the motion of a system relative to its center of mass, the principle of the conservation of angular momentum are easily derived from equation (Aa).

The main advantage of equation (Aa) over the corresponding equations of D'Alembert, so far as elementary mechanics is concerned, is derived from the fact that the only type of rotation which is studied in elementary mechanics is uniplanar rotation. For, in uniplanar motion $\dot{v}_p = \dot{v}$ and equation (Aa) reduces to the simpler form

$$\sum(G - mp\dot{v}) = 0,$$

which may be considered either as a vector or a scalar equation, and consequently does not have to be resolved into component equations. In the case of a rigid body the last equation is readily changed to

$$G = \frac{d}{dt}(I\omega),$$

where G denotes the resultant torque due to the external forces acting on the body.

The two action principles are equivalent, that is, each may be derived from the other. One is especially adapted to the mechanics of translation and the other to the mechanics of rotation. Each is so phrased as to emphasize the perfect analogy which exists between motion of translation and motion of rotation, and between the corresponding dynamical magnitudes in these two types of motion.

In equation (A) mass is supposed to be constant. If it is taken to be variable this equation is replaced by the more general equation

$$\sum\left[F - \frac{d}{dt}(mv)\right] = 0.$$

This does not involve, however, a change in the wording of either of the action principles, because when mass is variable the kinetic reaction equals $-(d/dt)(mv)$. In this case the angular kinetic reaction becomes $-p(d/dt)(mv_p)$, and equation (Aa) takes the form

$$\sum\left[G - p\frac{d}{dt}(mv_p)\right] = 0.$$

NUMERICAL SOLUTION OF LINEAR EQUATIONS BY VECTORS

By J. P. BALLANTINE, University of Washington

In two previous articles in this Monthly[1] I have developed certain well known theoretical facts about linear equations by the use of vectors. In the present note a new and practical method of obtaining numerical solutions of systems of linear equations is found to have as its basis the same notion of vectors.

The vector method of solution involves a large number of very simple steps. At the end of each step it is possible to read off an approximate solution together with a measure of precision. Thus the work can be carried to any desired degree of accuracy. At the end of each step a complete check is afforded against any possible mistakes of computation. When the coefficients are integers and the method of determinants yields exact rational values for the unknowns, the vector method also yields the exact solutions.

For the purpose of illustration, take the example:

$$\begin{aligned} 12x_1 - 3x_2 + 10x_3 + 6x_4 &= 12, \\ 21x_1 + 16x_2 - 8x_3 + 18x_4 &= 23, \\ 7x_1 + 18x_2 + 12x_3 - 5x_4 &= -15, \\ 18x_1 + 4x_2 + 17x_3 - 9x_4 &= 12. \end{aligned} \tag{1}$$

These four equations may be replaced by the single vector equation:

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = A_5, \tag{2}$$

where

$$\begin{aligned} A_1 &= (12, 21, 7, 18), \\ A_2 &= (-3, 16, 18, 4), \\ A_3 &= (10, -8, 12, 17), \\ A_4 &= (6, 18, -5, -9), \\ A_5 &= (12, 23, -15, 12). \end{aligned} \tag{3}$$

Numerical values of x_1, x_2, x_3, x_4, may be obtained by first solving the homogeneous equation:

$$A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 + A_5y_5 = 0, \tag{4}$$

and dividing the value of y_1, y_2, y_3, y_4, by $-y_5$.

An approximate solution of (4) can be obtained by finding a linear combination of the vectors A_1, A_2, A_3, A_4, A_5 which is at most a short vector. In the present case $-A_1+A_5$, which we shall call A_6, is a short vector, at least it is shorter than A_5. Hence A_5 is replaced by A_6, and from now on we work with the

[1] *A graphical derivation of Cramer's rule*, in vol. 36 (1929), pp. 439–441, *The theory of least squares by vectors*, in vol. 37 (1930), pp. 25–26.

set A_1, A_2, A_3, A_4, A_6. In a similar manner, $A_2+A_6=A_7$ is used to replace A_2. It is obvious that each new vector obtained is a linear combination of the original set, and as the vectors get shorter we obtain better approximate solutions of (4).

The information relating to each vector is given in one row of the following computation. First we give the number of the vector. Next the numbers of the two vectors added or subtracted to yield that vector. Next we give the coefficients of that vector when stated as a linear combination of the original set. Next the components of the vector. Next the norm, or square of the length, of the vector. It is useful to have the norms, for at every step one must decide which of two vectors is shorter. The norms, however, need not be computed exactly. In the final column we give the number of the vector which finally replaces the vector of that row. At every step we have five vectors, and each new one formed replaces a longer one obtained earlier.

		A_1	A_2	A_3	A_4	A_5						
1		1					12	21	7	18	958	9
2			1				-3	16	18	4	605	7
3				1			10	-8	12	17	597	12
4					1		6	18	-5	-9	466	8
5						1	12	23	-15	12	1042	6
6	$5-1$	-1				1	0	2	-22	-6	524	14
7	$6+2$	-1	1			1	-3	18	-4	-2	353	11
8	$7-4$	-1	1		-1	1	-9	0	1	7	131	
9	$7-1$	-2	1		0	1	-15	-3	-11	-20	755	10
10	$9+3$	-2	1	1	0	1	-5	-11	1	-3	156	20
11	$10+7$	-3	2	1	0	2	-8	7	-3	-5	147	22
12	$11+3$	-3	2	2	0	2	2	-1	9	12	230	13
13	$12+11$	-6	4	3	0	4	-6	6	6	7	157	15
14	$13+6$	-7	4	3	0	5	-6	8	-16	1	357	16
15	$13-8$	-5	3	3	1	3	3	6	5	0	70	
16	$15+14$	-12	7	6	1	8	-3	14	-11	1	327	17
17	$16+10$	-14	8	7	1	9	-8	3	-10	-2	177	18
18	$17-11$	-11	6	6	1	7	0	-4	-7	3	74	19
19	$18+15$	-16	9	9	2	10	3	2	-2	3	26	
20	$19+10$	-18	10	10	2	11	-2	-9	-1	0	86	21
21	$20+15$	-23	13	13	3	14	1	-3	4	0	26	
22	$21+11$	-26	15	14	3	16	-7	4	1	-5	91	23
23	$22+19$	-42	24	23	5	26	-4	6	-1	-2	57	24
24	$23+21$	-65	37	36	8	40	-3	3	3	-2	31	

Thus, reading opposite 11 in the first column, we find the notation $10+7$. This means that the vector A_{11} was computed by the formula $A_{11}=A_{10}+A_7$.

The numbers $-3, 2, 1, 0, 2$ indicate that A_{11} can be expressed as a linear combination of the original set of vectors by the formula

$$-3A_1 + 2A_2 + A_3 + 0A_4 + 2A_5 = A_{11}. \tag{5}$$

These coefficients were computed by adding the corresponding coefficients for the vectors A_7 and A_{10}. The numbers $-8, 7, -3, -5$, appearing next in the same row of the computation are the components of the vector A_{11}, and are computed by adding the components of the vectors numbered 10 and 7. They may also be checked by (5). The sum of the squares of the components of A_{11} is next shown to be 147. Of the two vectors A_7 and A_{10} used in forming A_{11}, A_7 is the longer, having a norm of 353, and is replaced by A_{11}. This fact is indicated by entering 11 in the last column opposite 7 in the first column. In a similar manner, A_{11} itself is later replaced by A_{22}.

The computation has been carried to a point where all the vectors have been replaced except A_8, A_{15}, A_{19}, A_{21}, A_{24}. The computation could be continued by forming $A_{24}-A_{15}$. However, this is not necessary as a good solution can now be read off by inspection. In equation (5), neglecting A_{11}, we see that $-3, 2, 1, 0, 2$ is an approximate solution of equation (4). It fails to satisfy equation (4) by the vector A_{11}. Dividing the first four values of this solution by the negative of the last one, i.e., by -2, we have $1.5, -1, -.5, 0$ as an approximate solution of (2). It fails to satisfy (2) by the vector $-A_{11}/2$, whose norm is $147/4$. A better solution is obtained by reading opposite 19. A_{19} is expressed as a linear combination of the original set by the coefficients $-16, 9, 9, 2, 10$, and since A_{19} is short, these coefficients are an approximate solution of (4). Dividing by the negative of the last of them, we obtain $1.6, -.9, -.9, -.2$ as an approximate solution of (2). It fails to satisfy (2) by $A_{19}/10$ which has a norm $26/100$.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE DIRICHLET FORMULA, AND INTEGRATION BY PARTS

By L. M. GRAVES, University of Chicago

It seems not to have been generally noticed that the calculus formula for integration by parts is a special case of the Dirichlet formula for interchange of order of integration in an iterated integral, namely,

$$\int_a^b \int_a^x F(x, s)ds\, dx = \int_a^b \int_s^b F(x, s)dx\, ds.$$

This relation holds true for either Riemann or Lebesgue integrals. The essential idea can be indicated by specializing the indefinite integrals as follows. Let

$$f(x) = \int_a^x f'(s)ds, \quad \int_b^x g(s)ds = h(x).$$

Then

$$\int_a^b f(x)g(x)dx = \int_a^b \int_a^x f'(s)g(x)ds\,dx$$

$$= \int_a^b \int_s^b f'(s)g(x)dx\,ds = -\int_a^b f'(s)h(s)ds.$$

Professor C. J. Coe has used the proof of this relation as an exercise for some of his classes at the University of Michigan. The relation first came to my attention through a calculus of variations problem in which the first variation could be transformed by the use of Dirichlet's formula, but not by the usual integration by parts.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematische Quellenbücher. IV, Infinitesimalrechnung. Heinrich Wieleitner. Berlin, Otto Salle, 1929. 160 pages. Price, 4.50 RM.

This little and inexpensive work by Dr. Wieleitner is the fourth mathematical source book in a series of handbooks edited by Drs. Elwald Wasserloos and Georg Wolff. The purpose of each of these works is to give a small number of the most important sources relating to some particular branch of knowledge. In this volume (No. 24 of the large series) upwards of twenty of the most interesting source items relating to the differential and integral calculus are given. These include such topics as the following: (1) The Archimedean axiom, with source quotations from Archimedes and Euclid. Substantially, Archimedes says (*De Sphaera et Cylindro*, I, 5) that of unequal magnitudes the greater exceeds the less by such a magnitude as, when continually added to itself, can be made to exceed any assigned magnitude of the same kind, and Euclid, V, Def. 4, involves the same assumption. (2) The ratio of the areas of two circles, from Euclid XII, 2. (3) The quadrature of the parabola, from Archimedes. (4) The refutation of the idea of indivisible parts of a line (*von den Atomlinien*, *De insecabilibus lineis*, *Atomos grammos*) by Aristotle. (5) The sum of square num-

bers, by Archimedes. (6) The volume of the spheroid, by Archimedes. (7) The volume of the sphere, by Archimedes. (8) The volume of the sphere in the 17th century (Luca Valerius). (9) The "apple-shaped body" of Kepler—the second great step, of which Guldin's was the third, towards the barycentric calculus. (10) A selection from Cavalieri's *Geometria Indivisibilibus* (1635; 2d. ed., 1653). (11) A selection from Torricelli's "De solido hyperbolico acuto," in the *Opera* of 1644. (12) The quadrature of hyperbolas in general, from Fermat. (13) Fermat's method of maxima and minima. (14) Pascal's "characteristic triangle" and its application to the problem of the quadrature of the circle and trigonometric integrals. (15) Newton's step towards considering differentiation and integration as inverse operations. (16) Leibniz's first printed rules for differentiation. (17) Newton's application of infinite series to the representation of the arc-sine, sine, and cosine. (18) Leibniz on the "arithmetic quadrature" of the circle. (19) Jean (I) Bernoulli's problem of tangents to the Archimedian spiral. (20) Newton's discovery of fluxions. (21) Euler's differential quotient of the sine.

It will be observed that Dr. Wieleitner has here indicated the most important steps taken in the development of the calculus up to about the close of the 18th century when, for practical purposes the theory was complete, although more remained in laying firm the foundations. He has not only given translations into German of important passages of the makers of the discipline, but he has added a commentary on each, explaining by the aid of modern symbols its significance. This has been done in a condensed form that will serve to give to students of the subject a good knowledge of the development of the method upon which rests so much of the advanced mathematics of the present day.

DAVID EUGENE SMITH

Elementary Theory of Finite Groups. By L. C. Mathewson. Houghton Mifflin Co., 1930. x+165 pages.

This text is planned for the beginning student of finite groups. The material is arranged for teaching purposes. Each unit of work is followed by a set of exercises and at the end of each chapter there are additional exercises of a more advanced nature which supplement the theorems of the text. The whole book is copiously illustrated with examples and easily read.

Perhaps a summary by chapters of the subjects discussed will indicate what field the text covers. The reader is introduced very gradually to the idea of a group. In fact the first chapter is devoted to simple examples of finite groups and to the set of postulates defining a group. The next two chapters take up permutations and permutation groups, defining such terms as transitivity and primitivity. A chapter each is devoted to abelian groups, to groups of isomorphisms, and to some of the elementary theorems of abstract group theory, whereas two further chapters are devoted to some more examples and definitions of abstract groups and certain special groups. The book closes with two chap-

ters on groups of linear substitutions and some applications of group theory, such as the Galois theory of equations, constructions by ruler and compass, and the Lie theory of one-parameter groups.

These last two chapters are very brief, but this is the intention of the author. The chapter on the applications is chiefly an outline, while the author states that the chapter on groups of linear substitutions "aims only to mention a few things in the Frobenius theory of groups of linear substitutions." No mention, except in a problem of a set of exercises, is made of the Frobenius group characters or characteristics. Consequently, some interesting theorems due to this theory are omitted. It would seem more valuable perhaps to the student to omit the chapters on the applications, which for lack of space are necessarily very sketchy, and to substitute a more detailed account of the theory of groups of linear substitutions and group characteristics. An omission of one of the methods of group theory seems unfortunate.

Some theorems, such as the one on page 44 on a k-ply transitive abelian group, seem somewhat trivial. In this theorem k is obviously restricted to one, for a permutation which fixes a letter a and displaces a letter b is not commutative with a permutation which replaces the letter a by the letter b. One of the elementary and recent theorems, namely P. Hall's extension of Sylow's theorem for solvable groups (Journal of the London Mathematical Society, 1928) might profitably be mentioned. However the elementary character and the brevity of the book cause inevitable limitations. On the whole the book is carefully written and well arranged.

MARIE J. WEISS

Infinite Series. By Tomlinson Fort. Oxford University Press, 1930. iv+253 pp.

There is a great need for an elementary text on infinite series since the lack of manipulative skill and of mathematical maturity on the part of most of the students make the books of Bromwich and of Knopp less suitable for introductory courses. Professor Fort's book fills this gap, to some extent at least.

The book covers extensive ground. The following extract from the table of contents shows what subjects are treated, the number of pages being given in parentheses after the title.

I. The number system (6); II. Sequences (11); III. Convergent and divergent series (7); IV. Series whose terms are positive (24); V. Series some of whose terms are positive and some negative (5); VI. Series whose terms are complex (7); VII. Transformations of series and operations with series (18); VIII. Multiple series (13); IX. Uniform convergence (27); X. Continuity and integrability. Quasi-, sub-, and infra-uniform convergence (8); XI. Power series (27); XII. Dirichlet series (21); XIII. Binomial coefficient series (2); XIV. Factorial series (4); XV. Generalized factorial series (6); XVI. Fourier series (15); XVII. Summation of divergent series (45); XVIII. Asymptotic series (7).

There are obviously some deviations here from the conventional plan, and

one or two of the inclusions are of doubtful value, but on the whole the book is well planned. It does not demand advanced knowledge or mastery of technique from the student. There is an ample supply of exercises and few if any of the tricky Tripos type. The author states clearly what are the hypotheses and what are the conclusions in every theorem, a helpful feature for the student, but slightly tedious when repeated 244 times. The book contains some interesting portions based on the author's own investigations, especially Chapter XV. This and Chapter XII add much to the value of the book.

A defect of the book is a certain indefinite monotony, and a lack of outlook and of historical perspective. The references to the literature are also far too scanty—less than a dozen books or papers on infinite series are mentioned from cover to cover. The author at times shows a surprising vagueness due no doubt to his endeavor to avoid certain advanced theories, such as measure, integration, even analytic functions. Much is said about differentiation and integration of functions of a complex variable throughout the book, but there is nowhere to be found a clear-cut definition of these operations, nor is the distinction between functions of a complex variable and analytic functions ever called to the reader's attention. One can understand the author wanting to avoid the Lebesgue theory in an elementary text, but he should then also avoid the subtleties of the Riemann integral and of continuity which are apt to be more difficult and less important than the Lebesgue theory. A couple of examples taken from Chapters IX and X will illustrate these points.

Thus theorem 124 on p. 111 deals with term by term differentiation of a series whose terms are functions of a complex variable. In the proof it is tacitly assumed that $u_n'(z)$ has a Riemann integral and is the derivative of its indefinite integral. This should be justified, or, if necessary, included among the hypotheses. Here, as in several other places, it is hard to tell if the author is exclusively concerned with analytic functions, or if he has also in mind general functions of a complex variable, or, in particular, functions of a real variable, the derivatives in question being taken along a curve or along the real axis as the case may be.

In theorem 133 on p. 124 the author wants to prove Arzelà's necessary and sufficient condition for term by term (Riemann) integration of a series of arbitrary functions. In trying to avoid sets of measure zero in the sense of Lebesgue, he introduces "discrete sets" instead, which are simply sets of Jordan content zero. His statements that the set of discontinuities of a Riemann integrable function is discrete, and that the sum of an enumerable set of discrete sets is also discrete, are obviously incorrect. The reviewer believes that it would have been much better to omit Chapter X altogether. The notions involved are rather complicated, and the author does not always treat them successfully. Even the terminology is at least debatable.

Various other points could be raised. Thus, the discussion of Fourier series is rather meager. On the other hand, there is no justification at the present time for devoting 11 pages out of 45 on the theory of summability, to Borel's integral

definition. It is an interesting method of great historical importance, but of comparatively little power. It would have been much better to devote some of this space to Cesàro's method for non-integral orders, and to give the rest to a study of analytic continuation by a method of summation which is really effective when applied to power series, e.g., that of LeRoy.

The printing is of the usual high quality which one associates with the Clarendon Press. The reviewer has found some misprints, but none of importance.

EINAR HILLE

College Algebra. By B. H. Crenshaw and D. C. Harkin. P. Blakiston's Son and Company, Philadelphia, 1929. xiii+224 pages.

The following are the chapter headings of this book in the order of their appearance: Review of fundamental concepts; Linear equations in two and three unknowns; Exponents; Quadratic Equations; Ratio and Proportion—Variation; Series, Binomial Theorem, and Mathematical Induction; Logarithms; Undetermined Coefficients and Partial Fractions; Numerical Equations; Theory of Equations; Theory of Numbers.

The first chapter is devoted to a review of fundamental concepts which according to the preface are "too often assumed to be well-understood." It is to be feared, however, that the first chapter may fail to relieve this condition.

In the preface the authors say: "In line with the dictum of Ernst Mach, that science tells not why but how, a large number of illustrative examples are offered, fully worked out and checked." Upon a first reading of the preface the reviewer interpreted this to mean that the theory was well illustrated by examples but a reading of the text convinced him at once that the authors meant quite literally what they said. Thus for the most part no logical development of a topic is attempted, the authors prefering to proceed at once to illustrative examples and then to direct the student to go and do likewise. The places where this policy has been deviated from convinces the reviewer that perhaps after all this was wise although in general he does not entirely agree with the authors' interpretation and application of Mr. Mach's dictum. Following the lead of the authors let us proceed with some illustrative examples.

After carrying out the ordinary elimination of y from two linear equations containing x and y, the authors proceed to relieve "determinants of the appearance of black magic" as follows: "By a detached coefficient method, we may, . . . , put the coefficients in a determinantal array and perform the same operations on its elements:

$$\begin{vmatrix} b & ax - e \\ d & cx - f \end{vmatrix} = 0.$$

The corresponding operations are then performed until zero is obtained in the upper left hand corner. Thus the term "determinant" appears for the first time

as an adjective and the actual concept introduced is the matrix of coefficients which, however, is set equal to a number. Without further explanation the "determinantal array" becomes on the next page a "determinant" and the expansion in terms of the elements of a row is (incorrectly) given. No explanation of the rule is attempted nor is one possible since no determinant has yet been defined.

Again in the chapter on series the terms series, sum of the series, and limit of the sum of the series are used in place of the almost universally adopted terms sequence, series, sum of the series, respectively. The comparison test is incorrectly stated inasmuch as the terms are not restricted to be positive. (Absolute and conditional convergence are defined for alternating series only.)

In the chapter on exponents the student is lead to believe that the meanings for non-integral exponents are derived from analogues in the operations of coefficients. In the discussion of irrational numbers are found the following two statements on the same page: "irrational number means an n-th root of a positive rational number which is not a perfect n-th power"; "rational numbers might be considered special cases of irrationals."

Illustrations and exercises on the four fundamental operations with complex numbers appear in the chapter on exponents. Thus, except perhaps for a chapter on "Permutations and Combinations," the book contains a satisfactory selection of material. There are plenty of illustrative examples throughout the book and numerous exercises to practically all of which answers are given. Historical notes are scattered throughout the text to give some indication of the development of the main notions of the science.

RAYMOND W. BARNARD

Solid Analytical Geometry and Determinants. By Arnold Dresden. John Wiley & Sons, Inc., New York, 1930. x+310 pages. $3.00.

The nature of this book is best explained by quotation from the preface: "Though books on plane analytical geometry frequently devote some chapters to the geometry of a space of three dimensions, the material covered in these chapters is with few exceptions, not intended to do more than provide a general introduction to the subject . . . it rarely goes far enough to acquaint them (the students) with the more interesting and valuable methods of this field . . . it has seemed to the author that, in the study of Solid Analytical Geometry, the young student of mathematics can find an excellent opportunity for an introduction to methods and principles which have an important part in various fields of advanced mathematics. Among these are the methods based on the theory of determinants and on the concept of the rank of a matrix . . . they find relatively simple application in the subject to which this book is devoted. . . . For these reasons the first chapter of this book presents an exposition of some of the properties of determinants and matrices, followed in Chapter II by a treatment of systems of linear equations. . . . With the basis thus provided

it becomes possible to deal with the geometrical questions of the later chapters in a way which lends itself readily to extension to problems of a more general character. . . . Chapters III to X deal with the loci of equations of the first and second degree in three variables from the point of view of real, metric geometry. Elements at infinity and complex elements are considered as non-existent. . . . In subject matter the last eight chapters follow largely the traditional content of introductory courses in Solid Analytical Geometry. . . . The exercises form an integral part of the course which this book presents. . . . A good many of the problems serve no other purpose than that of illustrating the material in the text. But there are other problems . . . which require a certain amount of original thinking."

Professor Dresden has, in our opinion, admirably carried out his purpose as outlined in his preface. The result is a conveniently small, attractive and readable book suitable for undergraduates of fairly high quality. We can however but wonder where such undergraduates are to be found in numbers sufficient to justify the regular giving of the course covered in this book. Doubtless the assimilation of its contents would be of great value to students intending to specialize in mathematics; it contains a large amount of valuable geometrical information with a thorough study and many applications of determinants and the elementary theory of matrices. We feel that the chief emphasis is on and perhaps the author's predilection is for the algebraic method rather than the geometrical fact. This is doubtless as it should be in view of the greater applicability of the former. There are eight appendices, pages 296–301, of which seven concern determinants. Number VI might be included with IV. There is a good index, pages 303–310.

The book is attractively made and is remarkably free of both misprints and errors. In a careful reading we have noted but one misprint, the omission of the exponent γ from the first symbol on page 22; two unimportant errors: the statement of example 21 on page 107 is not true; the incorrect statement on page 141 that conjugate hyperbolas have equal eccentricities. The style of the book is pleasing, the modesty of the author, continually in evidence, is winning. The frequent occurrence of the phrase, "it should be clear" provokes a smile—it is so much nearer the truth than the usual "it is clear." There follow a number of criticisms of minor importance. "It should be clear" that these criticisms are not intended to show the slightest condemnation of the book, which we thoroughly like and approve, but are only in the nature of the duty of a reviewer. The distinction between a determinant and the value of a determinant seems to us finespun and also unfortunate as tending to annoy the beginner; the definition of a determinant as "a square array of numbers to which a single number, called the value of the determinant, is attached" $\cdots$, page 1, is different from the definitions of Weber and Bôcher, both of whom define the determinant as what Dresden calls the expansion of the determinant. Further the definition is too much like that of a matrix, and finally, we do not see that the distinction is

of any use. We deprecate such a colloquial sentence as the following page 19: "In neither case will these operations kill off a non-vanishing minor of a nor bring a vanishing minor back to life." The usual way of giving the derivative of a determinant, $A(t) = |a_{ij}(t)|$, is $A'(t) = \sum a_{ij}'(t)A_{ij}$, where A_{ij} is the cofactor of a_{ij}, which seems to us a more compact and simple form than that given on page 32. It would perhaps be worth while to introduce into the book the idea of continuity, as is done in many books on Algebra. Some proofs might thereby be simplified, others presented which are not here given at all. See, for example, pages 42 and 125. In defining the direction angles of a directed line, page 55, it is necessary only to consider angles from 0° to 180°, not from −180° to +180°, since there is no question of the direction of measuring the angle. The proof of theorem 13, page 63, is far from being the simplest available. The distinction between right and oblique cylindrical surfaces, page 69, is not a distinction between different kinds of surfaces as might be supposed by the student, nor is it necessary to suppose the directrix to be a plane curve. It is not clear what the author means by "unsigned length," page 75, nor does the use of this term seem to be advantageous. In the introduction to Chapter V, "Other Coordinate Systems," page 108, the necessity of a choice of units is emphasized. The emphasis would be more appropriate at the beginning of Chapter III, page 49, where cartesian coordinates are first used. In the definition of spherical coordinates, page 108, the direction of measuring angles should be stated. In studying the shape of a surface, paragraph 70, pages 135–140, no mention is made of the traces of the surface on the coordinate planes. These traces are, we believe, of greater help to the student in visualizing a surface than the projections of the plane sections, which are here called by the author "contour lines," page 138. In the very interesting discussion of focal curves and directrix cylinders of central quadrics, paragraph 120, 121, pages 281–287, the use of the word "any" in definition IV., p. 281, is so confusing as to be actually misleading. It is to be regretted that some of the other interesting and important properties of focal conics are neither given in the text nor incorporated as exercises.

J. K. WHITTEMORE

Notions sur la Géométrie Réglée et sur la théorie du Complexe Quadratique (*Appendice au Cours de Géométrie analytique*). By Georges Bouligand. Paris, Librairie Vuibert, 1929. 84 pages. 11 francs.

The student ignorant of line geometry and lacking the text to which this little pamphlet is an appendix would find the brochure difficult reading. He would first encounter a smooth development of radial and axial coordinates, but would next find a discussion of a complex, and a complex cone, without finding a definition of these concepts. Similar lacunae are plentiful throughout. Frequent references to the *Géométrie Analytique* emphasize the fact that we are dealing with an appendix which the author doubtless did not intend to have stand on its own feet.

In rapid fashion are discussed one- and two-parameter families of linear complexes, Klein's interpretation of line geometry as that of points of a V_4^2 in S_5, six complexes in mutual involution, and the quadratic complex. In a scant eight pages, one finds the topics of special complexes, Chasles correlation, singular congruence, focal and developable surfaces of a congruence, etc. There is an equally brief space devoted to tetrahedral, Battaglini, and Painvin complexes, with indications of the Kummer surface.

Twenty-three pages are devoted to exercises and *questions d'agrégation.*

The reviewer feels the pace is far too swift for the beginner; the advanced student is perhaps surprised to see a reference to Koenigs and Jessop only in the addenda, and none at all to Sturm, Zindler, or Hudson's classic book on the Kummer surface.

C. A. RUPP

An Introduction to the Geometry of n Dimensions. By D. M. Y. Sommerville. E. P. Dutton, New York, 1930. xvii–196 pages.

Although the study of the geometry of n dimensions had its origin almost ninety years ago in a paper by Cayley, it had attracted comparatively little attention until recently. Much of the work so far accomplished in this branch of mathematics has been accomplished by Italian geometers. During recent years in England, the country of its origin, mathematicians have shown increasingly keen interest in the study and a fair number of articles have been written upon various aspects of the subject. Books in English dealing with this field of endeavor are scarce, and not much less so in other languages. An inquiring mind wishing an easy avenue of approach to an understanding of the nature of some of the beautiful hyperspatial configurations and relations in hyperspace is almost completely at sea. In view of this scarcity of books on the subject and also in view of the fact that a knowledge of hyperspace geometry is becoming more and more a necessity to a student of mathematics, the appearance of the book under review is timely.

Mr. Sommerville, the author, is a mathematician of high rank and quality and one is tempted to say that whatever he writes is worth reading. His book, *An Introduction to the Geometry of n Dimensions*, is certainly worth reading, worth studying. It is not a textbook nor a systematic treatise, but it is what it is, an introduction, introducing inquiring students, already having a fair knowledge of ordinary geometry, to the various representative topics of n-dimensional geometry.

The topics are well chosen and they appear in the ten chapters comprising the book as follows: Ch. I, Fundamental Ideas; Ch. II, Parallels; Ch. III, Perpendicularity; Ch. IV, Distances and Angles between Flat Spaces; Ch. V, Analytical Geometry: Projective; Ch. VI, Analytical Geometry; Metrical; Ch. VII, Polytopes; Ch. VIII, Mensuration: Content; Ch. IX, Euler's Theorem; Ch. X, The Regular Polytopes. At the end of every chapter valuable references are given.

All the chapters are well written, and the contents are clearly presented. Elementary and familiar ideas are first carefully explained and emphasized so as to lead the reader gradually to ideas of increasing difficulty. A thoughtful reader, familiar with geometrical reasoning both synthetic and analytic, will find it not too difficult to understand the book.

The book is well printed and is, to the reviewer's knowledge, free of misprints. On page 10, the terms *congruence* and *complex* are used in place of *complex* and *congruence* respectively but on other pages they are correctly used.

One may wish that other topics were included in the book but, the book, as it is, being a brief introduction to n-dimensional geometry, serves its purpose and should be in the hands of all those students of mathematics who have any interest in geometry at all.

B. C. WONG

Johannes Kepler in seinen Briefen. Edited by M. Caspar and Walter Von Dyck. R. Oldenbourg, Munich, 1930. xxvii+396+xvi+348 pages. 8 illustrations.

These two attractive volumes of letters selected from Kepler's correspondence make an important addition to the literature relating to one of Germany's great geniuses that is in published form at the three hundredth anniversary of his death.

While some of Kepler's letters have appeared previously they have been quite incidental to the presentation of some of his theories and have been used in fragmentary form as notes, appendices, and illustrative material. The fact that they were written in Latin or in the awkward ceremonious German of his time has been a serious handicap to their wide circulation. The present editors feel that they have done a favor to Kepler's admirers and all who appreciate contact with such intellectual strength as his in making available in easily readable German this highly valuable material.

The editors indicate that the selection has been made from some four hundred letters with painstaking care and with the purpose of making it of more general interest than is usually associated with Kepler. While technical subjects may have suffered by this treatment, a great deal of light is shed on Kepler's colorful character development from his school days at Tübingen to his death at Regensburg. His letters show in an unusually vivid way the greatness of his thoughts, the depth of his feelings, and the richness of his spiritual life.

Each of the two volumes contains five groups of letters corresponding rather closely to Kepler's residence in different places. The first of the Tübingen student letters is a request in refreshing youthful style for funds. The Grazian group concerns his first marriage, his calendar, which seems to have established his reputation as astrologer, and includes interesting correspondence with Tycho Brahe and Galileo. A group concerning his visit at Prague with Tycho is followed by the set incidental to his connection with the court of Rudolph II and

ion with its vertex at the center of a circle. He then trisected the subtended chord AB and constructed a hyperbola having AA' two thirds of the chord as its transverse axis and $\sqrt{3}$ times the transverse axis for the conjugate one.

Prove analytically that points joined to the ends of the chord such that one of the angles thus formed is double the other lie on the above hyperbola.

UNSOLVED PROBLEMS

Solutions are desired for the following unsolved problems.

238 [1916, 19]. *Proposed by Clifford N. Mills.*

Determine the rational value of x that will render x^3+px^2+qx+r a perfect cube. Apply the result to $x^3-8x^2+12x-6$.

461 [1916, 209; 1919, 414]. *Proposed by E. T. Bell.*

(1) Two events have probabilities p, q respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events where p^q is written for $p\times p\times\cdots\times p$, ($q$ factors p), in cases (i), (ii), and $p\times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) and (ii) can exist between two events? Upon what postulates is the answer to this based?

415, [1916, 301]. *Proposed by George Paaswell.*

If r is the distance from a fixed point (x, y, z) to a variable point (x', y'), in the plane $z=0$, determine the value of the integrals $\int\int r dx'dy'$ and $\int\int \log(z+r)\, dx'dy'$ for the two cases: (a) when the integration is extended over the surface of the circle of radius R; and (b) when the integration is extended over the surface of the rectangle of dimensions a, b.

These integrals are special cases of the direct and logarithmic potentials, the densities of the surface distributions being taken as unity.

343 [1917, 124]. *Proposed by J. Rosenbaum.*

Two bodies of equal masses and coefficients of friction μ_1 and μ_2 are connected by a light spring of stiffness k and placed on an inclined plane. Discuss the motion of each body when the angle between the non-stretched spring and the plane is θ.

344 [1917, 177]. *Proposed by J. Rosenbaum.*

Two bodies of equal masses, and coefficients of friction μ_1 and μ_2 are connected by a light, flexible string, and placed on an inclined plane. What is the angle, θ, between the string and the plane if the inclination, α, of the plane is a minimum when the bodies are on the point of motion?

429[1917, 231]. *Proposed by N. P. Pandya.*

Trace the curve given by the solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\frac{dy}{dx}+\frac{1-4x^2}{(1-x^2)^{3/2}}\cdot$$

432[1917, 287]. *Proposed by R. P. Baker.*

The expressions

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right)$$

and

$$x^{-(i+1)}\left(x^3\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right)$$

are formally equivalent for every integral value of i.

SOLUTIONS

3369[1929, 169]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two equilateral triangles one within the other, to construct a third equilateral triangle which shall be inscribed in the outer and circumscribed about the inner.

I *Solution by Otto Dunkel, Washington University*

Let ABC and $A'B'C'$ be the outer and inner equilateral triangles; it is required to inscribe in the first an equilateral triangle XYZ which also circumscribes the second. The results and notations in the solution [1931, 229] and discussion of 3441 [1930, 380] will be used in what follows. Since ABC is equilateral, there is no point O', and the point O must be its center and also the center of XYZ. Moreover, the sense of rotation of ABC and XYZ must be the same. Similarly, the centers of XYZ and $A'B'C'$ must coincide. It is supposed that A' lies on YZ, B' on ZX, and C' on XY. Hence the three triangles must be concentric and have the same sense of rotation. So we need to consider in the construction only one vertex of $A'B'C'$, which will be denoted by P.

The three parabolas for ABC are tangent to its sides at its vertices; the vertices of the parabolas are the mid-points of the altitudes of ABC; and the other intersections of the parabolas lie on these altitudes one ninth of their length from their respective bases. The regions I, II_a, II_b, II_c lie within ABC, while the regions IV lie outside. If P lies within I there is no construction; if it lies elsewhere within ABC there may be from one to four constructions.

If P lies within II_c an arc of a circle is described on OP so as to contain 30°, for example on the side of OP towards B. This arc cuts BC in two points, each of which is an X for the required triangle. No other side of ABC will be cut by this arc. If P lies within III_{ca} an arc of a circle is described on OP so as to contain 30°, towards C for example; this arc cuts BC in two points, each of which is

an X for two of the required triangles. The arc cuts also CA in two points, which are the Y's for the remaining two triangles. All of these triangles are inscribed in the strict sense.

Thus, if P is a point within ABC and the arc on OP containing 30° does not cut any side, P lies within I. If it cuts only one side in two points, P is within a II; if it cuts two sides each in two points P is within a III.

3374 [1929, 233]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

A Note by Otto Dunkel. The purpose of this note is to indicate the omission of a special case in the solution [1930, 160] of this problem. This special case in the notation of that solution is that in which $C'A'$ passes through C, $A'B'$ through A, $B'C'$ through B. For this situation of the inner triangle there are an infinite number of constructions of the desired triangles. The proof follows the method in the first part of the solution mentioned. Let $C'A'$ cut AB in P_1; $A'B'$ cut BC in Q_1; $B'C'$ cut CA in R_1. Then corresponding to the points P, Q, R, S, we find the three sets of points

$$A, Q_1, A, A; \quad B, B, R_1, B; \quad P_1, C, C, P_1.$$

Thus A, B, P_1 are self-corresponding points in the two projective ranges P and S on AB; and hence every point is self-corresponding; this means that S will always fall upon P. Here $A, B, C; A', B', C'; P, Q, R$ have the same sense of rotation, and the inner triangle is situated in this special position. This is the only way in which there can be an infinite number of constructions. Since this exceptional case has only recently been observed by the writer, it is probable that he also has overlooked it in the application of the related theorem to the solution of previous problems in the Monthly.

Passing now to the second part of the problem we see that the theorem there stated is not true, if we use the terms inscribed and circumscribed in the most general sense, unless we exclude this situation of the inner triangle as well as the case where ABC and $A'B'C'$ have opposite senses. In order to state the results for this second part it will be convenient to change the notation. The required triangle is denoted by XYZ, where X lies on BC, Y on CA, Z on AB; and A' lies on YZ, B' on ZX, C' on AB. The solution of 3369 in this issue shows the number of constructions of equilateral triangles XYZ for the several regions of the interior of ABC. In these constructions the sense of the three triangles is the same. With this requirement of sense fulfilled there are infinitely many triangles XYZ, if a vertex, say C', falls on the arc of a circle (C) through A, B and the center of ABC. We exclude the points A, B, and the center of ABC. In this case two vertices of XYZ lie within or at the ends of the corresponding sides of ABC while the third vertex lies on a prolongation of its corresponding

side. These triangles include special cases where two sides of XYZ are parallel and distinct or coincident. None of these triangles can be equilateral.

If ABC, XYZ have the same sense while $A'B'C'$ has the opposite sense, there cannot be an infinite number of constructions. But to each situation of $A'B'C'$ there are two distinct triangles XYZ; and here again one vertex lies on a prolongation of a side, while the other two lie on their respective sides. None of these triangles are equilateral. Hence there are always six possible constructions, as we see by renaming the points $A'B'C'$ in cyclic order. If a vertex, say C', lies on the circle (C) then one triangle degenerates into a triangle having two sides coincident. If a vertex of $A'B'C'$ falls within one of the III regions, then there are four constructions for equilateral triangles and six for non-equilateral triangles. Since this region does not contain in its interior any point of circles similar to (C) none of these triangles can have two sides coincident. But if a vertex lies on one of two certain hyperbolas passing through the center and a pair of mid-points of the sides of ABC, two sides of one triangle (of the six) are parallel. Hence there are regions for which we can construct ten true triangles, four of which are equilateral. Part of the arc AB of circle (C) lies in I and the other two parts in II_a and II_b, and the total number of constructions may be stated for these regions.

3413 [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The axis of a right circular cylinder of radius r coincides with a diagonal of a unit cube. Find the area of that part of the cylindrical surface which is inside the cube. Partial Solution (1930, 557) by V. F. Ivanoff, San Francisco, California.

II. *Solution by H. L. Rietz, University of Iowa*

Locate the cube with respect to rectangular coördinate axes as shown in the attached figure.

Let $P(x, y, z)$ be a point of the surface of the cylinder with OD as its axis. Then

$$x + y + z = 3m \tag{1}$$

is a plane normal to the axis OD of the cylinder cutting OD at $P_1(m, m, m)$, and

$$r^2 = (x - m)^2 + (y - m)^2 + (z - m)^2. \tag{2}$$

We shall find it convenient to locate P in another way by giving its distance p from the plane through O normal to OD, together with the plane polar coördinates (r, θ), where θ is the angle which P_1P makes with the line P_1N normal to OD and intersecting the z-axis at a point N.

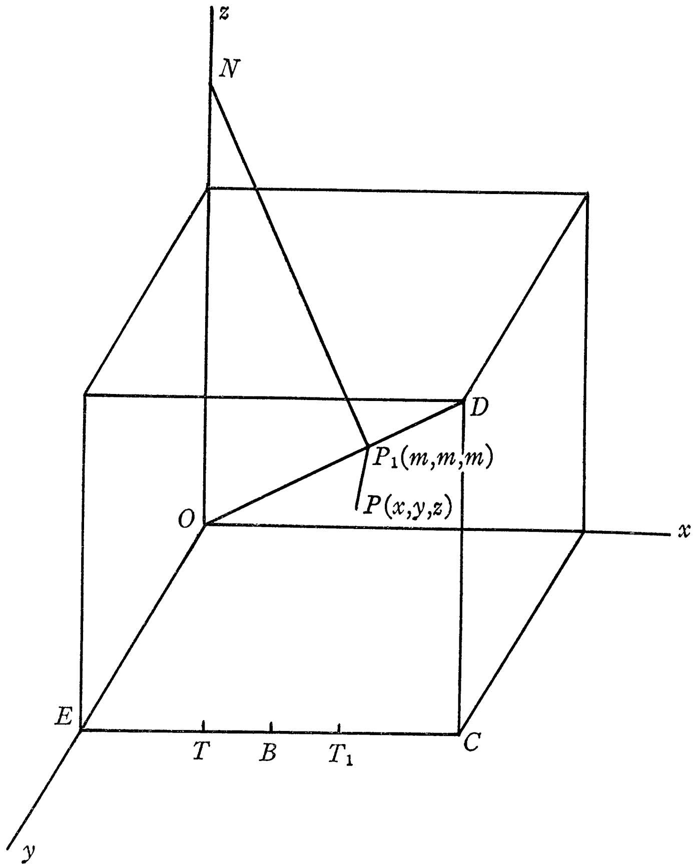
The direction cosines of P_1N being $-6^{-1/2}$, $-6^{-1/2}$, $2\cdot 6^{-1/2}$, and those of P_1P being $(x-m)/r$, $(y-m)/r$, $(z-m)/r$, we obtain

$$\cos\theta = \tfrac{1}{2}r^{-1}\cdot 6^{1/2}(z - m). \tag{3}$$

When $z=0$ in (3), we have $m=-2\cdot 6^{-1/2}r\cos\theta$, and

(4) $$p_1 = 3^{1/2}m = -2^{1/2}r\cos\theta,$$

where p_1 is the lower bound of p for an assigned θ.



To get the upper bound of p in the region[1] of the cube from $\theta=2\pi/3$ to π, we shall find p in terms of r and θ when $y=1$. For this purpose, we first make $y=1$ in equations (1) and (2). Next we solve for $z-m$, and then for m, using the value of $z-m$ from (3).

This gives

(5) $$m = 1 + 6^{-1/2}r\cos\theta \pm 2^{-1/2}r\sin\theta$$

and

(6) $$p_2 = 3^{1/2} + 2^{-1/2}r\cos\theta \pm 3r\cdot 6^{-1/2}\sin\theta,$$

[1] From symmetry this region from $\theta=2\pi/3$ to $\theta=\pi$ is obviously one-sixth of the total region with which we are concerned.

where p_2 is the upper bound of p for an assigned θ. It is easily verified that the negative sign should be used before $3r\cdot 6^{-1/2}\sin\theta$ in (6).

When $r\leqq 2^{-1/2}$, the surface of the cylinder is given by

$$(7)\qquad 6\int_{2\pi/3}^{\pi}(p_2-p_1)r\,d\theta = 2\cdot 3^{1/2}(\pi - 3\cdot 2^{1/2}r)r,$$

which checks with the result found by V. F. Ivanoff.

The foregoing would be unnecessary except as an introduction to the more complicated part of the solution.

For values of r in the remaining interval $2^{-1/2}$ to $2\cdot 6^{-1/2}$ within the cube, the cylinder will cut the edge $EC(y=1, z=0)$, and our integration will be separated into parts. Denote by S_1 the surface of the part for which the intersections of the cylinder with EC are along EB, say at T, the point B being the middle point of EC. Denote by S_2 the surface of the remaining part, that is, the part for which the intersections of the cylinder with EC are along BC, say at T_1.

In the integration to find S_1, we take $2\pi/3$ for the lower limit of θ, and the value of θ at a point T for the upper limit. In the corresponding integration to find S_2 we take the value of θ at a point T_1 for the lower limit and $\theta=\pi$ for the upper limit.

To find the values of θ that correspond to T and T_1, drop a normal[2] TP_2 from T to OD. At its foot, P_2, draw a normal P_2M where M is on OC. Then it follows by very simple geometry that

$$(8)\qquad -3^{1/2}r\cos\theta = 2^{1/2} - r\sin\theta.$$

Solving (8) for $\sin\theta$ and $\cos\theta$, we obtain

$$(9)\qquad \sin\theta = \tfrac{1}{4}r^{-1}\cdot 2^{1/2} \pm \tfrac{1}{4}r^{-1}(12r^2-6)^{1/2},$$

$$(10)\qquad \begin{aligned}\cos\theta = -\tfrac{1}{4}r^{-1}\cdot 6^{1/2} \pm \tfrac{1}{4}r^{-1}(4r^2-2)^{1/2} &= \lambda, \text{ when } + \text{ sign is used,}\\ &= \lambda_1, \text{ when } - \text{ sign is used.}\end{aligned}$$

The λ applies to θ corresponding to a point such as T on the left of the middle, B, of EC while λ_1 applies to θ for a point such as T_1 on the right of B. Moreover, it is easy to verify that the positive sign in (10) corresponds to the positive sign in (9). Now we may write

$$(11)\ S_1 = 6\int_{2\pi/3}^{\arccos\lambda}(p_2-p_1)r\,d\theta = 6r\left\{3^{1/2}\arccos\left[-\tfrac{1}{4}r^{-1}\cdot 6^{1/2} + \tfrac{1}{4}r^{-1}(4r^2-2)^{1/2}\right]\right.$$
$$\left. + (6r^2-3)^{1/2} - 2\pi\cdot 3^{-1/2} - 3r\cdot 6^{-1/2}\right\}.$$

$$(12)\ S_2 = 6\int_{\arccos\lambda_1}^{\pi}(p_2-p_1)r\,d\theta = 6r\left\{3^{1/2}\pi - 3r\cdot 6^{-1/2} - 3^{1/2}\arccos\left[-\tfrac{1}{4}r^{-1}\cdot 6^{1/2}\right.\right.$$
$$\left.\left. - \tfrac{1}{4}r^{-1}(4r^2-2)^{1/2}\right] + (6r^2-3)^{1/2}\right\}.$$

[2] Not shown in the figure.

From (11) and (12), we obtain

$$S_1 + S_2 = 6r[- 3^{1/2} \operatorname{arc} \cos (r^{-2} - 1) + 2(6r^2 - 3)^{1/2} + 3^{-1/2}\pi - 6^{1/2}r]$$

for the surface of the cylinder when

$$2^{-1/2} \leqq r \leqq 2 \cdot 6^{-1/2}.$$

Also solved by J. B. Reynolds.

A Note by Otto Dunkel. The analysis given in the partial solution of this problem [1930, 557] may be extended without difficulty to cover the remaining case. It seems quite likely that the solver was aware of this simple extension but did not give the result on account of its complicated form. Since this remaining part has appeared of interest, the rest of the analysis is given below using the numbered results in the reference above. The lateral area of that part of the cylinder outside the cube will be computed. The projection on the xy-plane of that half of a face considered in the solution is an equilateral triangle OAB with an edge of length $2^{1/2}\,3^{-1/2}$ and an altitude of length $2^{-1/2}$. Here the circle of radius r cuts AB in two points; let us call 2ψ the angle between the two radii to these points. Then $\cos \psi = 2^{-1/2}r^{-1}$, and the angles for these two radii are $\theta_2 = \frac{1}{6}\pi + \psi$, $\theta_1 = \frac{1}{6}\pi - \psi$. Inserting these limits in the integral in (5), and using the reduction $\sin \theta_2 - \sin \theta_1 = 3^{1/2} \sin \psi$, there results

$$(6) \qquad 12 \cdot 3^{1/2}r[\operatorname{arc} \cos (2^{-1/2}r^{-1}) - (2r^2 - 1)^{1/2}],$$
$$2^{-1/2} \leqq r \leqq 2^{1/2}3^{-1/2}, \quad 0 \leqq \operatorname{arc} \cos (2^{-1/2}r^{-1}) \leqq \tfrac{1}{6}\pi.$$

If the result in (6) is subtracted from (5) the difference is that part of the lateral area of the cylinder which is inside the cube.

3439 [1930, 315]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

In the *Pecqueur Bibasal Sextic Spur-Gear Train*, one of the 105 angular velocity ratios which are associated with it is

$$(1) \qquad N_{31}/N_{61} = (z_2z_4z_5z_6)/(z_3z_4z_4'z_5' - z_3z_4z_5z_6 - z_2z_3'z_4'z_5'),$$

in which the eight z's designate the numbers of teeth on the eight gears of the train. In designing such a train the z's must be chosen in accordance with the following fundamental requirements:

(A) All of the z's must be positive integers.

(B) No z may be less than a certain minimum, say 12, nor greater than a certain maximum, say 60.

Four questions now present themselves, in answering which it must be remembered that whatever special conditions may be imposed on equation (1), general conditions (A) and (B) must always be satisfied. The four questions are:

(I) Between what limits lie the possible numerical values of the ratio N_{31}/N_{61}?

(II) For an assigned, allowable value of the ratio N_{31}/N_{61},—say for the value 1,000,000, if that should prove to be an allowable value,—select the z's.

(III) If possible, so select the z's that the special condition

$$N_{31}/N_{61} = \pm z_2z_4z_5z_6$$

will be satisfied.

(IV) If possible, so select the z's that the following special requirements will be simultaneously fulfilled: (*a*) the ratio N_{31}/N_{61} is to have an assigned numerical value,—say 3,373,785 ($=1809\times 1865$) if that should prove to be an allowable value; (*b*) the sum $z_2+z_3+z_3'+z_4+z_4'+z_5+z_5'+z_6$ is to be a minimum.

Solution by Howard H. Mitchell, University of Pennsylvania

A slightly modified form of this problem was sent to me some time ago by Professor Louis O'Shaughnessy, a colleague of Professor Rasche, so that I had some familiarity with it before it was proposed for solution in the Monthly. While any method of solution would seem to involve a certain amount of trial, the following is suggested as possibly reducing that element to a minimum.

As the numerator of the given fraction is a product of four numbers each $\leqq 60$, it is a product of primes <60. If we restrict attention to possible solutions for which it assumes values between the limits $1{,}000{,}000\pm 100$, and divide 1,000,100 by suitable powers of the primes less than 60, we find the prime factors that are less than 60 of the numbers between these limits. The only such numbers that have no larger prime factors are $999{,}925=5^2\cdot 23\cdot 37\cdot 47$; $999{,}936=2^9\cdot 3^2\cdot 7\cdot 31$; $999{,}949=29^3\cdot 41$; $999{,}999=3^3\cdot 7\cdot 11\cdot 13\cdot 37$; $1{,}000{,}000=2^6\cdot 5^6$; $1{,}000{,}008=2^3\cdot 3^2\cdot 17\cdot 19\cdot 43$; $1{,}000{,}065=3\cdot 5\cdot 11^2\cdot 19\cdot 29$.

For each of these numbers there are possible choices for z_2, z_4, z_5, z_6. In order that the denominator of the given fraction shall have the value ± 1, the values assigned to z_2, z_4 can have no common factor. There are thus only two possible ways of choosing these four numbers if their product is to be 999,949.

For a particular selection of these four numbers we then seek possible values of z_3, z_3', z_4', z_5' that satisfy the equation

$$z_3z_4z_4'z_5' - z_3z_4z_5z_6 - z_2z_3'z_4'z_5' = \pm 1. \tag{1}$$

The possible values of z_4', z_5', in addition to being between 12 and 60, must evidently be prime to z_4, z_5, z_6. For values of the variables that satisfy (1), the coefficient of z_3 must be positive, and hence, if z_3 were replaced by 60, and z_3' by 12, the left member would be increased. Hence

$$z_4'z_5' > 5z_4z_5z_6/(5z_4 - z_2). \tag{2}$$

In view of the symmetry of (1) in z_4', z_5' we may assume that $z_4'\leqq z_5'$ and hence that z_5' must be greater than the square root of the right member of (2), while z_4' must exceed the right member divided by 60.

We observe also that the two congruences,

(3) $$z_3z_4z_5z_6 \equiv \mp 1, \bmod z_4'z_5',$$

(4) $$z_3z_4z_4'z_5' \equiv z_3z_4z_5z_6 \pm 1, \bmod z_2,$$

must each be satisfied, where the upper or the lower signs must be used in both cases. A table may then be constructed by use of (3) giving the possible values of z_3 for each of the possible values of z_4' or z_5'. For each possible value of z_5' the corresponding values of z_3 may be read from this table and then z_4' calculated from (4). By reference to the table it will then be found that (3) is not usually satisfied. If, however, (3) is satisfied, then z_3', as calculated from (1), may be an integer (*will* be, if the moduli of (3) and (4) are relatively prime), and, if within the prescribed limits, a solution of the desired sort will be obtained.

Some modification of the procedure may be desirable in case z_2, z_4', z_5' have factors in common. Thus, if z_2 and z_5' have a common factor, it may be desirable, after the calculation of z_3, to divide equation (1) by z_5' before transforming it into a congruence, mod. z_2.

For $z_2\ z_4\ z_5\ z_6=999{,}949$, solutions of the desired sort were found to exist. If we take z_2, z_4, z_5, $z_6=29$, 41, 29, 29, respectively, the inequality (2) gives $z_4'z_5'>979$, whence $z_5'>31$, $z_4'>16$. The congruences (3), (4) reduce to

$$34{,}481z_3 \equiv \mp 1, \bmod z_4'z_5';$$

$$z_3z_4'z_5' \equiv \mp 12, \bmod 29.$$

For $z_5'=35$ we find, in the case of the lower sign, $z_3\equiv 6$, mod. 35, whence $z_3=41$. For this z_3, we find $z_4'\equiv 5$, mod. 29, whence $z_4'=34$. For this z_4', the first congruence is found to be satisfied, and from equation (1) it is found that $z_3'=17$.

In a similar manner, it is found that a solution is given by $z_5'=51$, $z_3=41$, $z_4'=33$, $z_3'=29$.

These two solutions together with those obtained from them by interchanging z_4' and z_5' were all that were found to exist for z_2, z_4, z_5, $z_6=29$, 41, 29, 29. No solutions were found when the values of z_2 and z_4 were interchanged.

For each other possible value of $z_2z_4z_5z_6$ solutions could be sought in a similar manner. In general, however, the number of possible sets of values for these four variables would be considerably greater than in the case considered.

3444 [1930, 380]. *Proposed by Frank Morley, Johns Hopkins University.*

In an inversive plane, the general self-conjugate equation, $f(x, \bar{x})=0$, of degree three in x and $\bar{x}$ defines a bi-cubic curve, c. Since any circle has with such a curve six common points (intersections or common unique pairs) there are contact circles, touching thrice. There are, it is known, 120 contact circles. If we take three of these, the 9 points of contact either lie on a biquadratic b, or they do not. When they do, the circles are tied (or syzygetic); and the curve, b, meets c in the points of contact of a fourth circle, so that the circles are tied in sets of four. Prove that the four circles of a set touch a circle.

Solution by the Proposer

If a_1, a_2, a_3, a_4 be a set of contact circles of c, the 12 points of contact are on a curve b. We have then $a_1a_2a_3a_4+\mu b^2$ containing c as a factor. The other factor must be another circle a. Thus $a_1a_2a_3a_4+\mu b^2=ca$; whence, a will touch a_i where b meets it again.

The bi-cubic is an alternative to the space sextic of genus 4, according as one wishes to view things inversively or projectively. For the underlying theta-theory see Professor Coble's *Colloquium Lectures*.

3449 [1930, 447]. *Proposed by W. E. Buker, Leetsdale, Pa.*

(1) Find a number X such that X^2+5 and X^2-5 are each square numbers.

Note: This problem was proposed by John of Palermo and solved by Leonardo of Pisa about 1220 A.D. A solution is $X=41/12$; for $(41/12)^2+5=(49/12)^2$; $(41/12)^2-5=(31/12)^2$. How did he arrive at his result?

Reference: Florian Cajori, *History of Mathematics*, pp. 124.

Solution by J. D. Hill, University of California at Los Angeles

By subtracting the two equations $x^2+5=Q^2$ and $x^2-5=P^2$, there results $Q^2-P^2=(Q-P)\ (Q+P)=10$. If a solution exists, we may denote $Q-P$ by y and obtain the system

$$Q-P=y,\ Q+P=10/y.$$

The solution for P or Q, with the original relations, gives $x=(100+y^4)^{1/2}/2y$. Since y is assumed rational, we may write $y=a/b$ where a and b are integers, and we have $x=(100b^4+a^4)^{1/2}/2ab$. The problem now reduces to making the radicand a square number, which can be accomplished by Lagrange's method[1] and an infinite number of solutions obtained, provided (as shown in the reference) we can first find two integers A and B such that $100B^4+A^4$ is a square. In order to obtain this first solution we note that $100+y^4$ will be a square when[2] $y^2=-20t/(t^2-1)$, where t is a rational parameter. For y to be rational, it is sufficient that both numerator and denominator be squares, which is true for the former if $t=-m^2/20$, and for the latter if $t=(-1-n^2)/2n$ where m and n are arbitrary rational numbers. Equating these two values of t and solving for n in terms of m we obtain, $n=[m^2\pm(m^4-400)^{1/2}]/20$. For all values of m which make (m^4-400) a square, n and consequently y will be rational, and thus solutions are obtained. By inspection $m=5$ fulfills the condition and yields $y=20/3$ which gives $x=41/12$ or the solution obtained by Leonardo of Pisa. Placing $a=20$, $b=3$ we may now follow Lagrange's method and obtain as many solutions as desired.

Disregarding Lagrange's method, however, we may obtain a solution different from 41/12 as follows by considering m^4-400. Place $m^2+20=S^2$, m^2-20

---

[1] See Dickson, *History of Theory of Numbers*, vol. II, p. 628.

[2] See Wertheim, *Zahlentheorie*, p. 56.

$=R^2$ and subtract, obtaining $S^2-R^2=(S-R)\ (S+R)=40$. If we denote $(S-R)$ by z we have the equations

$$S - R = z, \quad S + R = 40/z.$$

The solution for S or R taken with our original relations gives $m=(z^4+1600)^{1/2}/2z$. By inspection, $z=3$ yields a rational value of m, namely $41/6$. For this value of m, $y=4920/1519$ which when substituted above gives $x=3,344,161/1,494,696$. The two resulting squares turn out to be

$$x^2 - 5 = (113,\ 279/1,\ 494,\ 696)^2$$

and

$$x^2 + 5 = (4,\ 728,\ 001/1,\ 494,\ 696)^2$$

The numerators of the last three fractions are relatively prime to their respective denominators.

This investigation, while not complete by any means, leads directly to Leonardo's solution and to as many others as desired. In the meantime certain leads suggested in the paper are being followed in hopes of securing a more general procedure.

Also solved by Mannis Charosh and J. Rosenbaum.

A Note by Otto Dunkel. The problem asks how Leonardo of Pisa arrived at his result. Since the method above as far as the derivation of the equation $a^4+100b^4=u^2$, where a, b, u are integers, would occur to anyone trying to solve this problem, it would require no great strain on the imagination to suppose that Leonardo proceeded in this way. A very few trials with the first integers would have led him to the solution $a=3$, $b=2$, and then to his result $x=41/12$.

It may be of interest to note that the method of Lagrange, alluded to above, which tells how to get a second solution a_1, b_1 from a first solution a, b leads precisely to the second solution above, if we start with $a=3$, $b=2$. For in that method $a_1=a^4-100b^4=-1519$, $b_1=2abu=492$. From these values of a_1 and b_1 follows the value of x given above.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Franklin Institute, Philadelphia, has awarded a Franklin medal to Sir James Jeans, " in recognition of his many fruitful contributions to mathematical physics, especially in the realms of dynamical theory of gases and the theory of radiation, and of his challenging explanations of astronomical problems and his illuminating expositions of modern scientific ideas."

The New York Academy of Sciences has awarded one of its A. Cressy Morrison prizes to Professor H. von Zeipel, of the Astronomical Observatory of Upsala, for his paper entitled *The evolution and constitution of stars.*

Professor George David Birkhoff, of Harvard University, will give a series of lectures on "Le dernièr théorème de géométrie de Poincaré, ses généralisations, et ses applications à la dynamique," at the Collège de France, during the last two weeks of April, under the Fondation Michonis.

Professor E. B. Stouffer has been elected a member of the section committee of Section A of the American Association for the Advancement of Science.

Dr. Nola L. Anderson has been appointed associate professor and acting chairman of the department of mathematics at Sophie Newcomb College.

Dr. John Jay Gergen and Dr. Albert Eldred Currier have been appointed Benjamin Peirce Instructors at Harvard University for the next academic year.

Associate Professor George W. Hess, of Howard College, has been promoted to a professorship of mathematics.

Dr. H. K. Hughes, formerly instructor at the University of Michigan, has been appointed assistant professor of mathematics at Purdue University.

Dr. Wolfgang Pauli, of the Zurich Technical School, and Dr. Arnold Sommerfeld, of the University of Munich, have been appointed lecturers in theoretical physics at the University of Michigan for the summer session of 1931.

President A. E. Whitford, of Milton College, Milton, Wisconsin, who resigned his office last June, is a lecturer in mathematics at the University of Wisconsin.

Professor David Vernon Widder, of Bryn Mawr College, has been appointed assistant professor of mathematics at Harvard University.

The following courses in mathematics are announced for the summer of 1931:

University of California at Los Angeles, July 1–August 10. By Professor Harriet E. Glazier: Foundations of arithmetic. By Professor E. R. Hedrick: The teaching of mathematics.

The following appointments to instructorships in mathematics are announced:

Columbia University, Mr. E. R. Lorch;

Harvard University, Douglas Payne Adams, Allen Emil Anderson, James Sutherland Frame, Alan Stuart Galbraith, Lester Turner Moston, Sumner Byron Myers, Griffith Baley Price, Frederick Henry Steen, George Booth Van Schaack.

Professor W A. Zehring, of Purdue University, died recently at the age of 55 years. He was a charter member of the Mathematical Association of America.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 6, JUNE-JULY

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The eleventh regular meeting of the Southern California Section was held at Occidental College, Los Angeles, on Saturday, March 21, 1931. Professor G. E. F. Sherwood presided.

The attendance was forty-two, including the following twenty-eight members of the Association: E. E. Allen, Harry Bateman, Clifford Bell, E. T. Bell, Jessie R. Campbell, Myrtie Collier, D. R. Curtiss, P. H. Daus, Iva B. Ernsberger, Raymond Garver, H. H. Gaver, Harriet E. Glazier, E. R. Hedrick, J. D. Hill, G. H. Hunt, Glenn James, G. R. Livingston, W. E. Mason, A. D. Michal, F. R. Morris, W. B. Orange, Lena E. Reynolds, G. E. F. Sherwood, Marcus Skarstedt, D. V. Steed, H. C. Van Buskirk, W. M. Whyburn.

The meeting began with a luncheon at the Student Union, after which it adjourned to Fowler Hall for a short business meeting and the program. The following officers were elected for the year 1931–1932: Chairman, H. C. Van Buskirk, California Institute of Technology; Vice-chairman, L. D. Ames, University of Southern California; Program Committee, G. R. Livingston, San Diego State Teachers College, G. F. McEwen, Scripps Institute of Oceanography. The next meeting was scheduled to be held at San Diego State Teachers College, March 26, 1932.

The following program was presented:

1. "The contributions of clergymen to the sciences of aerodynamics and hydrodynamics" by Professor H. Bateman, California Institute of Technology.

2. "The vector analytical determination of the radius of spherical curvature of a space curve" by Professor D. V. Steed, University of Southern California.

3. "Loci of the center and foci of the conic sections formed by a plane revolving about a line" by Professor Frank R. Morris, State Teachers College, Fresno.

4. "Equations implied by inequalities" by Professor W. M. Whyburn, University of California at Los Angeles.

5. "Some remarks on maxima and minima of functions of two variables" (by invitation) by Professor D. R. Curtiss, Northwestern University.

Abstracts of these papers follow:

1. This paper contained an account of the work of Derham, Willis, Whewell, Challis, Murphy, Bashforth and Ramus. The speaker illustrated his remarks by some simple interesting experiments.

2. The conditions for contact of order n or a surface with a curve were expressed in vector form, and then applied to obtain the formula for the radius of spherical curvature. Attention was called to errors in the discussion of this topic in certain current texts.

3. Assume a circular cone and a fixed line perpendicular to the axis of the cone. The fixed line and the axis of the cone may or may not be in the same plane. A plane revolving about the fixed line gives conic sections and it is the loci of the center and foci of these conics which were discussed. The center of the conic generates an hyperbola while the foci generate a fourth degree curve which, in general, has three double points, one of which may be isolated. Interesting degenerate cases arise with both curves if the fixed line is tangent to the cone or intersects the axis of the cone. Most of the conclusions may be reached either geometrically or analytically.

4. Inequalities of the type

$$F(x) \leqq \sum_{i=1}^{m} f_i(x) g_i(x),$$

where the functions involved are non-negative and continous on an interval X, were discussed with a view to determining continuous functions $h_i(x)$, such that $0 \leqq h_i(x) \leqq g_i(x)$,

$$F(x) \equiv \sum_{i=1}^{m} h_i(x) f_i(x),$$

and these functions take on assigned boundary values at points of X. The results were applied to a uniqueness proof for the differential equation $y' = f(x, y)$.

5. Professor Curtiss pointed out the reduction of more general cases to problems concerning polynomials and showed how propositions of elementary theory of equations (Rolle's Theorem, Budan-Fourier Theorem, Sturm's Theorem) can be extended to polynomials in two variables so as to apply to the maximum-minimum problem.

P. H. Daus, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The eighth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor on March 21, 1931 in conjunction with the Michigan Academy of Science, Arts, and Letters. Professor Theodore Lindquist, chairman, presided.

Fifty-seven persons registered, but over seventy were in attendance. The following forty members of the Association were present: N. H. Anning, W. L. Ayres, J. W. Baldwin, J. F. Barnhill, J. B. Brandeberry, Frances M. Brant, R. W. Clack, S. E. Crowe, Albertus Darnell, W. W. Denton, L. C. Emmons, C. M. Erikson, J. P. Everett, Peter Field, K. W. Folley, W. B. Ford, V. G. Grove, L. M. Graves, L. A. Hopkins, L. S. Johnston, D. K. Kazarinoff, K. D. Kelly, Theodore Lindquist, C. E. Love, E. B. Miller, A. L. Nelson, C. V. Newsom, Rufus Oldenburger, H. L. Olson, J. E. Powell, G. Y. Rainich, L. J.

Rouse, R. H. Schoonover, R. C. Shellenbarger, G. G. Speeker, C. E. Stout, A. G. Swanson, J. F. Thomson, C. C. Wagner, J. B. Winslow.

During intermission Professor Lindquist appointed as the nominating committee, Professor W. B. Ford, Professor R. C. Shellenbarger, Professor A. L. Nelson, and Professor L. A. Hopkins, who reported at the luncheon held at noon in one of the private dining rooms of the Michigan Union. Fifty-five persons attended this luncheon.

The following officers were elected for the ensuing year: Chairman, J. P. Everett, Western State Teachers College; Secretary-Treasurer, L. A. Hopkins, University of Michigan; Member of the Executive Committee, L. C. Emmons, Michigan State College.

Professor W. B. Ford, University of Michigan, announced a lecture to be given by Professor Harald Bohr of the University of Copenhagen on March 27 at the University of Michigan on "Almost Periodic Functions," and invited all present to attend.

The following eleven papers were read:

1. "Matrix representations of Cremona transformations" by Mr. Rufus Oldenburger, University of Michigan.

2. "On teaching number theory" by Professor G. Y. Rainich, University of Michigan.

3. "Some predictions of success in college from placement examination scores" by Professor L. C. Emmons, Michigan State College.

4. "Existence theorems for the absolute minimum in the calculus of variations" by Professor L. M. Graves, Visiting Professor from University of Chicago.

5. "Mathematical theory of heat conduction and convection from a hot vertical plate" by Professor W. S. Kimball, Michigan State College, by invitation.

6. "Mathematics in the training of Detroit teachers" by Professor A. L. Nelson, College of the City of Detroit.

7. "A construction of the unit angle" by Professor R. H. Schoonover, College of the City of Detroit.

8. "Results of examinations given to entering classes in Michigan colleges" by Professor J. P. Everett, Western State Teachers College.

9. "Perfect fluid motion and modern aerodynamics" by Professor M. J. Thompson, University of Michigan, by invitation.

10. "Singularities of a plane curve from the standpoint of its imaginary branches" by Mr. Hugh M. Ackley, Western State Teachers College, by invitation.

11. "Graduation of experimental data," by Professor H. C. T. Eggers, University of Minnesota (on leave), by invitation.

L. A. HOPKINS, *Secretary*

THE HUMAN ASPECT IN THE EARLY HISTORY OF THE AMERICAN MATHEMATICAL MONTHLY[1]

By B. F. FINKEL, Drury College

Since, by request of the program committee, the human aspect of the early history of the American Mathematical Monthly is to be emphasized, I hope I may be pardoned for speaking more often in the first person than a due sense of modesty would permit.

Why the American Mathematical Monthly? It may be of some interest to the younger members of the Mathematical Association of America, and perhaps to the older members as well, to know something of the early mathematical background of its founder. A number of the older members of this Association have been personally known to me throughout the existence of the Monthly, but I judge that most of the members here today have reached their majority since the founding of the Monthly. Many of these know very little of its early history and still less of its founder.

I have been attending these winter meetings quite regularly for the past ten years, and, on being introduced to some of the younger members, I am sometimes asked amusing questions. At a recent meeting one such member asked me if I was the son of the founder of the Monthly. At the Kansas City meeting in 1925 I chanced to meet a friend of the Monthly, a subscriber from the beginning. I had never had the pleasure of meeting him before. Overhearing someone asking me about Springfield, he leaned back over his chair and asked me if I knew that old man Finkel down there. I assured him that I had a somewhat intimate acquaintance with him. "Well," he said, "you know he published a Mathematical Solution Book some years ago. Recently I needed an authority for the numeration of a certain large number in connection with a law case, and I wrote to Professor Cajori who referred me to Finkel's book. A few weeks ago I bought a copy and I have been disseminating mathematical information from it among my friends ever since." Then he added, "I should like to meet him some time." "Well," I said, "you are talking to him right now." "Are you he?" he asked in surprise. "Why, I expected to see an old man, six feet tall, weighing two hundred pounds, with a long white beard reaching far below his vest."

By knowing something of the early life of the founder of the Monthly, a better understanding may be gained of the chief features characterizing the journal, as these are the realization of his early ideals. A superficial survey of the early volumes of the Monthly will, perhaps, readily disclose the fact that the founder was not a mathematical genius; for such, unlike Antaeus, loses his strength when touching earth, but shines with effulgent glory when soaring among the clouds of mathematical research. I did know, however, that the fundamental branches of mathematics were poorly taught in our high schools

[1] A paper read before the Mathematical Association of America at its meeting at Cleveland, Ohio, January 1, 1931, by invitation of the program committee.

and academies, that the *esprit de corps* and mathematical knowledge of the teachers of elementary mathematics were lamentably lacking.

My early knowledge of mathematics was very meagre and the outlook for future mathematical development very unpromising. Until I was seventeen years of age I had never seen a geometry or an algebra. I attended the Ridge country school in Fairfield County, Ohio, until I was eighteen, giving scant attention to the acquisition of knowledge of any kind. The older boys felt it encumbent upon them to make life as varied, active and uncomfortable for the teachers as possible. Disorder reigned supreme. When I was about twelve years old, R. V. Allen, a young man with grit and courage, was employed to teach the school and, after he had introduced himself by taking some of the older boys, who had bullied his predecessors, by the nape of the neck and shaking them as a dog shakes a rat, the atmosphere clarified, discipline followed, and order came out of chaos. Thus it was made possible for teachers following him, who had less muscle but more ability to impart knowledge, to devote themselves to the purpose for which they were employed, namely, the dispelling of the darkness of ignorance and the creation of the light of knowledge.

What were the influences in such environment leading one to think of establishing a mathematical journal? Shakespeare answers, "There is a divinity that shapes our ends." A number of years ago I received from the publishers a brochure on astrology in which I found the answer: "Your stars are favorable." In cataloging the chief elements of character entering into the lives of people born in Gemini, my constellation, the author says among other things, "These people are sensitive, intuitional, imaginative, idealistic, very fond of scientific knowledge, consequently often very studious, ambitious, aspiring, curious, given to investigation and experimentation, excellent reasoners, generally good writers, and they have the capacity to engage in two occupations at the same time. They make successful lawyers, lecturers, school teachers, and journalists, and as editors they have splendid facilities." There you have the answer in the stars, every star millions of miles from where it was when I was born!

To continue seriously: When I was fifteen years old, it was my good fortune to come under the influence of a very superior country school teacher. This was George W. Bates, and to him, next to my mother, I owe more than to any person who ever influenced my life. Though small in stature and crippled in limb, he was a man of unusual courage, unswerving honesty, unfailing firmness, and accurate judgment. It was due to his fine discipline and leadership that more teachers, preachers, lawyers and judges came from that school during the two years he taught there than in all the years since. My interest in mathematics was aroused at this time and it came about in this way. A certain problem was going the rounds of the country at that time as many problems have the habit of doing. This problem gained the attention of the fellows, young and old, who in those days gathered at the village store for the discussion and solution of questions of religion and politics. An older half-brother of mine, hearing it discussed at one of these grocery-store sessions, brought it home to me to think

about. I took it to this teacher of mine and he explained how it could possibly be solved by geometry. But since I had never seen a geometry, it gave me little satisfaction to know that the problem could be solved by means of it. Nevertheless, having studied Ray's Third Part Arithmetic, I attempted to solve the problem by applying the rules of mensuration contained in that book. My efforts were crowned with success several years after my first attempt. This problem reads as follows:

> There is a ball 12 feet in diameter on top of a pole 60 feet high. On the ball stands a man whose eye is six feet above the ball. How much ground beneath the ball is invisible to him?[1]

This perfectly senseless problem with no value whatsoever from the standpoint of modern educational theory, nevertheless was the borax in the mortar which retarded mental hardening until a time arrived when other elements could play their part in the active materials of a life, and it seems to me that such a result should be the test by which the value of a problem should be gauged. The so-called practical problems so sonorously insisted upon, if any problems at all are insisted upon by modern educators, are often the most impractical from the standpoint of interest and consequent mental development of the pupil. The perfectly senseless problem, "If the third of six be three, what will the fourth of forty be," is much more likely to arrest the interest of the pupil than the more practical concrete problem, "If one-third of six apples sell for three cents, what will one-fourth of forty apples sell for?" But modern educators have ruled against placing any such problems as the former in our elementary mathematical text books, and the authors of such books have heard their voice.

When I was in my eighteenth year I left the country school to attend the Ohio Normal University at Ada, Ohio, now called the Ohio Northern University, a name suggested by me. After spending a year at that school I began teaching in the country schools. It was at this time that a fellow teacher put into my hands a copy of the *School Visitor*, edited and published by John S. Royer. In this little magazine was conducted a Mathematics Department to which some very good mathematicians of the day often contributed. Among them I may mention Dr. Artemas Martin, editor and publisher of the *Mathematical Magazine* and the *Mathematical Visitor*, Washington, D.C.; Dr. William Hoover, professor of mathematics at the Ohio University at Athens; and Professor E. B. Seitz of the Kirksville, Missouri, Normal School, the world-renowned problem solver of his time. Professor Seitz contributed problems and solutions to every magazine in England and America in which was conducted a mathematical department. *The Educational Times* of London contains many of his finest solutions of some of the most difficult problems in the theory of probability, and the more difficult a problem seemed to the average mind, the more ardently he attacked it.

---

[1] See Finkel's Mathematical Solution Book, page 367.

Having become a subscriber to the *School Visitor* in 1884, I began contributing problems to its pages. Later, as I gained in knowledge, I contributed solutions of the easier problems. A contemporary contributor to the *Visitor* at that time was the late Professor Wayland Dowling. In a conversation with him not long ago, I learned from him that he too had gained a love for mathematics through his contributions to that journal. Later I learned of Dr. Martin's mathematical publications, purchased all the back numbers, and became a subscriber and contributor to both. Through these two publications I learned of *The Educational Times* of London, and became a subscriber and contributor to it. While again a student in the Ohio Normal University I conducted a mathematical department in the *University Herald* from 1887 to 1888, and for several years longer after graduating from that institution.

While teaching in a country school in Union County, Ohio, in 1887 I began the writing of my Mathematical Solution Book, designed to aid in improving the teaching of elementary mathematics in the rural schools, high schools and academies, and got it ready for publication the following year. It was in connection with this enterprise that I met my first financial reverse. The printer who had undertaken the publication of the book failed after printing eighty-eight pages and borrowing two hundred dollars from me. It was not until 1893 that I was able to bring out the first edition of one thousand copies.

During these years from 1884 to 1893 I was teaching in the rural schools of Ohio, in the Fostoria Academy, and in the Gibson Male and Female Academy, Gibson, Tennessee. At the same time, at my leisure, I was contributing problems and solutions to the journals already mentioned and, in addition, to the *Ohio Educational Monthly*, *The School Messenger*, and the *Davenport, Iowa, Monthly*. In 1890–91 I was superintendent of schools at North Lewisburg, Ohio, and in 1891–92 at West Middleburg, Ohio. I became thoroughly discouraged and disheartened because of the dishonorable political methods used in securing positions in most of the city schools in Ohio, and decided to quit the public school and join my good friend, Professor G. W. Shaw, Principal of Kidder Institute, Kidder, Missouri. It was while Professor Shaw was principal of Fostoria Academy that he invited me to become a member of his faculty there. The remuneration at Kidder Institute was not very lucrative, the routine work of teaching quite heavy—amounting often to forty-five three-quarter-hour periods per week. The management of the school, however, was free from every form of petty politics so deadening to intellectual honesty and spiritual development. In an atmosphere of that sort one may often ascend to the mountain heights of imagination and get glimpses of things unseen rather than have one's attention focussed on the sordid things of earth.

The mathematical journals with which I was acquainted and to which I contributed, and for which I waited anxiously at the stated times of their appearance, were for the most part irregular in reaching me on schedule time. Dr. Martin's two journals appeared only "semi-occasionally." In passing, I wish to say that everything connected with the publication of the *Mathematical*

Magazine and the *Mathematical Visitor*, save the making of the wood-cuts, was done by Dr. Martin himself. The type-setting, the press work, binding and mailing were all done by his own hand. He owned a small hand press and a full line of mathematical type. This equipment he offered to give to me just before his death, but he died before he gathered them together for shipment. The typographical composition of these two journals was pronounced by competent judges to be unsurpassed by any other mathematical typographical work done anywhere in the world. The *School Visitor*, edited and published by John S. Royer, was still making its appearance quite regularly.

Knowing that the status of the mathematical teaching in our high schools and academies was very deplorable and even worse in the rural schools, I had the ambition to publish a journal devoted solely to mathematics and suitable to the needs of teachers of mathematics in these schools. With that idea in mind, I talked to Mr. E. J. Chubbuck, the editor and publisher of the local paper in Kidder, with reference to the publication of such a journal. Mr. Chubbuck was a young man about my age who also had ambition and courage, and he agreed to print, bind, and place in the mail the journal for just what he could make out of it after deducting postage and the cost of the wood-cuts. I decided to call the new publication THE AMERICAN MATHEMATICAL MONTHLY, a most ambitious title, as my friend Dr. E. H. Moore afterwards told me. With the agreement with Mr. Chubbuck consummated, I began looking around for an associate. Having known of Professor John M. Colaw of Monterey, Virginia, through his contributions to the *School Visitor*, I laid my plans before him by correspondence and invited him to join me in the enterprise. He readily accepted the invitation and became co-editor. These arrangements were made in the fall of 1893.

Our next task was to solicit subscribers and contributions, a task to which Professor Colaw devoted himself whole-heartedly. I wrote to a number of high school teachers of mathematics, asking for their cooperation in the way of taking subscriptions and sending in contributions for publication. The first person to respond to my solicitation was the distinguished, scholarly, and eminently successful superintendent of the Kansas City schools, Professor J. M. Greenwood. He enclosed his check for $2.00 in payment of his subscription for one year—the first money received with which to found the Monthly—and he assured me that he would call the attention of all his mathematics teachers to the new venture. I also wrote many letters to professors of mathematics in our universities and colleges, asking their cooperation by securing subscribers and sending in contributions for publication. The first person from these sources to respond to my appeal was Professor George Bruce Halsted, professor of mathematics in the University of Texas, the "Stormy Petrel" in the mathematical world, one who was in his element when in the midst of a violent verbal storm initiated by himself or otherwise. In answer to my letter asking his cooperation in the establishing of the Monthly, Dr. Halsted was very enthusiastic in his approval of our undertaking and pledged his support by sending me a check for

$30.00, which amount he paid annually as long as he was in the University of Texas. He also promised contributions for publication. These were numerous and of a high order, though chiefly confined to the non-Euclidean geometry, on which subject he was America's chief expounder. He continued his contributions, though less frequently, during the whole time the Monthly was under my personal management. While I met Dr. Halsted personally but twice—once at Columbus, Ohio, in 1900 and once in St. Louis in 1904—he continued until his death to be one of my most loyal and devoted friends. Many other professors in our leading universities became subscribers, not because of any benefit to be derived personally, but merely to encourage and support the Monthly in getting under way. I wish I had time to mention all of them. I shall take time to name Professor E H. Moore of the University of Chicago, Professor W. E. Byerly of Harvard, Professor Edwin S. Crawley of the University of Pennsylvania, Professor Robert J. Aley of Indiana University, Professor Irving Stringham of the University of California, and Professor H. A. Newton of Yale. Professor Moore lent encouraging assistance and prestige by contributing several articles to the early volumes. Professor Aley answered my letter by sending me a list of sixteen subscribers and a check covering their subscriptions. For several years afterwards he kept sending subscriptions. I think he sent in more subscriptions than any one else. He also was a valued contributor to the Monthly's pages.

It soon became apparent to me that the teachers of mathematics in our high schools and academies and normal schools felt no need for such a journal as the Monthly. Whether these teachers were unaware of their lack of equipment for the work they were attempting to do, or whether they scorned the Monthly because of the presumption on the part of its editor that they were in need of such a stimulus, I never was able to learn. Nevertheless I think I am safe in saying that during the nineteen years when the Monthly was in my possession not more than a dozen high school teachers were on our subscription list at any one time. Thus it came to pass, in due process of time, that the field which the American Mathematical Monthly was designed to cultivate for the benefit of high school mathematics teachers particularly, became occupied by a more virile race of mathematicians, namely the teachers of college and university mathematics, particularly the former. The Monthly soon adapted itself to the needs of the field of collegiate mathematics and in that field it has made its most noteworthy contributions. It was my purpose from the first that the Monthly should become a sort of repository of mathematical material of lasting value, and with that in view that it should not contain reports or other material which diminished in value with time. Thus in the early volumes no reports of any kind appear.

The first number of the Monthly appeared the latter part of January, 1894. The introduction in this number sets forth the purposes of the Monthly. In it we read:

> It has seemed to the editors that there is not only room but a real need for a mathematical journal of the character and scope of the Monthly. At the present time there is no

mathematical journal published in the United States sufficiently elementary to appeal to any but a very limited constituency, and that comes to its readers at regular intervals. Most of our existing journals deal almost exclusively with subjects beyond the reach of the average student or teacher of mathematics or at least with subjects unfamiliar to them, and little, if any, space is devoted to the solution of problems. While not neglecting the higher fields of mathematical investigations, the American Mathematical Monthly will also endeavor to reach the average mathematician, devoting regular departments to the important branches of mathematical science.

It is recognized that those improvements in the science are most fruitful which lead to improvements in the elementary treatises, and yet it must be admitted that little has been accomplished by previous mathematical journals in this line, as the crudities and solecisms handed down from one textbook to another bear witness. While realizing that the solution of problems is one of the lowest forms of mathematical research, and that in general it has no scientific value, yet its educational value cannot be over-estimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem. The American Mathematical Monthly will, therefore, devote a due portion of its space to the solution of problems, whether they be the easy problems in arithmetic, or the difficult problems in the calculus, mechanics, probability, or modern higher mathematics. Papers and other interesting features will be presented, including the portraits of prominent mathematicians with their biographies, a column of queries and information in which readers may have information furnished and their doubts cleared up by aid of the contributors and editors, a column of notes, and book reviews.

No pains will be spared on the part of the editors to make this the most interesting and popular journal published in America. In order to do this, we must have the earnest cooperation of our readers, teachers and students, and all lovers of mathematics are, therefore, cordially invited to contribute problems, solutions, and papers on interesting and important subjects in mathematics. We will be pleased to note your successes, and all information of interest in regard to our contributors will be cheerfully received and noted.

It is very gratifying to me to relate that one of the contributors to the first number of the Monthly was a young man in his nineteenth year doing graduate work under Dr. Halsted in the University of Texas, and giving promise at that age of becoming one of the foremost mathematicians of the world. The title of his article is, "Lowest integers representing the sides of a right-angled triangle." The author is Leonard E. Dickson. Young Dickson also contributed an article on "The simplest model for illustrating the conic sections" in the August number of Volume I, and one on "The inscription of regular polygons" in the October, November, and December numbers. We shall have more to say of him later. Professor Robert J. Aley contributed a "Bibliography on the history of mathematics" and a list of mathematical periodicals. In the March number of Volume I, David Eugene Smith of the State Normal School, Ypsilanti, Michigan, later to become the well-known historian of mathematics and one of the leading teachers of mathematics in Teachers College, Columbia University, and author of many books on mathematics, contributed a critical note on "J. K. Ellwood's Remarks on Division," thus opening up the first controversy to be published in the American Mathematical Monthly. In this number also Professor Halsted began his series of articles on "Non-Euclidean Geometry, Historical and Expository," a series continued through many succeeding numbers.

Professor George B. McClelland Zerr of Staunton, Virginia, a versatile problem solver, began in the first number his contributions of problems and solutions and continued contributing until his untimely death in October 1910. One of his great admirers asked me once what kind of food Professor Zerr lived on to enable him to solve so many difficult problems in so many different branches of mathematics. In my course in Byerly's Fourier's Series and Spherical Harmonics in the University of Pennsylvania we encountered the definite integral on page 79 of that text. Byerly merely refers to Bierens de Haan, Tables of Definite Integrals, for its value. The professor giving the course expressed the desire to know how the integral was evaluated. That night I telephoned to Professor Zerr, who was then teaching chemistry in Temple College, Philadelphia, and asked him if he could evaluate the integral. The next morning almost before I was out of bed he brought me a complete solution. I gave this solution to the professor in charge of the course, who was then anxious to meet the man who could produce such a satisfactory solution in so short a time.

For the early numbers of the first volume of the Monthly the wood-cuts were made by a Chicago engraver. In order to reduce expenses in publication of the Monthly, I undertook the making of the wood-cuts myself. My first effort in this new field was the wood-cut on page 71, Volume I, Number 3, which I made with a penknife. I made practically all the wood-cuts used to the end of the first nineteen volumes, except those appearing in the numbers issued during the two years in which I was doing graduate work at the University of Pennsylvania. In making arrangements for the wood-cuts in my absence from Drury College, I called on a Philadelphia wood-engraver. When I informed him that I had been making my own wood-cuts without ever having seen one made, he doubted my veracity. But when I told him that I made my first wood-cut with a penknife, he was certain that I was an impostor, for he assured me that a wood-cut could not be made with a penknife. The next time I went to see him I took several copies of the Monthly with me. I showed him cuts made by the Chicago engraver, asking him if they were all right. He said they were good. Then I showed him some of those I had made, and he assented that they, too, were good. As I was not a trained engraver I thought he could readily pick out my work by its defects, but he admitted that he could not tell any difference. I think I can pick out most of my work by its imperfections, for my work was often done at times when I ought to have been doing several other things. I am sorry that I did not have a copy of the cut I made with a penknife to show him. However, I employed this engraver to make the wood-cuts for the Monthly during the two years I was in Philadelphia.

The publication of the second volume was entered upon with all contributors and subscribers maintaining their allegiance to the Monthly. Beginning with the May number of Volume II, the first article on the Theory of Groups appeared. The author of the article has since become recognized as the greatest expounder of Group Theory in America. It would be interesting to know just how many articles on Group Theory have been published in Europe and Amer-

ica since that time by our good friend Dr. G. A. Miller. Later Professor Miller became an associate editor of the Monthly.

In June 1895, through the influence of Dr. Henry Hopkins, pastor of the First Congregational Church of Kansas City, Missouri, a member of the Board of Trustees of Kidder Institute and Drury College, I was elected to the professorship of mathematics and physics at Drury College, Springfield, Missouri. This event foreshadowed a possible change in the affairs of the Monthly. On coming to Springfield, by an unusual combination of favorable circumstances, I found another young man in the person of S. A. Dixon, a printer by trade and owner of a printing office. Mr. Dixon was born and reared in Cleveland, I think, and was easily the best printer in Springfield at the time I went there. I took a copy of the Monthly with me to show him what it was. He said that, while he had never done any mathematical type-setting and had no mathematical type in his office, he thought if Mr. Chubbuck of Kidder could print it, he could. As a consequence, Mr. Dixon agreed to print, bind, and put in the mail the thirty-two-page issue of the Monthly for $30.00. But the price of labor and material rapidly rose and we were obliged to increase the price of publication, and this price always exceeded the income from subscriptions. We were thus forced either to increase the subscription price or to decrease the number of pages per issue; we chose the latter alternative. During all these more than seventeen years Mr. Dixon never received commercial prices for the work he did on the Monthly. Being an unusually swift and accurate type-setter himself, he set the type for the Monthly with his own hands and thus he was enabled to carry on its publication without serious financial outlay. Had he been obliged to put hired help on the type-setting, he could not have done the work at the price he received. Could we have paid him commercial rates so that he could have supplied himself with a larger and more varied assortment of type, he would have been able to turn out work in the way of typographical neatness and style that could not have been excelled by the most elaborately equipped establishment in the country.

Having been assigned a graduate scholarship in the University of Chicago in the summer of 1895, I attended the second summer session, and it was then that I became personally acquainted with Leonard Eugene Dickson. Dickson had been appointed to a University Fellowship and was doing graduate work towards the degree of Doctor of Philosophy. During this and the following summer Mr. Dickson and I had many friendly conversations about the Monthly and its future. I speculated with him at that time as to the possibility of the University's taking it over, thus to insure its permanency. After taking his degree at the University of Chicago, studying in Europe, teaching in the University of California and in the University of Texas, Dr. Dickson was called in 1900 to the University of Chicago as assistant professor of mathematics. On my way home to Springfield in September, 1902, I went through Chicago to call on him and to invite him to join me in the editorship of the Monthly. Not seeing his way clear at the time to give me a definite answer, he withheld his reply until

he could consider the matter more fully. After some meditation he wrote me saying that he would accept the co-editorship with me. The day of his decision was a red-letter day in the history of the Monthly. His official connection with the Monthly began with the October, 1902, number.

During the two or three years previous to 1902, Professor Colaw and J. K. Ellwood were preparing a series of textbooks in arithmetic and geometry and thus Professor Colaw's time was pretty well occupied so that he was able to do very little for the Monthly. Wishing to hold his connection with the Monthly and hoping that later he would be able to give his usual amount of time to it, I had kept his name on the cover as Assistant Editor. In order to consummate my arrangements with Dr. Dickson, Professor Colaw's name was omitted entirely, and his connection was never renewed. Dr. Dickson maintained his official connection with the Monthly until the completion of Volume XIII, 1906, and in an advisory way until the end of Volume XV. This change in the affairs of the Monthly was brought about on account of his increased duties in the University and his desire to devote all of his leisure moments to investigations in a most attractive field of research. During the first year of my graduate study at the University of Pennsylvania, Dr. Dickson was editorially assisted by Dr. Saul Epstein, and the second year by Dr. O. E. Glenn.

During Professor Dickson's connection with the Monthly, the University of Chicago contributed a subsidy of $50.00 per year, thus helping to meet the expense of publication. All cost of publication over receipts from subscriptions and this University subsidy was borne by me personally. No help of any kind was employed and no expenses incurred except those of printing, binding and mailing the Monthly. Mrs. Finkel helped me read practically all of the proof and often addressed all the wrappers herself. For the three summer numbers of Volume II she read the proofs alone. It was some task to proof-read type matter set up by the inexperienced type-setters at Kidder. The first galley sheet was so full of marks that no room was left on the margin to make further corrections. This often necessitated my going to the printing office as many as three times for each galley proof in order to insure comparative freedom from typographical errors.

In severing his official connection with the Monthly, Professor Dickson suggested that his mantle be placed upon the shoulders of the aggressive, indomitable, and persevering Professor H. E. Slaught. This move was strongly supported by Professor E. H. Moore. Both Professor Slaught and I agreed to the suggestion, and thus was inaugurated a second red-letter day in the history of the Monthly. Soon after his connection with the Monthly began, Professor Slaught secured through the influence of Professor Townsend an annual subsidy of $50.00 from the University of Illinois and Professor G. A. Miller as editorial representative.

One of the objects set forth in the founding of the Monthly was that it should reach its readers regularly each month. During the five or six years preceding 1909 I was often obliged to state that an issue was delayed for one reason or

another. The chief reason was that our printers took on more work at times than their office force could handle, and as a consequence the Monthly was laid aside until the rush was over. This happened frequently even though I had a contract that drew a forfeit of five dollars for each day the Monthly was delayed beyond the scheduled mailing date. As the type-setting for the Monthly was done by Mr. Dixon himself, and could not be done by anyone else in his office, any indisposition on his part for any reason whatsoever delayed the type-setting and consequently the mailing of the issue. When the Monthly was delayed thus on account of illness or other unavoidable causes, I was very lenient and did not demand the forfeit. On one occasion on account of the illness of Mr. Dixon I remitted the forfeit. Later in the same year when the Monthly was delayed a week or two I demanded the forfeit. Its payment, however, was refused on the ground that I had violated the contract by having remitted the previous forfeit. I then asked Mr. Dixon if that was the basis on which he was going to transact business and he said it was. I said, "All right, I think we understand each other perfectly." I assured him that there would be no more remitting of forfeits under future contracts and there never was, even though in several cases the forfeit covered the cost of printing the delayed number. Mr. Dixon was a fine gentleman, and he and I transacted all our business on the most friendly basis, never having had any unfriendly words during the whole of the more than seventeen years of business relations. However, the exaction of the forfeits did not improve materially the regularity of the publication of the Monthly, and it became apparent to me that the time was approaching when Mr. Dixon could no longer afford to publish the Monthly. If, then, the Monthly was to continue its existence, some arrangement would have to be made to forestall its discontinuance.

In view of this situation, I had been making some efforts to keep the Monthly permanently at Drury College. Few of the college authorities had very much appreciation of the value to the college of such a publication. Furthermore, their energies were too much consumed in providing means to keep the college a going concern to consider seriously the value of the Monthly. A few may have had the feeling that it was only a mathematical journal which was read by a few cranks here and there, and that it was of no practical value to the college anyway, certainly of no commercial value. But there were some very wise and farsighted men on the Board of Trustees and a few on the faculty who recognized its value far beyond the power of dollars and cents to determine. For example, Dr. Henry Hopkins, whom I have already mentioned, pointed out to his colleagues on the Board of Trustees the immense value of such a publication to the college and, from the time the Monthly took up its abode in Drury College until he became president of Williams College, he was a staunch supporter of the Monthly. It was he who secured for it an annual subsidy from Drury College. Another fine friend on the Board was Dr. Albert Bushnell, pastor of the First Congregational Church in St. Joseph, Missouri. In speaking of the Monthly he compared its value to the college with that of *Popular Astronomy*,

published at Carleton College. Among my colleagues on the faculty was my good friend, Dr. E. M. Shepard, professor of geology and at one time acting president. He was most appreciative of the value of the Monthly as a means of bringing the college to the attention of the educational world, and was active and enthusiastic in advertising it on all suitable occasions.

In order to make a permanent abode for the Monthly in Drury College, I took the matter up at one time with President J. H. George, who lent a very appreciative ear. Dr. George made several suggestions, one of which seemed feasible. Since the college was unable to finance the publication of the Monthly, he suggested the organization of a stock company and offered to take five shares of $100.00 each. He also offered to provide suitable quarters for the printing equipment and office on the college campus free of rent and with heat and light furnished without cost. It was agreed to invite Mr. A. S. Dixon to take charge of the plant, using student labor as far as possible and giving instruction in the art of printing to such students as wished it. In attempting to carry out this plan we encountered several obstacles, the chief of which was that Mr. Dixon could not have used student labor to do commercial printing off the campus without running counter to the labor union. Without being able to take in such commercial jobs, Mr. Dixon could not afford to head the enterprise. After this effort failed I fortified myself against the event of discontinuing the Monthly, in case Mr. Dixon should give it up, by finding another printer in Springfield who agreed to carry on the publication, however, at a considerably higher price.

Thus matters went on until in the summer of 1912 I called on Professor Slaught at his home in Chicago and explained to him the possibility of Mr. Dixon's giving up the publication of the Monthly at any time. In consequence of this conference, Professor Slaught later approached the authorities of the University of Chicago with a view of having them take over the Monthly as one of the regular publications of the University. After making some inquiries of the University of Chicago Press as to the possible cost, the authorities decided this was prohibitive and so declined. Because of the encouragement he had previously received from the University of Illinois, Professor Slaught's next effort was to enlist the cooperation of other institutions. The first response came from Colorado College through the influence of Professor Cajori, and the second came from the University of Missouri through the influence of Professor E. R. Hedrick. Then followed the universities of Minnesota, Nebraska, Kansas, Indiana and Iowa, with Professors Bussey, Brenke, Ashton, Carmichael and Baker as editorial representatives. Beginning with Volume XX the Monthly was published under the auspices of twelve universities and two colleges, each institution contributing toward a subsidy, which was sufficient to meet the deficit in the expense of its publication. Under this arrangement each institution furnished an associate editor, and Professor Slaught acted as editor-in-chief.

This arrangement worked very well, but it still lacked the element of permanence. Professor Slaught next approached the American Mathematical Society, asking the officers to take over the Monthly and publish it under the

auspices of that Society; but the committee to whom he made his appeal in April 1915, while respectful and considerate, turned to him a deaf ear. They did, however, recognize the importance of the mission which the Monthly was trying to fulfill and expressed approval of any steps which might be taken to carry on the work in this field. Determined not to fail in the achievement of his purpose, Professor Slaught conceived the idea of founding a new mathematical organization, the purpose of which should be the establishment of the Monthly on a sound financial basis by having it become the official organ of the new association. To that end he wrote hundreds of letters to professors of mathematics in the colleges and universities of the United States and Canada setting forth his plan. In this way he secured the cooperation of about four hundred men and women agreeing to organize such a society. Professor Slaught proposed that these men and women have a meeting in Columbus, Ohio, in connection with the American Association for the Advancement of Science. This meeting was held, December 30, 1915, and an organization was effected, the name of the organization to be the Mathematical Association of America. The subscription book for charter membership was kept open until April 1, 1916, and at that date the charter membership numbered 1097, including 52 institutional members. The person to whom chief credit is due for attaining this phenomenal charter membership was Professor E. R. Hedrick, first president of the Association, whose optimism and undaunted determination to see this worthy cause succeed abundantly never flagged or hesitated. The third member of what has been facetiously called the Association's Triumvirate was Professor W. D. Cairns, the first and only secretary of the Association, without whom (or his like) the early accomplishments of this young organization could never have been so effectively consummated. From this date to the present time the history of the Monthly has been an open book well read by all its members. I gave the Monthly life, but Professor Slaught has, I hope, given it immortality.

Now what had the Monthly accomplished in the first nineteen years of its existence? This question cannot be answered completely. Incomplete answers are numerous and varied. One answer, and that a fairly accurate one, is that the Monthly had much to do with the organizing and systematizing of the teaching of mathematics in colleges in the United States. Personally, I hope that answer is substantially true. It did another thing. It was instrumental in finding a few young men with fine mathematical ability and putting them in touch with fountains of aid and inspiration. I remember very well about 1905 receiving contributions from a young man in a college in Alabama. His contributions were varied, though more often confined to problems and papers along the line of number theory. These contributions bore the earmarks of originality and I was glad to receive them and publish them as soon as possible. From the value I attached to his contributions, I formed a picture of him similar to the one formed of me by my friend in Kansas City, though as a matter of fact he was only a very young man, twenty-four years old! I was surprised to receive a letter from him inquiring as to opportunities for study in the larger universities

of the country. I wrote him telling him about the scholarships and fellowships offered in our larger universities to those men who showed ability to do research work, and suggested that he make application. I was delighted when later I learned that he was a graduate student in Princeton University. He was destined later to be recognized as one of the foremost authorities in America on the Theory of Numbers. I refer to Professor R. D. Carmichael. While preparing this paper I wrote to Professor Carmichael asking him to give me a statement as to the influence of the American Mathematical Monthly on the development of his mathematical career. In reply I received the following letter from him:

University of Illinois
December 18, 1930

DEAR MR. FINKEL:

It is a pleasure to me to respond to your request for a statement concerning the relation of the American Mathematical Monthly to my early mathematical development. You may feel free to use the whole of this letter or any part of it in any way that you see fit.

You remember, of course, my original interest in mathematics arose at a time in my life when I was entirely isolated from all mathematical centers and had no connection whatever with any one who had any knowledge of mathematics beyond the barest rudiments of the high school course. It was almost through an accident that I learned of the existence of the Mathematical Monthly. I had sent a letter of inquiry to a person, whose name and address had come to my attention, asking some questions about mathematical matters. From the response to this letter I learned of the existence of the American Mathematical Monthly, as well as of one or two other mathematical journals. In response to my request to you, I received a sample copy of the Monthly, and from that date to the present time I have been greatly interested in its progress.

In those early days I received a marked and distinct stimulation from having my problems and solutions and some short articles published in the Monthly. The stimulation came just at that time of my life when I was considering the question whether or not I should turn from other lines of activity into mathematical study and investigation. A stimulation coming at such a time as this I have no doubt left a marked influence on my decisions. When I decided to become a member of the American Mathematical Society, I remember distinctly that I had never yet had the opportunity of meeting with any one who was in any sense a mathematician. In response to my request you kindly agreed to sponsor my application for membership, and with your signature and that of a mathematician of the University of Alabama I was admitted to the American Mathematical Society. When I desired some information later concerning the places for mathematical study, I turned to you and to a very few other people, who in the meantime I had come to know by correspondence, for advice concerning the steps to take.

It is difficult to assess the influence which was exerted upon my decisions by the stimulus and encouragement of those early days, but I believe that it is probably true that they had no small part in my final decision to undertake a serious study of mathematics; and that decision I have never regretted.

The following is perhaps the main thing which I wish to say in this connection: To Professor Benjamin F. Finkel I would extend an expression of my gratitude, with a hearty appreciation of the stimulus which I received from him and through him. Though I have seen him infrequently in the flesh, and have not any time been closely associated with him, I have, nevertheless, for years looked upon him as a sort of father in the spirit.

Very truly yours,
(*Signed*) R. D. Carmichael

This letter alone is worth more to me than all the labor and the sacrifice the Monthly cost me, and I here acknowledge my deep appreciation of Professor Carmichael's expression of gratitude. Space forbids my mentioning others in this connection.

In the prosecution of my work as editor, I have been both blamed and praised. In editing a journal, as in other affairs of life, one cannot please everybody. It was my aim to deal generously and justly with our contributors, keeping in mind that truth and accuracy should not be sacrificed under any circumstances. My correspondence with circle-squarers, angle-trisectors, cube-duplicators, and Fermat's Last Problem solvers was voluminous, varied and interesting, and if compiled would add an interesting chapter to De Morgan's *Budget of Paradoxes*. I regret that I did not keep all my correspondence in this connection. I shall give two samples:

> A few years ago I received in pamphlet form a demonstration of the trisection of an angle. The author informed me that if I could show him that his demonstration was wrong he would send me a turkey for my Christmas dinner. I pointed out the weak spot in this demonstration, but I never received the turkey, since the author was judge, advocate, and jury in the matter.
>
> Another angle-trisector in 1927 wrote me as follows: "I have solved the problem of trisecting a rectilinear angle, of trisecting the arc of a circle, . . . This despite the theories and quack formulae advanced by half-baked mathematicians in an attempt to prove the possible impossible." He stated that he had referred his construction to professors at Northwestern University and the University of Michigan, one of whom admitted that he found no flaw in the construction and another passed the buck by suggesting that the construction be sent to me. This trisector goes on to say that "the skepticism that exists as to the possibility of trisecting a rectilinear angle is largely due to the number of people who have submitted what they thought to be solutions, but which proved not to be. To a person who has any real knowledge of mathematics this seems strange, but human experience notes that in most mathematics classes there is not even one mathematician, and that in a class in which there is *one*, there are generally one, two or three others. But those who are not mathematicians 'get by' and some of them become teachers of mathematics." He then asks the question, "But is it not assuming a great deal to claim that a rectilinear angle, or the arc of a circle, cannot be trisected?" As this angle-trisector had all his drawings copyrighted and was considering propositions from publishing houses, he said he would be willing to loan them to me on my agreeing not to publish them. I let the whole matter drop at this stage.

If the Monthly did not accomplish all that it should have done during the first nineteen years of its existence, Goldschmid's criticism of a noted painting is apropos: "The painting could have been improved if the painter had taken more pains." In the case of the Monthly, be it remembered that the work on it was my avocation. During most of the time while I was conducting the publication of the Monthly I was teaching from nineteen to twenty-seven hours per week; for the first seven years in Drury College I was secretary of the faculty and registrar of the college; I served on various committees; was director of summer sessions for three years; and was nominally librarian of the college for ten or twelve years. No consideration was ever given me by the college because of the work the Monthly entailed and I never asked for any such consideration.

I acted on the theory of David Harum's philosophy that "the fleas on a dog are good for him for they keep him from musing on the fact that he is a dog." So in my case, the extra loads I was carrying kept me from realizing that I was carrying extra burdens. My apology is for the incompleteness and imperfections of my accomplishments

In bringing to a close this history of the American Mathematical Monthly, covering the period of the publication of the first nineteen volumes, it should be said that whatever success it had, and whatever influence it may have exercised in the teaching and advancement of mathematics in the United States, are attributable to all the agencies contributing to its growth. Editors, contributors and printers all shared in the early history of its development.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

CALCULATION OF NUMERICAL ROOTS

By IRWIN ROMAN, West Orange, New Jersey

The problem of finding a desired root of a number is usually avoided, or solved by logarithms, except in the case of square root, for which two convenient methods are available. In this special case, the commercial computing machine manufacturers suggest a method which involves successive subtraction of odd integers. The division method has been known and used for many years and is preferred by some persons. For cube root, the method indicated by the machine manufacturers is awkward, while the division method is extended with only minor changes.

The present note makes the extension of the division method to all integral and some fractional roots. The method involves successive approximation, starting with an estimate. The closer the estimate, the more rapid the convergence to the correct value and the more convenient the method. A table of approximate values for the root sought is valuable, and should be prepared in advance when many cases are to be computed.

Let it be desired to find the n-th root of P, i.e. to find

$$a = P^{1/n}. \tag{1}$$

Let D be an estimated value for a. If the correction to D is e, it is apparent that

$$D = a - e. \tag{2}$$

Let

$$Q = P/D^{n-1}. \tag{3}$$

Then

$$Q = P/(a - e)^{n-1} = a[1 - (1 - n)e/a + (e/a)^2\phi],$$

where ϕ is finite. If the estimate D is reasonably close to a, e/a is small and we may write

$$Q = a + (n - 1)e = a + (n - 1)(a - D) = na - (n - 1)D; \tag{4}$$

whence

$$a = [Q + (n - 1)D]/n. \tag{5}$$

The value of a obtained by (5) is usually a better approximation than D. The method is repeated using the calculated value of a as the new value of D, as often as may be necessary. The method is applicable whenever the $(n-1)$ th power of D is available.

For the square root, this reduces to the usual division rule:

Step 1) Estimate the desired root.

Step 2) Divide the number by the estimate.

Step 3) The average of the estimate and the quotient is a better approximation.

Step 4) Repeat the preceding steps until estimate and quotient agree sufficiently. With a computing machine, the method is rapid and simple.

To illustrate the method in unusual cases, consider two cases, $n=5$ and $n=3/2$. The method is more complicated in the calculations, but the rule is the same as for the square root, except for steps 2 and 3, which become:

Step 2′) Divide the number by the $(n-1)$th power of the estimate.

Step 3′) The weighted mean of the estimate, weight $(n-1)$, and the quotient, weight unity, is a better approximation.

In the case of $n=3/2$, it is obviously possible to calculate the cube root of the square, but the calculation of the cube root is of the same order of difficulty as that of the two thirds power directly. In the case of the fifth root, no other simple method has come to the attention of the writer.

Example 1. Find $(1728.526)^{1/5}$.

Here, $n=5$, $P=1728.526$, $Q=P/D^4$ and $a=(Q+4D)/5$. If we estimate the fifth root as 4, initially, we have the calculations of Table 1.

Table 1

$(1728.526)^{1/5}$		
D	D^4	Q
4.	256.	6.75
4.55	428.593	4.033 024
4.446 605	390.943 689	4.421 419
4.441 568	389.175 282	4.441 510
4.441 556	389.171 100	4.441 557

Thus the fifth root of 1728.526 is 4.441 556, correct to the last figure.

Example 2. Find $(1728.526)^{2/3}$.

Here $n=3/2$, $P=1728.526$, $Q=P/D^{1/2}$ and $a=(2Q+D)/3$. If we start with the obviously poor estimate, $D=100$, we have the calculations of Table 2.

Table 2

$(1728.526)^{2/3}$		
D	$D^{1/2}$	Q
100.	10.	172.852 600
148.568 400	12.188 864	141.811 903
144.064 069	12.002 669	144.011 803
144.029 225	12.001 218	144.029 214
144.029 218	12.001 217	144.029 226

Thus $(1728.526)^{2/3}$ is 144.029 22, correct to the last figure.

On the Characteristic Equations of Products of Square Matrices

By H. S. Thurston, University of Alabama

A few months ago, while "playing" with square matrices, the writer noted a very interesting property of the products of three matrices of the same order, namely, that associated with the six products formed from the matrices a, b, and c, were two distinct characteristic equations, one common to the products abc, bca, and cab, and the other to the products acb, bac, and cba. Further investigation led to the proof of a general theorem, and although no claim is made that the theorem is new, it was felt that it might be of sufficient interest to readers of the *Monthly* to merit a short note.

On page 283 of Bôcher's *Introduction to Higher Algebra*, we find this theorem:

If a_1 and a_2 are two matrices independent of λ, a necessary and sufficient condition that a non-singular matrix p exist such that

$$a_2 = pa_1p^{-1}$$

is that the characteristic matrices A_1 and A_2 of a_1 and a_2 have the same invariant factors—or, if we prefer, the same elementary divisors. For our purposes, we do not require both the necessity and the sufficiency of the condition, and may modify the enunciation of the theorem to read as follows: If there exists a non-singular matrix p such that

$$a_2 = pa_1p^{-1}$$

then a_1 and a_2 have the same characteristic equation.

Now if a and b are non-singular square matrices of the same order, a non-singular matrix p can be found such that

$$ba = pabp^{-1},$$

for in this case p may be taken as b. It follows from the above theorem that the products ab and ba have the same characteristic equation.

Next, let a or b, say b, be singular. Since a determinant is a continuous function of its elements, one element of b may be increased by an infinitesimal ϵ, thus obtaining a non-singular matrix b_1, and the characteristic equation of ab_1 is the same as that of b_1a. As ϵ approaches zero, the characteristic equations of ab_1 and b_1a remain equal, so that we may conclude that ab and ba have the same characteristic equation. Should both a and b be singular, a similar argument may be made.

Finally, consider three matrices a, b, and c, singular or non-singular. It follows from the above argument that $a(bc)$ and $(bc)a$ have identical characteristic equations and the same is true of $b(ca)$ and $(ca)b$, and since multiplication of matrices is associative, it is seen that abc, bca, and cab have a common characteristic equation. This argument may obviously be extended to include the case of any finite number of square matrices, and we are led to the following:

Theorem. *If $a, b, c, \ldots q$ are square matrices of the same order, the product $abc \ldots q$ and all products formed by cyclicly permuting its factors, will have the same characteristic equation.*

As an immediate consequence of this theorem, we may state a corollary: *If $a, b, c, \ldots q$ are square matrices, n in number, then associated with the $n!$ products formed by permuting the factors of the product $abc \ldots q$, there are, in general, $(n-1)!$ characteristic equations.* The phrase "in general" is used advisedly, for should any of the given matrices be equal, or should multiplication be commutative with respect to any of them, there will be fewer than $(n-1)!$ distinct equations.

A very simple illustration of the above theorem follows. Using as factors the three matrices

$$a = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix},$$

there may be formed six products. Three of these, namely,

$$abc = \begin{pmatrix} -1 & -4 \\ -1 & 0 \end{pmatrix}, \quad bca = \begin{pmatrix} 2 & 1 \\ -2 & -3 \end{pmatrix}, \quad cab = \begin{pmatrix} 3 & 2 \\ -4 & -4 \end{pmatrix},$$

have a common characteristic equation $\lambda^2+\lambda-4=0$, while the others,

$$acb = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}, \quad bac = \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}, \quad cba = \begin{pmatrix} -2 & -7 \\ 0 & 2 \end{pmatrix},$$

have in common the characteristic equation $\lambda^2-4=0$.

A Well Known Theorem of Integral Calculus

By H. L. Krall and J. D. Tamarkin, Brown University

The purpose of this note is to prove the inequality

$$(1) \qquad \left| \int_a^b f(t)dt \right| \leqq \int_a^b | f(t) | \, dt, \quad b > a,$$

for the case in which $f(t)$ is integrable (Lebesgue) on (a, b) and is a complex valued function of the real variable t, i.e. $f(t)=u(t)+i\,v(t)$ where $u(t)$ and $v(t)$ are real and integrable. This simple truth, which is obvious in the case of a real valued function or of a Riemann integrable function is widely used but the proof does not appear to be given explicitly in standard treatises. We can divide the proof into three steps.

(1) *Approximation of $f(t)$ by a continuous function.* By a theorem[1] in the theory of functions of a real variable, we can approximate $u(t)$ by a continuous function $\phi(t)$ and $v(t)$ by another continuous function $\psi(t)$ so that for any prescribed $\epsilon>0$

$$\int | u - \phi | < \epsilon \quad \int | v - \psi | < \epsilon.$$

(Since the limits of integration (a, b) and the differential dt are the same throughout, we shall drop them in order to simplify the notation.)

If we use a sequence of ϵ's, $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots \to 0$ and corresponding sequences $\phi_1, \phi_2, \cdots \phi_n, \cdots, \psi_1, \psi_2, \cdots \psi_n, \cdots$, then these inequalities can be written in the form

$$\lim_{n\to\infty} \int | u - \phi_n | = 0, \quad \lim_{n\to\infty} \int | v - \psi_n | = 0,$$

and

$$\int \phi_n \to \int u \quad \int \psi_n \to \int v.$$

[1] See Hobson's *Theory of Functions of a Real Variable*, vol. 1, 3rd edition, p. 632.

Let

$$\omega_n(t) = \phi_n(t) + i\psi_n(t);$$

then

$$\lim_{n\to\infty}\left|\int f - \int \omega_n\right| = \lim_{n\to\infty}\left|\left(\int u - \int \phi_n\right) + i\left(\int v - \int \psi_n\right)\right| = 0.$$

(2) An approximating function for $|f(t)|$ is the absolute value of the function which approximates $f(t)$. This follows from the obvious inequality

$$\big||f| - |\omega_n|\big| \leqq |f - \omega_n|,$$

whence

$$\left|\int |f| - \int |\omega_n|\right| \leqq \int \big||f| - |\omega_n|\big| \leqq \int |f - \omega_n|$$

$$\leqq \int |u - \phi_n| + \int |v - \psi_n| \to 0.$$

(3) *Proof of* (1): We now have

$$\left|\int f\right| = \left|\lim_{n\to\infty}\int \omega_n\right| = \lim_{n=\infty}\left|\int \omega_n\right|.$$

But, ω_n being a continuous function, $\int\omega_n$ and $\int|\omega_n|$ coincide with the Riemann integrals of the respective functions so that

$$\left|\int \omega_n\right| \leqq \int |\omega_n| \to \int |f|.$$

Hence we have

$$\left|\int f\right| \leqq \lim_{n\to\infty}\int |\omega_n| = \int |f|.$$

A Note by the Editor

An alternative proof using only elementary properties of Lebesgue integrals might be as follows: Put

$$L = \left|\int_a^b f(t)dt\right| \text{ and } R = \int_a^b |f(t)|\,dt. \text{ Let } f(t) = u(t) + i\,v(t) \text{ and}$$

$$A + iB = \int_a^b f(t)dt = \int_a^b [u(t) + i\,v(t)]dt.$$

Then

$$A - iB = \int_a^b [u(s) - i\,v(s)]ds$$

and

$$L^2 = A^2 + B^2 = \int_a^b [u(t) + i\,v(t)]dt \int_a^b [u(s) - i\,v(s)]ds$$

$$= \int_a^b dt \int_a^b \{[u(t)u(s) + v(t)v(s)] + i[v(t)u(s) - v(s)u(t)]\}ds$$

$$= \int_a^b dt \int_a^b [u(t)u(s) + v(t)v(s)]ds,$$

since the coefficient of i on the right must vanish. Now

$$R = \int_a^b |f(t)|\,dt = \int_a^b \sqrt{u^2(t) + v^2(t)}dt$$

and

$$R^2 = \int_a^b \sqrt{u^2(t) + v^2(t)}dt \int_a^b \sqrt{u^2(s) + v^2(s)}ds$$

$$= \int_a^b dt \int_a^b \sqrt{[u^2(t) + v^2(t)][u^2(s) + v^2(s)]}ds$$

$$= \int_a^b dt \int_a^b \sqrt{[u(t)u(s) + v(t)v(s)]^2 + [u(t)v(s) - v(t)u(s)]^2}ds.$$

Hence

$$R^2 \geqq \int_a^b dt \int_a^b |u(t)u(s) + v(t)v(s)|\,ds \geqq \int_a^b dt \int_a^b [u(t)u(s) + v(t)v(s)]ds,$$

$R^2 \geqq L^2$ and $R \geqq L$.

R.E.G.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Elements of Analytic Geometry. By Clyde E. Love. New York, Macmillan, 1931. xii+150 pages. $1.60.

Elementary Theory of Tensors. By T. Y. Thomas. New York, McGraw-Hill, 1931. x+122 pages. $2.50.

Probabilités des Statistiques. By R. de Montessus de Ballore. Paris, Hermann et Cie., 1931. x+212 pages. 60 fr.

Plane Trigonometry with Tables. By W. W. Burton. New York, T. Y. Crowell Co., 1931. x+126 pages; tables, viii+94 pages.

Komplexe Reihen. By Hans Falckenberg. Sämmlung Göschen 1027. Berlin, De Gruyter, 1931. 140 pages.

The Theory of Ruled Surfaces. By W. L. Edge. Cambridge, The University Press, 1931. x+324 pages. $7.00.

The Theory of the Volterra Integral Equation of the Second Kind. By H. T. Davis. Indiana University Studies, 88, 89, 90. Bloomington, Indiana, 1931. 76 pages. Paper, $1.00, cloth $1.25.

Advanced Calculus. By George A. Gibson. Macmillan, London, 1931. xviii+510 pages. $6.50.

Solid Geometry. By W. H. Macaulay. Cambridge, The University Press; American agents, Macmillan, New York, 1931. xii+304 pages. $4.75.

Il passato e il presente delle principali teorie geometriche, storia e bibliografica. By Gino Loria. 4to ed., totalmente rifatta. A. Milani, Padova, 1931. xxiv+468 pages.

Mathematics for Junior High School Teachers. By W. L. Schaaf. Richmond, Johnson Publishing Company, 1931. xiv+440 pages.

La Géométrie. By Lucien Godeaux. Paris, Hermann et Cie., 1931. 182 pages. Paper, 15 fr.

Vorlesungen über allgemeine naturliche Geometrie und Liesche Transformationsgruppen. By Gerhard Kowalewski. Goschens Lehrbucherei, Reine und angewandte Mathematik, 19. Berlin, De Gruyter, 1931. 280 pages. RM 15.50; bound RM 17.

Electricity and Magnetism, The Mathematical Theory. By Vincent C. Poor. New York, John Wiley, 1931. x+184 pages. $2.25.

Tutorial Exercises in Trigonometry. By Raymond W. Brink and Ella Thorp. New York, The Century Company, 1931. 104 pages.

REVIEWS

Fouriersche Reihen. By Dr. Werner Rogosinsky. Sammlung Göschen, Berlin-Leipzig, No 1022, 1930. 135 pages.

The author succeeded in collecting interesting and useful material in this little book. The reader will find there, together with indispensable classical results, many facts of the modern theory of Fourier series, including some results due to the author.

The book contains seven chapters followed by an appendix. Chapter 1 serves as an introduction. Chapter 2 is devoted to the "theory of representation" (Darstellungstheorie). The aim of this chapter, which is not quite clear from the title, is to characterize extended classes of functions that are represented everywhere by their Fourier series expansions. The Riemann-Lebesgue theorem concerning the Fourier coefficients and the Lebesgue theorem concerning the term-

wise integrability of a Fourier series of an absolutely integrable function are derived and applied to the expansion theorem of a piece-wise differentiable function. The theory is illustrated by important examples (Bernoulli polynomials, vibrating strings, Weierstrass approximation theorem). Chapter 3 deals with completeness properties of trigonometric functions, together with Parseval's identity and an application to the solution of the isoperimetric problem for the circle. Chapter 4 treats of the convergence theory of Fourier series. We find there Riemann's localization theorem, criteria of Dini and Dirichlet-Jordan, an estimate (due to Hardy) of the n-th partial sum of a Fourier series, some properties of Lebesgue's constants. The chapter closes with an exposition of some properties of the conjugate of a Fourier series. Chapter 5 contains some important applications: Fourier integral theorem, Euler-Maclaurin summation formula, Stirling series. In Chapter 6 the reader will find interesting material concerning the summability of divergent Fourier series. After a short sketch of the general Toeplitz method of summation of divergent series, the methods of Fejér, of Abel-Poisson, of Riemann-Lebesgue are given. The end of the chapter is devoted to a careful exposition of the Gibbs phenomenon and to the related theory of the "Abschnittskoppelung" of Fourier series, which was originated and successfully pursued by the author. In Chapter 7 the uniqueness theorem for general trigonometric series is treated. Theorems of Cantor and Du Bois-Reymond are proved. The Appendix contains proofs of some necessary facts of the theory of functions of a real variable.

The exposition throughout the book is elegant and clear. The reviewer was not able to find any slips and but very few misprints, except on the last page 135, where the theorems on term-wise differentiation of sequences and series are stated in a too general form and hardly can be considered as correct. One general remark should be made in conclusion: the author avoids the notion of Lebesgue integral and manages to get along by an adroit manipulation with improper absolutely convergent Riemann integrals. This should be considered as a weak point in the book. We have now reached the stage where the notion of Lebesgue integral should belong to the elements, since it is simpler, more general and more useful than that of Riemann. At the cost of a couple of additional pages in the Appendix the author would enable himself to avoid some clumsy passages and to include some additional important results, without which no adequate idea of the modern theory of Fourier series is possible.

J. TAMARKIN

The Zeta-Function of Riemann. By E. C. Titchmarsh, M. A. No. 26 of the Cambridge Tracts. London, Cambridge University Press, 1930; New York, The Macmillan Co. 104 pages. Price $2.25.

The Riemann zeta-function characterized by Gram as "une des plus remarquables acquisitions de l'analyse moderne" is defined as a function of the complex variable $s=\sigma+ti$ by the Dirichlet series $\zeta(s)=\Sigma n^{-s}$, $n=1, 2, \cdots$. It was first studied by Riemann in a paper on prime numbers published in 1859 that "has

become famous for the number of ideas it contains which have since proved fruitful and it is by no means certain that its riches are even now exhausted" (p. 43 of the tract before us).

The zeta-function has a twofold interest, due to the difficult function theory problems it raises as well as to its prominent role in the analytic theory of numbers.

The valuable tract by Professor Titchmarsh presupposes some knowledge of $\zeta(s)$ from both of these points of view and (except for a brief sketch of elementary properties) confines its limited space to the present form of the theory as developed during the last twenty years. In fact, all of the one hundred memoirs listed in the bibliography were published since 1910 and the majority of them from 1920 on.

The main objective of the theory cannot be better stated than in the first lines of Chapter I: "The fundamental problem which emerges from the attempt to determine the distribution of the prime numbers is: Where are the zeros of the zeta-function? We also encounter the problem of the asymptotic behaviour of $\zeta(s)$ as $t\rightarrow\infty$, for given values of σ. The two problems are closely connected."

The present tract will be very welcome to workers in this field and even the general reader will find interest in looking over the theorems and reading the illuminating comments scattered through the text. Professor Titchmarsh has very ably unified and reduced to a moderate space a large amount of difficult material and given it a clear and systematic development. The titles of the six chapters are: The asymptotic behaviour of $\zeta(s)$; Mean value theorems; The distribution of the zeros; The general distribution of the values of $\zeta(s)$; Consequences of the Riemann hypothesis; Lindelöf's hypothesis.

J. I. HUTCHINSON

Mathematische Volkswirtschaftslehre. By Otto Weinberger. Leipzig und Berlin, 1930. xiv+241 pages.

In this interesting book problems of political economy are considered exclusively from the mathematical view-point. As a result all questions which do not show any point of contact with the mathematical method, such as the problems of the character of population, the historical antecedents of a given economic order and so forth, are omitted. The mathematical prerequisites for intelligently reading this book are analytical geometry and calculus. Knowledge of fundamental economic concepts is presupposed, and, as a result, the book is not suited for a first reading in economics by mathematicians. It will, however, serve as a splendid book to follow the reading of G. C. Evans's book, *Mathematical Introduction to Economics*. Historical settings and references are given which are not included in Evans' book.

The first chapter explains the quantitative nature of fundamental economic concepts and in some detail considers the difficulties which proponents of the mathematical method have had to overcome. Irving Fisher is repeatedly mentioned as a champion of the mathematical method. The second chapter, pages

31–118, is a history of contributions to mathematical economics. Here we find references to the research works of Daniel and Jacob Bernoulli, Poisson, Laplace, Gauss, Edgeworth, Evans, Roos, Schultz, Cournot, Pareto, Amoroso, Irving Fisher, Walras, Karl Menger, Schumpeter, Böhm-Bawerck, H. L. Moore, Weinberger, Auspitz, Lieben, A. Marshall, Gossenn, Thünen and others. In the third chapter the author briefly considers some of the principal problems of mathematical economics, such as, the statical problems of utility and value, price, monopoly, wages, rent, production and capitalization, and the dynamical problems of demand, production, profit and speculation. The fourth and concluding chapter is in the nature of a statistical appendix.

As a reference book this work fills an important gap in economic literature. Several other books give extensive bibliographies and some cite a few references in foot-notes, but none give the settings of problems and their associated bibliographies nearly so well.

CHARLES F. ROOS

Lebendige Mathematik. By Felix Auerbach. Ferdinand Hirt, Breslau, 1929. 355 pages. Price ca $2.50.

Of the many strange things called forth by the mathematical genius of the nineteenth century none was more baffling to its discoverers than the idea that the validity of the stately structure of mathematics does not depend upon the "self-evidence" of the basic axioms of the science. To their further great amazement mathematicians arrived at the conclusion that they are free to give any meaning they please to the very entities their science is to deal with, if only these entities satisfy the basic truths the mathematicians have promulgated as such. Small wonder that having, so unexpectedly, found himself clothed with such dictatorial power, the mathematician lost all sense of reality.

Postulational mathematics is undoubtedly one of the great discoveries of the human mind. The insight it has given us into the nature of mathematics, and of logical reasoning in general, is just as profound as it is unexpected. But the trifling attitude towards the connection between mathematics and the world around us, which became popular in some quarters on account of this great discovery, is both unwarranted and harmful.

Signs are not wanting that this attitude is on the wane. The book under review is one of them. The author undertakes to show the reader how deeply mathematics penetrates into our daily life, into all industrial activities, as well as into all our preoccupations with philosophy and art, in the broader sense of these terms. But this is only one side of his amibition. He has another, and a far greater one. He wants to show the lay reader that the truths mathematics arrives at are fascinating; that the objects of mathematical speculation, that these very speculations, far from being as abstruse and as repulsive as they are reputed to be, are full of interest, and that they can and should be a part of the mental equipment of any educated person who is willing to make the effort necessary to grasp them.

The book covers both elementary and higher mathematics. It contains no proofs. It begins with geometry, then takes up the theory of numbers, and ends with analysis. It is written in an easy style and the whole thing is admirably well done. Any layman who will read it through will have learned a great deal of mathematics. But the book should be read by professional mathematicians as well, especially teachers of mathematics. They will find there interesting methods of approach to various branches of mathematics, suggestive connections between the branches, and valuable hints as to the practical applications of the various mathematical theories.

The author missed an excellent opportunity to give some historical background to the subject matter he so skillfully deals with. But perhaps this would have made the book too large.

It is rather humiliating that mathematicians had to wait until a physicist did this work for them.

NATHAN ALTSHILLER-COURT

Geschichte der Elementar-Mathematik. Erster Band, Rechnen. By Johannes Tropfke. 3. Auflage. Berlin and Leipzig, 1930. vii+222 pages. Price 12 RM.; bound, 13.20 RM.

Dr. Tropfke's series of histories, in seven volumes, begun nearly thirty years ago, is so well known in the mathematical world as to need little comment with respect to the general purpose and method. The plan of treating in separate volumes the large topics of (1) Rechnen, (2) Allgemeine Arithmetik, (3) Proportionen, Gleichungen, (4) Geometrie, (5) Trigonometrie, (6) Analysis, and (7) Stereometrie, has much to commend it, and students find the arrangement more convenient than that of Cantor.

In this revision the general arrangement of topics is as in the first edition. There is no index, this being left for the seventh volume. The "Inhalt" answers the purposes fairly well, but much use of the work has led this reviewer to regret that an index for each volume has not been provided. Nevertheless this is a small matter compared with the general excellencies of the text. The revision under review is upwards of 25% larger than the first edition and has brought the bibliography, notes, and necessary changes of text up to the year 1929. Since the notes form one of the most valuable features of the work, it should be mentioned that the number has been increased from 874 to 1343, a gain of more than 50% over the earlier editions.

The chief value of the revision, however, is in the text itself, for this has been greatly extended. There have been included items of much importance the absence of which was very noticeable in the first edition. For example, the section on numerals (*Die Ziffern*) has been rewritten so as to include some of the latest information relating to the Babylonian, Egyptian, Greek, Mayan, Hindu, and medieval forms, and the table showing the development of our common system has been enlarged to include 44 instead of 15 different stages, beginning with the Aśoka inscriptions and ending with Widman's work of 1489. Similarly

a much-needed summary of recent developments in sexagesimal fractions has been made under the caption of angle measure (*Die Winkelmasse*), although it brings us no nearer the solution of the real question of origin than we have been since classical times. We know that the Sumerians used 60 for various purposes, but why they used it, or why 360° was later used for the circle is still a matter of speculation. Knowing the difficulties that the ancients had with fractions and the tendency to avoid them by the use of submultiples (as with the inch and foot), it is not unreasonable to think that 120 was taken as the diameter of a circle because of its list of factors (2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60), whence 60 was the radius and 3×60 (using the common value of π in ancient times) was the circumference. At any rate, this seems a rather better guess than the others.

The treatment of computation has been improved in various ways, notably by the inclusion of matter from Peet's work on the Rhind Papyrus (the more complete Chace edition not being available before the copy was sent to press), and by the improvement of Müller's Greek vocabulary of terms used. The section on the "Eigenschaften der ganzen Zahlen" has been improved by additions relating to cardinal and ordinal numbers, to squares and non-squares, and certain recent discoveries by Dickson and others. Considerable new material has been added with respect to the history of common fractions, especially that relating to the Egyptian and Arabic fields.

Assuming that the revisions of the other volumes will be of merit commensurate with that of the first one, the work will continue to stand as the most helpful one of its kind, for beginners in the study of the history of mathematics, that we have. The particular value lies not in its style or in the grasp of the large problems so much as in its mass of detail and its very extensive bibliography. It is a work that every student of the history of mathematics should own and annotate.

Having spoken of its general value, it is proper to refer to two other features—the question of the treatment of proper names and of titles, and the too-frequent misprints and lack of care in the notes.

As to the spelling of proper names in general, Dr. Tropfke seems to incline to use the modern German form, although in this he is not at all uniform nor is he consistent in certain specific names. For example, the German form appears in such names as Adrian for Adriaen (Metius), properly using the Latin form as well. The name *Widman* (pp. 38, 41) appears more frequently as *Widmann*, although the former seems to have been the preferred one at the time the man lived. It is, of course, of no great moment that the form Koebel should be used (p. 12 et pass.) instead of Köbel. In the various copies of his books with which I am familiar, however, either the two dots or the small *e* is printed over the *o*. What is more serious, however, at least from the bibliographic standpoint, is that the title of Köbel's work of 1514 is incorrectly copied. Instead of *Ain Newgeordnet Rechen biechlin mit den zyffern*, Augsburg, 1514, quoted from F. Müller in note 57, the title should read . . . *Rechen biechlin auf den linien mit Rechen*

pfennigen (see facsimile in *Rara Arithmetica*, p. 103), unless it should turn out that there were two impressions the same year at Augsburg (not Oppenheim, see note 93).

As to the combination of languages in a proper name, it is difficult for anyone to be consistent. For example, shall we who speak English say Leonardo of Pisa, Leonard of Pisa, Leonardo Pisano, Leonardo Fibonacci, or Leonardus Filius Bonacii, or shall we use some other form? Dr. Tropfke says Leonardo (which is Italian) von (which is German) Pisa (which is Italian but is generally international except for the French Pise). It would seem as if the most sensible plan would be to spell a man's name as he himself generally spelled it in the vernacular which he generally used. In the case of Latin names of the Renaissance, it would seem reasonable to use these if they are the forms by which the man is at present generally known, which rule of course introduces an element of uncertainty, as in cases like Viète and Vieta. As to a name like Newton's, it hardly seems justifiable to use the form J. Newton instead of I. Newton, even though a German might write Jsaac instead of Isaac. (See p. 90 et pass.)

As to the oriental names, it is very desirable to have some kind of international authority. To one who has given the matter no attention it is confusing to see one author use Alḫwārizmī and another use Al-Khowârizmî. Each has authority for his transliteration, but the lack of uniformity is confusing. Probably Suter's list is the most widely used of any.

As to the titles of books and the names connected with them, a number of questions arise. First, why in copying a title should anyone change the spelling and the diacritical marks unless this is necessary? For example, why should Dr. Tropfke print *Sūma* as *Summa*, and & as *et* in the title of Pacioli's work? To be sure they mean the same, but in general he rightly uses precise forms. In the same way, if Piero Borgi wished to spell his name thus in his book which he called *arithmethica* in 1484, or Piero or Pietro Borghi later, why should we spell the name Piero Borghi and the title *arithmetica* when quoting the first edition? Why should anyone use the form *Liber Abbaci* (p. 13) in speaking of Fibonacci's work? The MS used by Boncompagni reads "Incipit liber Abaci Compositus a leonardo filio Bonacij Pisano In Anno M°cc°ij°," and the fact that he and others used the Italian plan of doubling the *b* (*abbaco*) is no good reason for changing from the first edition and using poor Latin in this one, when Leonardo himself used the correct form. In the matter of titles, too, those who are working in the field of bibliography will regret to see defects in those which assume to be correctly transcribed, as in the case of Tartaglia's *General trattato nvmeri et misvre* instead of ...*di nvmeri, et misvre*. A similar situation arises with respect to Colebrooke's work (note III) in which there are six errors in copying the title. Of course these are of little moment to one casually consulting the work, but to a bibliographer they seem at least undesirable. This tendency to inaccuracy in titles is often apparent, a single further illustration appearing in respect to Boncompagni's *Trattati d'arithmetica* (in note 35) and *Trattati d'Aritmetica* (in note 389), and to the precise form of the title of Stevin's work, *L'Arithmetiqve* (all capitals and no diacritical mark over *e*, and *v* for *u*; compare note 498).

Since this reviewer is quoted (n. 54) in the assertion that the Treviso arithmetic was the first to appear in print, it is proper that he should say that the editors of the new Gesamt-Katalog have found that it had one or more predecessors, and that the statement will need to be changed in our histories to read "the first printed arithmetic with date."

As to the matter of misprints and lack of uniformity, these features are of little importance in a review. The reader will easily find them. It must be said, however, that the revision is itself in need of revision, even though the points at issue are of little consequence. It detracts from the pleasure in consulting the book to find at least four different ways of referring to the *American Mathematical Monthly* (pp. 7, 22, 38, 73), and at least three in the case of the *Zeitschrift für Mathematik und Physik* (pp. 12, 36, 78). The proof could not have been read with due care when it has such misprints as Canton for Cantor (note 95) and Heiberb for Heiberg (note 543), and numerous others of the same kind.

DAVID EUGENE SMITH

Mechanics for Students of Physics and Engineering. By H. Crew and K. K. Smith, The Macmillan Co., New York, 1930. xvi+371 pages.

This excellent text had a predecessor "Principles of Mechanics" written twenty-five years ago by Professor Crew. We must acknowledge, however, that the present volume is sufficiently different in its treatment to be considered as an entirely new work. It is well adapted both to give the engineering undergraduate a working knowledge of mechanics, and to prepare the future physicist with a good grounding in the fundamentals of the subject. Its method of presentation carries over the secrets of original search in that the subject matter is given in the order of its historical development, with adequate reference to the efforts of the first investigators, and some suggestion as to the genesis of the problems considered. The abundance of historical material is to be welcomed not merely for its cultural value to the average student, but for its depicting mechanics, to the possible specialist among the undergraduates, as a product of human effort.

The early chapters of the book, dealing with the statics of a particle and of a rigid body, virtual work, and the equilibrium of a rigid body, are developed in an elementary fashion with no appeal to the calculus. This is intended to permit the simultaneous study of calculus and mechanics. Vector methods are, however, introduced at the outset, and further developed as need arises. The vector point of view is held to consistently, and the advantages accruing therefrom are especially evident in the chapter on kinematics and kinetics.

There is an interesting short chapter on the units of mechanics where the notion of dimensional analysis is touched upon. Among other topics treated are properties of elastic bodies, hydromechanics, and wave motion, which serve to bring the student in contact with the newer physics.

Throughout the text application of the theoretical principles to concrete problems is illustrated by well pointed examples. There is a large choice of exer-

cises to tax the ingenuity and industry of the serious student. In closing comment, Professors Crew and Smith are to be credited with presenting to the younger student of mechanics a treatment of the subject characterized by its clarity and simplicity, though sophisticated in its scientific point of view.

S. B. LITTAUER

Plane and Spherical Trigonometry. By Leonard M. Passano. Revised edition. The Macmillan Company. New York, 1930. XVIII+155 pages.

This revision of this well-known text presents a more open faced page due to the use of a slightly different font of type, a more numerous set of new problems, and a careful rephrasing of portions of some paragraphs. This, together with one redrawn figure, results in fewer infelicities and ought to augur well for the continued usefulness of the book, which is supplied either with or without tables.

C. F. CRAIG

A LETTER FROM R. A. FISHER TO THE EDITOR

In your issue of December, 1930, you print a long review of the third edition of my book, *Statistical Methods for Research Workers*, from the pen of Mr. C. C. Grove. Reviews from writers familiar with aspects of the literature somewhat distant from those under discussion are usually of great value and interest, and this case is no exception. Mr. Grove is, however, principally concerned—and to this point he gives three-quarters of his review—with his fear that workers who have in recent years made use of the methods I have put forward, are inclined to deceive themselves, or their readers, as to the novelty of their results. A careless reader might in fact easily conclude that an accusation of consistent and widespread plagiarism had been launched, and it is this circumstance which induces me to put before your readers some facts which Mr. Grove has evidently overlooked, the more especially as the examples he quotes are not from my own works, but from those of six other writers who have made use of them.

The features of modern statistics which are touched upon are:—

(a) The use of the exact distribution in "small" or finite samples, for the sampling errors of quantities calculated from them, in place of the "large sample" distributions, which are approximations based on the limiting form of the distribution as the sample is increased indefinitely. Mr. Grove seems to have misunderstood the meaning of the term "small" sample in this connection, which merely means that we take account of the fact that it is not infinitely great, but which he seems to take to imply that the users of the theory of small samples are advocates of paucity of data. It is largely because a sample of few observations contains less information than a sample of many that it is more necessary to use the exact, in place of the older approximate, distributions.

(b) The development of the theory of estimation, with the distinctions it draws, on the basis of their sampling properties, between estimates that are

consistent or inconsistent, efficient or inefficient, etc., either for small or, in the limit, for infinitely large samples. Mr. Grove would probably be less contemptuous of the "method of maximum likelihood," the supreme value of which has been established by this theory, if he realised that the method (in spite of its "weighting the observations in an unusual way") dates back at least to Gauss, who should surely belong to his "galaxy of astronomers and actuaries"; though by Gauss it was derived by a proof, which is neither necessary nor sufficient, from the ill-fated theory of inverse probability.

(c) The introduction of a new series of rational and integral symmetric functions of the sample values, known as the k-statistics, to replace those used by Thiele and Tchouproff, in developing the general relationships between the moments of such statistics and those of the population sampled. Mr. Grove supports his contention that insufficient credit has been given to Thiele by quoting from a paper by Wishart the definition of the first four k-statistics, of which the fourth is:—

$$k_4 = n^2(n-1)^{-1}(n-2)^{-1}(n-3)^{-1}\{(n+1)m_4 - 3(n-1)m_2^2\},$$

and quotes from Thiele a series of expressions for statistics denoted by λ, of which the fourth, written with the substitution suggested by Mr. Grove, is

$$\lambda_4 = n^3(n-1)^{-1}(n^2-6n+6)^{-1}\{m_4 + 6(n-1)^{-1}m_2^2\}.$$

He remarks that

"The chief difference in the formulas seems to be that R. A. Fisher has replaced the letters λ, μ, and m in Thiele's notation by the letters k, m, and n," a remark which is unintelligible, or at least highly captious, when the expressions are compared with this substitution. In pointing this out I should add that Mr. Grove appears to have introduced an unnecessary confusion by identifying Thiele's μ with my m; these two statistics being in reality as different in their definitions as are Thiele's λ and my k. Thiele was evidently not using, and as far as I know never did use, statistics so defined that their mean sampling values were equal to the semi-invariants of the population; and it is this choice which makes the algebra manageable, and has yielded the flood of new relationships now available in addition to the few given by Thiele.

The second quotation for which Mr. Grove takes Wishart to task is even more strange as a basis for complaint. In explaining the combinatorial notation, which I had introduced for the compact expression and derivation of the new results, Wishart remarks:—

"Thus the well-known formulae for the moments of the distribution of the mean in samples are summed up, in the notation of this paper, by the formula

$$k(1^r) = k_r/n^{r-1},"$$

with which Mr. Grove compares Thiele's statement of the same property in his own notation, and adds:—

"One, of course, may be dense, but it is difficult to see in what essential way

these older formulas differ from those attributed to the reputedly "new" work by the small sample experts."

One would have thought that Wishart's phrase "the well-known formulae" would have been sufficient to obviate the mis-representation that he was putting forward the result as new. On the contrary he is obviously explaining a new notation by expressing it in an old and familiar result. And why is Thiele in particular supposed to be injured? The result is one which has been familiar to readers of Laplace for over a century. Even if, as is quite possible, Thiele recognized the property of these functions entirely independently, it would, I imagine, come as a surprise to him that, by giving them a name, he was claiming to be mentioned whenever their properties were alluded to.

(d) The use of orthogonal polynomials in least square fitting was introduced by Tchebycheff, and until recently little attention has been paid to his treatment of this point, not only in England and America, but, *pace* Mr. Grove, in Scandinavia also. At all events Esscher working at the Lunds Observatory in Sweden, and myself in England, both developed the same method independently in papers published in 1919 and 1920 respectively. Certainly I have, and I think Esscher also has, never put this forward as mathematical discovery. Indeed I imagine that no mathematician considering the same problem, with the analogy of Fourier analysis in mind, could have failed to introduce orthogonal functions, in order to reduce a set of simultaneous to a set of simple equations. I have indeed recently examined, in the French translation, the several papers of Tchebycheff, among which his development of the theory of orthogonal polynomials is scattered, with a view to ascertaining whether any useful purpose could be served by referring modern students of statistics to them. The conclusion I have arrived at is that, at the expense of much labour, a good student could in this way improve his powers of algebraic manipulation, but not his grasp of statistical method. If this is the only "ground for feeling that Dr. Fisher is too parsimonious in references to other workers outside the fellowship of his own countrymen" Mr. Grove is welcome to it; but his real grievance seems to be that I do not, in this connection, mention Gram, who admittedly did not introduce these functions, although to Mr. Grove they are still "the Gram polynomials."

For the history of the subject nothing but good can come of the precept "to delve more" into the early literature, both English and foreign. The student, however, who wishes primarily to obtain a good grasp of effective methods, should be warned that he will meet, as in all subjects which have developed rapidly, with much that is obsolete, in thought, and in notation. Regarding in particular, statistical literature, even of the recent past, he must remember:—

(a) not to be misled by the widespread confusion between that which is estimated, and our estimate of it, to the extent of using the same symbol for these different quantities,

(b) that methods of statistical estimation were habitually chosen solely on

the basis of the criterion of consistency, without reference to their efficiency, which was often very defective;

(c) that prior to "Student's" work of 1908 no attempt was ordinarily made to obtain the exact sampling distribution in finite samples of the estimate chosen; and even the calculation of the sampling variance, in its limiting form for an infinitely large sample, was regarded as something of a feat, and was too often performed by extremely crude and laborious methods.

Finally, there is, perhaps, something to be said for "delving more" into the modern literature, even if it happens to be written in English. This at least would have spared Mr. Grove the error of supposing that Dr. Irwin was making a comparison "between Pearson's test for goodness of fit and another test devised by Irwin himself." The very live interest now shown for the newer work in English, throughout Northern Europe, and particularly in the Scandinavian countries, should certainly reassure Mr. Grove as to the non-existence of such intellectual isolation as might obscure the real course of scientific advance. It constitutes, also, perhaps the best answer to the theory that there is nothing (at least since Mr. Grove's favourite authorities) new under the sun.

R. A. Fisher

A Letter from G. O. Irwin to the Editor

In a review of R. A. Fisher's "Statistical Methods for Research Workers" published in the "American Mathematical Monthly" of December, 1930, Mr. Charles C. Grove pays me the compliment of saying that I have devised a test for "goodness of fit" which (I quote his words) "appears superior to the Pearsonian test."

I feel therefore that it is incumbent on me to say that I have never devised any test for "goodness of fit" at all.

The object of the little paper in the Journal of the Royal Statistical Society to which Mr. Grove refers was to make an attempt to clear up the confusion which I felt at that time was prevalent between the underlying hypotheses on which are based Pearson's and Fisher's methods of applying Pearson's test for goodness of fit. Pearson's method was published in 1900 (Phil. Mag.) and Fisher's method of application in 1921 (Journal of the Royal Statistical Society).

It would be interesting if Mr. Grove could tell us how Thiele's "Fejlkritik" anticipated Pearson's χ^2 test.

Both Pearson's and Fisher's methods are so well known that I never imagined any one would do me the honour of attributing either of them to me.

I concluded my paper by saying: "These remarks have been made after a careful perusal of the available literature on both sides of the question, and it is hoped that they may be of use to readers of the "Journal of the Royal Statistical Society."

This was I think enough to show that I made no claim to originality but that

my paper was in the nature of a commentary, which I hoped might be useful, on what had been done by others.

With the reviewer's estimate of the value of the Scandinavian work I am in complete agreement.

Yours faithfully,

J. O. IRWIN

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3491. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The internal bisectors of the angles subtended by the six edges of a tetrahedron at the centroid of the tetrahedron are such that the three lines joining the points on the pairs of opposite edges are concurrent.

3492. *Proposed by J. B. Reynolds, Lehigh University.*

One hundred circular disks 1/10 in. thick are piled in a cylindrical vertical pile which is free to turn about a horizontal fixed axis at its bottom. If the coefficient of static friction of one disk on another is 1/4, between which two disks will sliding first occur as the pile rotates under gravity from a vertical position? (Neglect the radius of the disks.)

3493. *Proposed by Paul Wernicke, Washington, D.C.*

An opaque circular circumference with a small transparent gap G rotates about its center C with uniform velocity. A luminous point L outside illuminates through G a wire soldered to the circumference at two points and forming a curve going through C. The point of this curve illuminated by the ray LG is always to lie on a perpendicular to CL at C. (a) Give the equation of the curve formed by the wire and the points where it is soldered to the circumference, and (b) Give the velocity of the illuminated point on CL.

3494. *Proposed by L. M. Graves, The University of Chicago.*

In how many ways can n objects be distributed into k different groups $G_1, \cdots, G_k$, containing $g_1, \cdots, g_k$ objects, respectively, if n_1 of the objects are

alike, n_2 are alike, $\cdots$, n_h are alike, where $\Sigma_1^h n_i = n = \Sigma_1^k g_j$? For example, in how many ways can 100 mathematicians and 300 economists be distributed among three hotels containing respectively 500, 300, and 200 rooms?

3495. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N.Y.*

Prove that the $\phi(m)$ integers less than m and prime to it constitute an automorphic group, with respect to multiplication and reduction to least positive residue modulus m.

Prove that if each member of any group in (1) be multiplied by r $(r \neq m)$, a new group will be formed which will be automorphic and also isomorphic with the first group. The modulus in this case is rm.

3496. *Proposed by H. E. Stelson, Kent State College.*

In the triangle ABC, a normal is drawn to the circumcircle cutting the lines BC and AC at X and Y respectively. Find the locus of intersection of AX and BY.

3497. *Proposed by J. Rosenbaum, Milford, Conn.*

Denoting the coefficients of the expansion of $(x+y)^n$, n being a positive integer, by $a_0, a_1, \cdots, a_n$; find the sum of $a_0, a_3, a_6, \cdots, a_{3k}$, where $(n-2) \leqq 3k \leqq n$.

3498. *Proposed by Robert E. Moritz, University of Washington.*

There are eight bridge tables, with eight men and eight women from Tacoma playing in couples against eight men and eight women from Seattle, also playing in couples. It is desired to arrange the seating so that each man plays at each table once and never has the same partner nor the same opponents twice. In other words, no one of the four persons playing a table has played at the same table before nor with any of the other three persons at the table.

Can the seating be arranged, and if so, how?

3499. *Proposed by Otto Dunkel, Washington University.*

In a text on advanced mathematics occurs an argument similar to the following: By means of the transformation $x = \phi(X, Y)$, $y = \psi(X, Y)$, we have $f(x,y) = F(X, Y)$. Hence

$$\frac{\partial F}{\partial X} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial X} = \frac{\partial F}{\partial x}\phi_X + \frac{\partial F}{\partial y}\psi_X = \frac{\partial f}{\partial x}\phi_X + \frac{\partial f}{\partial y}\psi_X.$$

Is this reasoning correct, assuming the existence of all the derivatives involved?

3500. *Proposed by Emma Gibson, Springfield High School, Springfield, Mo.*

Show that the primitive of the differential equation, $p^2(1-x^2) = (1-y^2)$ is $x^2+y^2-2Axy = 1-A^2$ and derive this equation by taking the sine of the sum of two angles, both of which are arcsines.

3501. *Proposed by E. B. Seitz, from "The Mathematical Visitor."*

Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken anywhere within the circle, find the average area of the triangle formed by joining the three points.

3502. *Proposed by B. F. Finkel, Drury College.*

The centroid of a triangle is joined to its vertices and a point is taken at random in each of the three parts. Find the average area of the triangle formed by joining the three random points.

UNSOLVED PROBLEMS

Solutions are requested for the following unsolved problems.

266 [1917, 288]. *Proposed by the late J. L. Riley.*

In how many ways can a given number be polygonal?

351 [1917, 328]. *Proposed by G. Paaswell.*

A transition curve is one such that its curvature varies directly with the distance measured along the curve from its point of zero curvature, that is, from the tangent. Its intrinsic equation is given by $d\alpha/ds=ks$, the constant being determined from the fact that for a given length of transition the final radius of curvature, i.e., the radius of the circle into which the transition runs, is given together with the length of the transition. In making a turnout from the transition curve there is as yet no direct way of computing the functions which would completely locate this turnout. In Figure 1, the point of switch is at B and the frog point at C. The angle F is given, termed the frog angle, and either the location of C or B is given, whence it is required to find either B or C and the radius of the turnout. Note that all these data must be referred finally to the center lines of the tracks and not to the individual rails. In attempting approximate solutions do not replace the transition by the cubic parabola as that is not always a good approximation. (See Crandall, *The transition curve* for a discussion of the properties of this curve.)

434 [1917, 328; 1918, 119]. *Proposed by E. W. Chittenden.*

Evaluate $\int_0^1 f(x)dx$ where

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\operatorname{sgn}(x - x_n)}{2^n}.$$

The function $\operatorname{sgn} x=-1, 0, +1$ according as x is negative, zero or positive. The numbers x_n form the series

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \cdots$$

for which the general formula is $x_n=(2h+1)/2^k$, where k is the greatest integer such that $2^{k-1}\leqq n$ and $h=n-2^{k-1}$.

Note: This problem is reprinted because of a misprint in the original statement and also because of a lack of definiteness in the definition of the series of values x_n. An inquiry has also been received in regard to the origin of the notation "sgn x." This notation seems to be due to Kronecker; cf. Werke, vol. II, p. 500.

270 [1917, 389]. *Proposed by G. N. Carmichael.*

Does there exist a fraction p/q in its lowest terms such that the ratio of the sum of the divisors of p to the sum of the divisors of q is equal to p/q? Give a method of finding such fractions not in their lowest terms.

272 [1917, 427]. *Proposed by C. C. Yen, Tangshan, North China.*

How many integers prime to n are there in each of the sets:

(a) $$1\cdot 2,\ 2\cdot 3,\ 3\cdot 4,\ \cdots,\ n(n+1);$$

(b) $$1\cdot 2\cdot 3,\ 2\cdot 3\cdot 4,\ 3\cdot 4\cdot 5,\ \cdots,\ n(n+1)(n+2);$$

(c) $$\frac{1\cdot 2}{2},\ \frac{2\cdot 3}{2},\ \frac{3\cdot 4}{2},\ \cdots,\ \frac{n(n+1)}{2};$$

(d) $$\frac{1\cdot 2\cdot 3}{6},\ \frac{2\cdot 3\cdot 4}{6},\ \frac{3\cdot 4\cdot 5}{6},\ \cdots,\ \frac{n(n+1)(n+2)}{6}?$$

275 [1917, 467]. *Proposed by V. M. Spunar, Chicago, Ill.*

A square, side $2a$, is represented by the equation $x^n+y^n=a$ $(n=\infty)$. Find a like formula for an equilateral triangle.

SOLUTIONS

3443 [1930, 380]. *Proposed by Warren A. Rees, Houston Junior College.*

To construct a quadrilateral, given the four feet of the perpendiculars from the point of intersection of the diagonals upon the four sides.

Solution by A. Pelletier, Montreal, Canada.

Let us suppose the problem solved, and let $ABCD$ be the required quadrilateral with the diagonals BD and AC intersecting in O. Let the feet of the perpendiculars from O to AB, BC, CD, DA be A', B', C', D', respectively. Then

$$\begin{aligned}\angle DOC &= \angle DD'C' + \angle CB'C' = \angle AOB = \angle AD'A' + \angle BB'A',\\ &= 180^\circ - \tfrac{1}{2}(\angle A'D'C' + \angle A'B'C').\end{aligned}$$

Also

$$\begin{aligned}\angle A'OC' &= \angle A'B'C' + \angle DOC,\\ &= 180^\circ - \tfrac{1}{2}(\angle A'D'C' - \angle A'B'C').\end{aligned}$$

Similarly

$$\angle B'OD' = 180^\circ - \tfrac{1}{2}(\angle B'C'D' - \angle B'A'D').$$

The angles $A'OC'$ and $B'OD'$ are therefore known, and arcs of circles containing these angles can be described on $A'C'$ and $B'D'$, respectively, as chords. The intersections of these arcs determine O, and the rest of the construction follows at once.

3447 [1930, 381]. *Proposed by Vladimir F. Ivanoff, California.*

Prove that

$$2R = R_x \cos\alpha + R_y \cos\beta + R_z \cos\gamma,$$

where R is the radius of curvature of a given curve at the point (x, y, z); R_x, R_y, R_z are the radii of curvature of the projections of this curve on the co-ordinate planes, YOZ, XOZ, and XOY at the points (y_1, z_1), (x_1, z_1), and (x_1, y_1), respectively; and α, β, γ are the angles between R and R_x, R_y, R_z, respectively.

Solution by Robert E. Moritz, University of Washington.

Let A, B, C denote the angles which the direction of R, taken from the point (x, y, z) to the center of curvature, makes with the positive directions of the x-, y-, and z-axes respectively: $\alpha_x, \beta_x, \gamma_x$; $\alpha_y, \beta_y, \gamma_y$; $\alpha_z, \beta_z, \gamma_z$ the corresponding angles for R_x, R_y, and R_z, respectively.

If the equations of the curve are given in terms of the arc-length, s, as parameter, and if differentiation with respect to s be indicated by accents we have the well known expressions

$$R = [(x'')^2 + (y'')^2 + (z'')^2]^{-1/2}, \quad \cos A = Rx'',$$

which may be written in the equivalent forms,

$$(1) \qquad R = [(z'y'' - y'z'')^2 + (x'z'' - z'x'')^2 + (y'x'' - x'y'')^2]^{-1/2}$$

$$(2) \qquad \cos A = R[y'(y'x'' - x'y'') - z'(z'x'' - x'z'')],$$

with analogous expressions for $\cos B$ and $\cos C$.

It is obvious that the corresponding expressions for R_x, R_y, R_z, and their direction cosines can be obtained from (1) and (2) by putting successively x, y, and z equal to zero. Thus

$$(3)\ R_z = \frac{(s_z')^3}{y'x'' - x'y''}, \quad \cos\alpha_z = R_z\frac{y'(y'x'' - x'y'')}{(s_z')^4}, \quad (s_z')^2 = (x')^2 + (y')^2,$$

with analogous expressions for $\cos\beta_z$ and $\cos\gamma_z$.

It is now easy to compute the values of the angles α, β, and γ between R and R_x, R_y, and R_z, respectively. We have

$$\cos\gamma = \cos A \cos\alpha_z + \cos B \cos\beta_z + \cos C \cos\gamma_z,$$

whence, on substituting the values given by (2) and (3) we obtain, after proper reduction,

$$R_z \cos\gamma = RR_z^2(y'x'' - x'y'')^2/(s_z')^4;$$

and on substituting the value of R_z^2 from (3) we find

$$R_z \cos\gamma = R(s_z')^2 = R[(x')^2 + (y')^2], \quad R_x \cos\alpha = R[(y')^2 + (z')^2],$$
$$R_y \cos\beta = R[(z')^2 + (x')^2];$$

and finally

$$R_x \cos\alpha + R_y \cos\beta + R_z \cos\gamma = 2R.$$

Also solved by V. F. Ivanoff, B. F. Kimball, and Paul Wernicke.

A Note by the Editors: The formulae in the solution may be simplified. Let λ, μ, ν be the direction cosines of the curve, and λ_1, μ_1, ν_1 those for R. Then $R\lambda' = \lambda_1$, $R\mu' = \mu_1$, $R\nu' = \nu_1$ where the accents mean derivatives with respect to s. Also

$$R_z = \pm (\lambda^2 + \mu^2)^{3/2}(\mu'\lambda - \lambda'\mu)^{-1} = \pm R(\lambda^2 + \mu^2)^{3/2}(\mu_1\lambda - \lambda_1\mu)^{-1};$$
$$\cos\gamma = \pm (\mu_1\lambda - \lambda_1\mu)(\lambda^2 + \mu^2)^{-1/2},$$

where the same sign is used in both expressions. Hence

$$R_z \cos\gamma = R(\lambda^2 + \mu^2),$$

and, by adding to this the two other similar expressions, we obtain the desired result.

3451 [1930, 447]. *Proposed by R. Goormaghtigh, Bruges (Belgium).*

If two conics circumscribed to a triangle are orthogonal at a given point M, their tangents at M being the axes of the conic conjugate to the triangle and having M as center, the product of their normal chords at M is equal to four times the product of their radii of curvature at M.

Solution by Otto J. Ramler, The Catholic University of America

Let the equations of three lines in normal form be

$$L_i \equiv x \cos\phi_i + y \sin\phi_i - p_i = 0 \qquad (i = 1, 2, 3).$$

Then

$$A \equiv \sum_1^3 a_i L_j L_k = 0, \quad B \equiv \sum_1^3 b_i L_j L_k = 0 \qquad (i \neq j \neq k)$$

are the equations of two conics A and B circumscribing the triangle formed by L_1, L_2, and L_3.

The conditions that the conics A and B be tangent to the y and x axes respectively, at the origin M, are,

$$(1) \qquad \begin{aligned} &\sum_1^3 a_i p_j p_k = 0, &&\sum_1^3 b_i p_j p_k = 0, \\ &\sum_1^3 a_i p_j \sin\phi_k = 0, &&\sum_1^3 b_i p_j \cos\phi_k = 0 \quad i \neq j \neq k. \end{aligned}$$

The normal chords are then the lengths of the x and y intercepts, and their product is readily found to be

$$\left(\sum_1^3 a_i p_j \cos\phi_k \Big/ \sum_1^3 a_i \cos\phi_j \cos\phi_k \right)\left(\sum_1^3 b_i p_j \sin\phi_k \Big/ \sum_1^3 b_i \sin\phi_j \sin\phi_k \right).$$

The equations of conics A and B are of the types

$$A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x = 0,$$
$$A_2x^2 + 2B_2xy + C_2y^2 + 2E_2y = 0.$$

The radii of curvature R_1 and R_2, at the origin are found to be $-D_1/C_1$ and $-E_2/A_2$, respectively.

Hence

$$4R_1R_2 = \sum_1^3 a_i p_j \cos\phi_k \cdot \sum_1^3 b_i p_j \sin\phi_k \Big/ \sum_1^3 a_i \sin\phi_j \sin\phi_k \cdot \sum_1^3 b_i \cos\phi_j \cos\phi_k.$$

Our problem then reduces to showing that

$$\sum_1^3 a_i \cos\phi_j \cos\phi_k \cdot \sum_1^3 b_i \sin\phi_j \sin\phi_k = \sum_1^3 a_i \sin\phi_j \sin\phi_k \cdot \sum_1^3 b_i \cos\phi_j \cos\phi_k.$$

The left member equals

$$\sum_1^3 a_i b_i \sin\phi_j \cos\phi_j \sin\phi_k \cos\phi_k + \sum_1^3 a_i b_j \sin\phi_k \cos\phi_k \sin\phi_i \cos\phi_j.$$

The right member is obtained by merely interchanging a_i and b_i in this result. We must therefore show that

$$\sum_1^3 a_i b_j \sin\phi_k \cos\phi_k \sin\phi_i \cos\phi_j = \sum_1^3 a_j b_i \sin\phi_k \cos\phi_k \sin\phi_i \cos\phi_j.$$

Transposing all terms to the left member we note that the coefficient of $\sin\phi_k \cos\phi_k$ is

$$a_i b_j \sin\phi_i \cos\phi_j + a_j b_i \sin\phi_j \cos\phi_i - a_i b_j \sin\phi_j \cos\phi_i - a_j b_i \sin\phi_i \cos\phi_j$$
$$= \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \begin{vmatrix} \sin\phi_i & \sin\phi_j \\ \cos\phi_i & \cos\phi_j \end{vmatrix}.$$

Hence we must prove, taking i, j, k in cyclic order only, that

$$\sum_1^3 \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \begin{vmatrix} \sin\phi_i & \sin\phi_j \\ \cos\phi_i & \cos\phi_j \end{vmatrix} \sin 2\phi_k = 0.$$

But from (1)

$$\frac{1}{p_i} = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\|$$

Hence the required relation is equivalent to

$$(2) \qquad \begin{vmatrix} \sin 2\phi_1 & \sin 2\phi_2 & \sin 2\phi_3 \\ p_1 \sin \phi_1 & p_2 \sin \phi_2 & p_3 \sin \phi_3 \\ p_1 \cos \phi_1 & p_2 \cos p_3 & p_3 \cos \phi_3 \end{vmatrix} = 0.$$

A conic conjugate to the triangle formed by L_1, L_2, L_3 has an equation,

$$\sum_1^3 k_i L_i^2 = 0.$$

The conditions that this conic shall have the origin as its center and the coordinate axes as its principal axes, are

$$\sum_1^3 k_i \sin 2\phi_i = 0, \quad \sum_1^3 p_i k_i \sin \phi_i = 0, \quad \sum_1^3 p_i k_i \cos \phi_i = 0.$$

These conditions will simultaneously exist only if relation (2) is true.

3453 [1930, 447]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given the radical axis of two circles, their circle of similitude, and the length of their line of centers, construct the two circles.

Solution by Otto J. Ramler, The Catholic University of America

Let the centers of the required circles be O and O'. Then the distance OO' is known. Call it $2a$. Let S,S' be the centers of similitude. Then the circle of similitude has O,O' as inverse points and is orthogonal to the circle on OO' as a diameter. Thus the midpoint of OO' is readily found on the line SS' produced, i.e., the points O and O' are located. Let the radical axis meet the line SS' at X. With X as center draw the circle orthogonal to the circle of similitude. The lengths of the tangents from O and O' to this circle are the radii of the required circles having O and O' as centers.

3454 [1930, 447]. *Proposed by M. S. Knebelman, Princeton University.*

Deduce the explicit expression for $P(m, n)$ from the relation

$$P(m, n) = P(n, m) = \sum_{i=1}^{i=m} P(i, n-1), \; P(1, 1) = 1,$$

where m and n are positive integers.

Solution by Robert E. Moritz, University of Washington

By definition

$$(1)\quad P(1, n) = P(1, n-1) = P(1, n-2) = \cdots = P(1, 2) = P(1, 1) = 1,$$

$$\begin{aligned}(2)\ P(m, n) &= \sum_{i=1}^{i=m} P(i, n-1) = \sum_{i=1}^{i=m-1} P(i, n-1) + P(m, n-1) = P(m-1, n) \\ &\quad + P(m, n-1) \\ &= P(m-2, n) + 2P(m-1, n-1) + P(m, n-2) \\ &= P(m-3, n) + 3P(m-2, n-1) + 3P(m-1, n-2) + P(m, n-3) \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= P(m-k, n) + C_1^k P(m-k+1, n-1) + C_2^k P(m-k+2, n-2) \\ &\quad + \cdots + C_{k-1}^k P(m-1, n-k+1) + P(m, n-k)\end{aligned}$$

Since by definition $P(m, n) = P(n, m)$ there is no loss in generality in assuming $m \geqq n$, say $m = n + p - 1$, where p is a positive integer. With this value of m, and with $k = n - 1$, equation (2) becomes

$$\begin{aligned}P(m, n) &= P(p, n) + C_1^{n-1} P(p+1, n-1) + C_2^{n-1} P(p+2, n-2) + \cdots \\ &\quad + C_{n-2}^{n-1} P(n+p-2, 2) + P(n+p-1, 1) \\ &= P(p-1, n) + P(p, n-1) + C_1^{n-1}[P(p, n-1) + P(p+1, n-2)] \\ (3)\qquad &\quad + C_2^{n-1}[P(p+1, n-2) + P(p+2, n-3)] + \cdots \\ &\quad + C_{n-2}^{n-1}[P(n+p-3, 2) + P(n+p-2, 1)] + P(n+p-1, 1) \\ &= P(p-1, n) + C_1^n P(p, n-1) + C_2^n P(p+1, n-2) + \cdots \\ &\quad + C_{n-2}^n P(n+p-3, 2) + C_{n-1}^n,\end{aligned}$$

because

$$C_s^r + C_{s+1}^r = C_{s+1}^{r+1}, \text{ and } P(n+p-2, 1) = P(n+p-1, 1) = 1.$$

On repeating this process $k-1$ times we obtain

$$\begin{aligned}(4)\quad P(m, n) &= P(p-k, n) + C_1^{n+k-1} P(p-k+1, n-1) \\ &+ C_2^{n+k-1} P(p-k+2, n-2) + \cdots + C_{n-2}^{n+k-1} P(p-k+n-2, 2) + C_{n-1}^{n+k-1}.\end{aligned}$$

Finally let $k = p - 1$; then

$$\begin{aligned}(5)\qquad P(m, n) &= P(1, n) + C_1^{m-1} P(2, n-1) + C_2^{m-1} P(3, n-2) + \cdots \\ &\quad + C_{n-2}^{m-1} P(n-1, 2) + C_{n-1}^{m-1}.\end{aligned}$$

In (5) put $n = 2$; then

$$P(m, 2) = P(1, 2) + C_1^{m-1} P(2, 1) = C_1^m.$$

When $n=3$,

$$\begin{aligned} P(m, 3) &= P(1, 3) + C_1^{m-1}P(2, 2) + C_2^{m-1}P(3, 1) \\ &= P(1, 3) + C_1^{m-1}[P(1, 2) + P(2, 1)] + C_2^{m-1}P(3, 1) \\ &= (1 + C_1^{m-1}) + (C_1^{m-1} + C_2^{m-1}) = C_1^m + C_2^m = C_2^{m+1}. \end{aligned}$$

Similarly

$$P(m, 4) = C_3^{m+2}, \quad P(m, 5) = C_4^{m+3}.$$

Hence, by induction,

$$P(m, n) = C_{n-1}^{m+n-2} = C_{m-1}^{m+n-2}.$$

Also solved by H. W. Bailey, F. Underwood, and Morgan Ward.

A Note by Otto Dunkel: The proposer of this problem stated that it was equivalent to the problem of finding the number of ways of making the shortest journey from the point (1, 1) to the point (m, n) by going always a number of units to the right and then a number of units vertically up, and so on until (m, n) is reached. He desired, however, a solution independent of this interpretation and derived from the analytical definition of $P(m, n)$. It is easy to see how this definition follows from the interpretation. For the paths may be grouped so that the ith group contains all paths from (1, 1) to $(i, n-1)$ and then a journey one unit up and $(m-i)$ units to the right. The number of paths in this group is $P(i, n-1)$, and the sum of such terms for $i=1, 2, \cdots m$ is $P(m, n)$. Also $P(m, n) = P(n, m)$ and we naturally define $P(1, 1)=1$. From this interpretation the solution follows directly. For a path may be described by the notation $a_1a_2a_3b_1b_2$ $a_4a_5a_6a_7b_3 \cdots$, where there are $m-1$ of the a's and $n-1$ of the b's, each a meaning a unit to the right, each b a unit up. The total number of paths is then the permutations of $m+n-2$ divided by those of $m-1$ and those of $n-1$. In this form the problem seems to have appeared some years ago in this Monthly.

A generation function is often useful in such problems. Let

$$\begin{aligned} f(x, k) &= \sum_{i=1}^{k+1} P(k + 2 - i, i)x^{i-1}, \\ &= \sum_{i=1}^{k} P(k + 1 - i, i)x^{i-1} + x\sum_{i=2}^{k+1} P(k + 2 - i, i - 1)x^{i-2}, \end{aligned}$$

where (1) and the first line of (2) in the above solution have been used. Then we have in turn

$$f(x, 1) = 1 + x; \; f(x, k) = (1 + x)f(x, k - 1); \; f(x, k) = (1 + x)^k.$$

Hence

$$P(k + 2 - i, i) = \binom{k}{i - 1}; \quad P(m, n) = \binom{m + n - 2}{n - 1}.$$

We may also apply the methods of the difference calculus. Let the difference operator Δ apply to m alone. Then

$$\Delta P(m, n) = P(m+1, n-1);\ \Delta^i P(m, n) = P(m+i, n-i);$$
$$\Delta^{n-1} P(m, n) = P(m+n-1, 1) = 1.$$

Integrating once the last equation, we have $\Delta^{n-2}P(m, n) = m+f(n) = P(m+n-2, 2)$. From the symmetry of $P(m, n)$ we have $f(n) = n+c$; but for $m+n=3$, we find $P(1, 2) = 1 = 3+c$, or $c=-2$. Integrating now $\Delta^{n-2}P(m, n) = m+n-2$, we have $\Delta^{n-3}P(m, n) = (m+n-2)^{(2)}/2+f_1(n) = P(m+n-3, 3)$, where $(m+n-2)^{(i)} = (m+n-2)(m+n-3)\cdots(m+n-i-1)$. Here $f_1(n)$ must be a constant, and by setting $m+n=4$ we find that the constant is zero. Continuing in this way we obtain at the end

$$P(m, n) = \frac{(m+n-2)^{(n-1)}}{(n-1)!} = \binom{m+n-2}{n-1}.$$

3455 [1930, 447]. *Proposed by Solomon Kullback, Brooklyn, N. Y.*

A triangle ABC, with AB constant, is inscribed in a circle. On AC and BC, construct equilateral triangles, ACD and BCE. Find the locus of the midpoint of ED as C moves along the arc ACB.

Solution by Eugene M. Berry, Lynchburg College

Let O and R be the circumcenter and circumradius respectively of the triangle ABC. As C traces the circle (O), the locus of D is a circle with radius R, whose center O' makes an equilateral triangle with A and O; for $\angle ADO = 30°$ and subtends the chord $OA = R$. The locus of E is also an equal circle, whose center O'' makes an equilateral triangle with B and O.

There are two loci for each of the points D and E since the triangle ACD may be outside or else may overlap the triangle ABC. In any case the circles (O') and (O'') intersect at the points O and Q, where $OO''QO'$ is a parallelogram. Let M be the intersection of the two diagonals $O'O''$ and OQ.

Case 1: The triangles ACD and BCE are both outside or both overlap the triangle ABC. In this case Q forms an equilateral triangle with A and B, with Q and C on the same or opposite sides of AB according as the triangles ACD and BCE are both outside or both overlap the triangle ABC; for $QA = QB$, and $\angle AQO = \angle ADO = 30°$, and similarly $\angle BQO = 30°$.

Triangles AQD and ABC are congruent since they have two sides of one equal to two sides of the other, and $\angle ADQ$ and $\angle ACB$ intercept equal arcs on equal circles. Hence $DQ = BC = CE$. In like manner, $EQ = AC = CD$. From this we see that $DCEQ$ is a parallelogram whose diagonals are ED and CQ. Hence P, the midpoint of ED, is also the midpoint of CQ. As Q is fixed and C describes the circle (O), P describes a circle homothetic to (O) with the homothetic ratio $1/2$. Then the center of this circle is at M, the midpoint of OQ and the radius is $R/2$. This case has two solutions.

Case 2. Triangle ACD overlaps the triangle ABC while the triangle BCE is outside the triangle ABC.

$O'O''=AB$, for the triangles AOB and $O'O''O$ are congruent. $DE=AB$ for the triangles ACB and DCE are congruent, also $O''E=O'D$, whence $O'O''DE$ is a parallelogram. Since M and P are the midpoints of $O'O''$ and ED respectively we have $MP=O'D=R$. That is, P describes a circle whose center is M and whose radius is R.

If triangle ACD is outside and triangle BCE overlaps the triangle ABC we get another solution for case 2. This gives us four solutions in all.

Also solved by W. R. Church, Rufus Crane, H. D. Grossman, F. Underwood, Roscoe Woods, and G. A. Yanosik.

3458 [1930, 508]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, a tangent is drawn to the incircle cutting the lines BC and AC at X and Y respectively. Find the locus of the intersection of AX and BY.

Solution by H. E. Stelson, Kent State College.

The solution of this problem is simplified by using trilinear coordinates as given in Chapter XIII of C. Smith's Conic Sections. Let ABC be the triangle of reference, then the coordinates of the vertices will be $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$. The equation of the incircle is

$$(1) \qquad (P\alpha)^{1/2} + (Q\beta)^{1/2} + (R\gamma)^{1/2} = 0,$$

where $P=a(s-a)$, $Q=b(s-b)$, $R=c(s-c)$ and the equation of the tangent XY at any point $(\alpha_1, \beta_1, \gamma_1)$ on the circle is

$$(2) \qquad \alpha(P/\alpha_1)^{1/2} + \beta(Q/\beta_1)^{1/2} + \gamma(R/\gamma_1)^{1/2} = 0.$$

Solving (2) with the equation $\alpha=0$ of the side BC, we have as the coordinates of X, $(O, (R/\gamma_1)^{1/2}, -(Q/\beta_1)^{1/2})$ and likewise the coordinates of Y are $((R/\gamma_1)^{1/2}, 0, -(P/\alpha_1)^{1/2})$.

The equations of AX and BY are respectively,

$$(3) \qquad \beta(\gamma_1/R)^{1/2} + \gamma(\beta_1/Q)^{1/2} = 0$$

$$(4) \qquad \alpha(\gamma_1/R)^{1/2} + \gamma(\alpha_1/R)^{1/2} = 0.$$

The intersection, O, of AX and BY has coordinates $(\alpha_1/P)^{1/2}$, $(\beta_1/Q)^{1/2}$, and $-(\gamma_1/R)^{1/2}$. Now the line joining the points of tangency of BC and AC has the equation

$$(5) \qquad P\alpha + Q\beta - R\gamma = 0.$$

If we substitute the coordinates of O in (5) we have $(P\alpha_1)^{1/2}+(Q\beta_1)^{1/2}+(R\gamma_1)^{1/2}=0$ which is the condition that the point $(\alpha_1, \beta_1, \gamma_1)$ lie on the incircle. Therefore the locus of O is a straight line joining the points of tangency of AC and BC.

A Note by the Editors: From Brianchon's theorem it follows that the two diagonals AX, BY of the circumscribed quadrilateral $ABXY$ and the two chords of contact of opposite sides meet in a point. Hence the locus of the intersection of AX and BY is the chord of contact of AC and BC. The circle may be replaced by an ellipse in this proof.

Also solved by Frank Ayers, H. A. Do Bell, R. P. Johnson and T. L. Smith, A. Pelletier, F. Underwood, A. W. Randall, G. A. Yanosik, and the Proposer.

3459 [1930, 508]. *Proposed by Norman Anning, University of Michigan.*

It is observed that $3003=\binom{15}{5}=\binom{14}{6}$. Solve in positive integers the equation

$$\binom{x+1}{y}=\binom{x}{y+1}.$$

Solution by F. Underwood, University College, Nottingham, England

The equation of the problem reduces to

$$x^2-3xy+y^2-2y-1=0. \tag{1}$$

Solving this equation for x we obtain

$$2x=3y\pm m,\quad m^2=5y^2+8y+4, \tag{2}$$

where m is to be a positive integer. Solving the second equation in (2) for y and rejecting the negative root, we find

$$5y=t-4,\quad t^2=5m^2-4, \tag{3}$$

where t is to be a positive integer. Solutions of the second equation in (3) which are even integers may be obtained by setting $m=2r$, $t=2s$; and after this substitution the equation reduces to

$$5r^2-s^2=1. \tag{4}$$

Since y must be an integer, $s-2$ must be divisible by 5. A simple solution of the second equation of (3) in odd integers is given by $m=1$, $t=1$, and so the equation may be written

$$5m^2-t^2=-(5\cdot 1^2-1^2)(5r^2-s^2),$$

where r and s are solutions of

$$5r^2-s^2=-1. \tag{5}$$

With the same conditions we may write

$$5m^2-t^2=5(s\pm r)^2-(5r\pm s)^2.$$

Hence

$$m=s\pm r,\quad t=5r\pm s,$$

where r and s are solutions of (5). Here, in order that y may be an integer, $s\mp 4$ must be divisible by 5.

Now, as in *Higher Algebra* by Hall and Knight, §§369–376, the solutions of (4) and (5) depend upon the continued fraction

$$5^{1/2} = 2 + \frac{1}{4+}\ \frac{1}{4+}\ \frac{1}{4+}\ \cdots$$

The convergents for this fraction may be computed from the relations

$$p_n = 4p_{n-1} + p_{n-2}, \quad q_n = 4q_{n-1} + q_{n-2};$$

or they may be obtained from

$$p_n = \tfrac{1}{2}(\alpha^n + \beta^n), \quad q_n = \tfrac{1}{2}5^{-1/2}(\alpha^n - \beta^n),$$
$$\alpha = 2 + 5^{1/2}, \qquad \beta = 2 - 5^{1/2}.$$

A few of these are tabulated below in readiness for the last stage of the solution

n	1	2	3	4	5	6	7
p_n	2	9	38	161	682	2889	12238
q_n	1	4	17	72	305	1292	5473

The odd convergents furnish a solution of (4) $r = q_{2n-1}$, $s = p_{2n-1}$; whereas the even convergents give a solution of (5).

Case I. Odd convergents. Solutions of the second equation of (3) are given by $m = 2q_{2n-1}$, $t = 2p_{2n-1}$, but they are subject to the restrictions that y and x in (2) and (3) must be positive integers such that x is greater than y. For $n = 1$ we find $r = 1$, $s = 2$, $m = 2$, $t = 4$, and then $y = 0$, $x = 1$. This may be considered a trivial solution of the equation of the problem; it is a solution since

$$\binom{2}{0} = \binom{1}{1} = 1.$$

For $n = 3$ there results $r = 17$, $s = 38$, $m = 34$, $t = 76$, and we have to reject this case since the resulting values of x and y are fractional. For $n = 5$, $r = 305$, $s = 682$, $m = 610$, $t = 1364$, leading to the solutions $y = 272$, $x = 713$ and $x = 103$, $y = 272$. Only fractional values of x and y result from $n = 7$.

Case II. Even convergents. Solutions are given by $m = s \pm r$, $t = 5r \pm s$, where $r = q_{2n}$, $s = p_{2n}$, but they are subject to the same restrictions as before. For $n = 2$, $r = 4$, $s = 9$, and for the pair (m, t) we have the two sets of values (13, 29) and (5, 11). The second pair of values for m, t give fractional values for x and y; while the first pair gives $y = 5$, $x = 14$, which is the solution mentioned in the problem. For $n = 4$, $r = 72$, $s = 161$, $(m, t) = (233, 521)$ and (89, 199). The first pair gives fractional values for x and y, but the second pair yields the solution $y = 39$, $x = 103$. For $n = 6$, $r = 1292$, $s = 2889$, $(m, t) = (4181, 9349)$ and (1597, 3571). The second pair gives fractional values, but the first gives the solution $y = 1869$, $x = 4894$. A table is given below of these solutions including also those for $n = 8, 9, 10$. Evidently the method may be used to obtain all the solutions of (1) in positive integers.

n	1	2	3	4	5	6	7	8	9	10
x	1	14	—	103	713	4894	—	33551	229969	1576238
y	0	5	—	39	272	1869	—	12815	87840	602069

A Note by Otto Dunkel: The cases which do not lead to solutions may be sorted out before beginning the computation by an examination of the formulae. Thus in (2) we must reject the case $2x=3y-m$. For, since m and y are to be positive, we must have from the second equation of (2) $m>5^{1/2}y$. Hence $2(y-x)=m-y>(5^{1/2}-1)y>0$, or $y>x$. If $2x=3y+m$, it is obvious that x is greater than y. In order to set aside other useless cases the following theorem will be proved:

With respect to the modulus 5 the following congruences are true

$$p_{4i+1}\equiv 2,\ p_{4i+2}\equiv -1,\ p_{4i+3}\equiv -2,\ p_{4i+4}\equiv 1, \tag{1}$$

where $i=0, 1, 2, \cdots$.

Proof: We have in general $p_{n+2}=4p_{n+1}+p_n\equiv -p_{n+1}+p_n$ (mod. 5). Assume that the first two congruences of (1) are true. Then $p_{4i+3}\equiv -p_{4i+2}+p_{4i+1}\equiv -2$; $p_{4i+4}\equiv 2-1=1$; $p_{4i+5}\equiv -1-2\equiv 2$; $p_{4i+6}\equiv -2+1=-1$. Now $p_1=2$ and $p_2=9\equiv -1$, and hence the theorem is always true.

Even convergents. When $m=s-r$, $t=5r-s$,

$$y=r-\frac{s+4}{5},\quad x=r+\frac{s-6}{5}\cdot$$

Hence it is necessary and sufficient to have $s\equiv 1$ (mod 5) in order to have x and y integral. Hence the only solutions in this case are given by

$$y=q_{4i}-\frac{p_{4i}+4}{5},\quad x=q_{4i}+\frac{p_{4i}-6}{5},\quad 5q_{4i}^2-p_{4i}^2=-1. \tag{2}$$

For $m=s+r$, $t=5r+s$, we have

$$y=r+\frac{s-4}{5},\quad x=2r+\frac{4s-6}{5}\cdot$$

Hence it is necessary and sufficient to have $s\equiv -1$ in order to obtain integral values for x and y. The only solutions in this case are given by

$$y=q_{4i+2}+\frac{p_{4i+2}-4}{5},\quad x=2q_{4i+2}+\frac{4p_{4i+2}-6}{5},\quad 5q_{4i+2}^2-p_{4i+2}^2=-1. \tag{3}$$

Odd convergents. Here we have $m=2r$, $t=2s$, and hence

$$y=\frac{2(s-2)}{5},\quad x=r+\frac{3(s-2)}{5}\cdot$$

For integral solutions it is necessary and sufficient that $s\equiv 2 \pmod 5$. The $(4i+1)$th convergents are the only ones which can be used. Hence

$$(4)\qquad y=\frac{2(p_{4i+1}-2)}{5},\quad x=q_{4i+1}+\frac{3(p_{4i+1}-2)}{5},\quad 5q_{4i+1}^2-p_{4i+1}^2=1.$$

It is easily verified that (2), (3), (4) are solutions; for the given equation may be written in forms corresponding to each solution, and in these forms the substitution is easy:

$$5\left[\frac{x+y}{2}+1\right]^2-\left[5\left(\frac{x-y}{2}\right)+1\right]^2+1=0,$$

$$5\left[\frac{4y-x}{2}+1\right]^2-\left[5\left(\frac{x-2y}{2}\right)-1\right]^2+1=0,$$

$$5\left[x-\frac{3}{2}y\right]^2-\left[\frac{5y}{2}+2\right]^2-1=0.$$

Also solved by J. C. Brixey, Paul Capron, P. A. Caris, Rufus Crane, H. E. H. Greenleaf, H. G. Green, D. F. Gunder, D. H. Lehmer, R. E. Moritz, A. Pelletier, C. A. Rupp, E. E. Whitford, Roscoe Woods, and T. R. C. Wilson.

3461 [1930, 508]. *Proposed by Boyd C. Patterson, Hamilton College.*

Given a triangle $A_1A_2A_3$, its circumcircle (O), and a line L in its plane. To prove the following construction of the point P on (O) whose Simson line with respect to $A_1A_2A_3$ is parallel to L: through the circumcenter O draw a line OX parallel to L. If M is one of the points where OX cuts (O) then P may be located by the relation

$$\text{arc } MP\equiv \text{arc } MA_1+\text{arc } MA_2+\text{arc } MA_3 \pmod{4\pi}.$$

Solution by A. Pelletier, Montreal, Canada

Let the feet of the perpendiculars from P, on the circle (O), to the sides A_1A_2, A_2A_3, A_3A_1, be R, T, S, respectively; and suppose that R lies outside the segment A_2A_1. Then R, S, T lie on a straight line. Draw the diameter $A_3'A_3''$, where A_3' is the other intersection of PR with (O). Let the diameter $M'M$ be parallel to RST, and suppose that it cuts A_1A_3 in V. Join PA_1 and PA_3''. Then angle $V=$angle $RSA_1,=$angle RPA_1, where the last equality results from the fact that P, R, A_1, S are concyclic. The angles V and RPA_1 are measured by the arcs $\frac{1}{2}(A_1M'-A_3M)$ and $\frac{1}{2}A_1A_3'$, respectively. Hence the two arcs $A_3'M'$ and A_3M are equal; also $A_3'M'=MA_3''$. Hence M is the mid-point of arc A_3A_3''. Also, since A_1A_2 is parallel to PA_3'', the mid-points K and L of the arcs A_1A_2 and PA_3'' lie on the ends of a diameter. If we consider MA_2A_1P to be in the order written, we shall suppose that all arcs are taken in this direction. Then arc MA_1+arc $MA_2=2$ arc MK; arc $MP+$arc $MA_3''=2$ arc ML; arc $ML-$arc $MK=\pi$; arc $MA_3''=2\pi-$arc MA_3. Combining these results we have

$$\text{arc } MP = \text{arc } MA_1 + \text{arc } MA_2 + \text{arc } MA_3,$$

If the reverse direction is used, 4π must be subtracted from the right side.

3468 [1930, 552]. *Proposed by C. A. Rupp, Pennsylvania State College.*

Show that the determinant of n^2 elements in the upper left corner of the Pascal triangle

$$\begin{matrix} 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 2 & 3 & \cdot & \cdot & \\ 1 & 3 & \cdot & \cdot & & \\ 1 & \cdot & \cdot & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \end{matrix}$$

has the value unity.

Solution by H. T. R. Aude, Colgate University

Denote by D_n the determinant in question and by a_{ij} the element in the ith row and the jth column. Then from the law of formation for the elements we have

$$a_{ij} = a_{ij-1} + a_{i-1j}, \quad a_{1j} = a_{i1} = 1.$$

Subtract each row of D_n from the row following it, beginning the process with the last pair of rows. After the $n-1$ subtractions the above equality shows that the element a_{ij} is replaced by a_{ij-1}, and all the elements, in the first column, except $a_{11}=1$, become zeros. Now subtract each column from the one following it, beginning with the last pair. After this process the element a_{ij-1} is replaced by $a_{i-1\ j-1}$, as shown by the above relation. The result of the two operations is to replace a_{ij} by $a_{i-1\ j-1}$, and to reduce each element in the first row and in the first column, except $a_{11}=1$, to zero. Hence $D_n=D_{n-1}$; and consequently

$$D_n = D_{n-1} = D_{n-2} = \cdots D_2 = 1.$$

Also solved by Frank Ayers, A. G. Clark, F. A. Carey, H. R. Cooley, W. H. Erskine, Raymond Garver, H. M. Gehman, R. E. Moritz, Y. V. Maizlish, J. Lipschitz, J. H. Neelley, Oscar J. Peterson, A. Pelletier, B. D. Roberts, H. S. Thurston, H. E. Stelson, F. Underwood, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The summer meeting of the Mathematical Association of America will be inaugurated by a joint session of the Association with the Society for the Promotion of Engineering Education on the evening of Friday, September 4, 1931. The officers of the Mathematical Association are glad to be able to present this opportunity to its members for meeting the engineering group as they close their important summer term which has been already announced in these columns (April issue, page 236). There will be two addresses. The first will be given by Dean C. S. Slichter of the University of Wisconsin on the subject "Some great engineer-mathematicians"; the second will be given by Professor W. H. Roever of Washington University on the subject "Some frequently overlooked mathematical principles of descriptive geometry."

The separate sessions of the Mathematical Association will be held on Monday afternoon and Tuesday morning, September 7–8. On Monday afternoon there will be two addresses, one by Professor E. P. Lane of the University of Chicago on "The present situation in projective differential geometry," and one by Professor H. C. Carver of the University of Michigan, by invitation, on a topic connected with the mathematical theory of statistics. On Tuesday morning there will also be two addresses, one by Professor J. H. Van Vleck of the University of Wisconsin, by invitation, on the topic "Some mathematical aspects of the new physics"; the second will be the retiring presidential address by President J. W. Young of Dartmouth College on the topic "Functions of the Mathematical Association of America." The complete program and full details of the meeting, together with reservation cards, will be sent to members of the Association in July.

The program of the American Mathematical Society provides for meetings on Tuesday afternoon, Wednesday morning and afternoon, and Thursday and Friday mornings. The Colloquium Lectures by Professor Marston Morse of Harvard University have been tentatively scheduled for Tuesday afternoon and Wednesday, Thursday, and Friday mornings. There will be an address on "Ideals in linear algebras" by Professor C. C. MacDuffee of Ohio State University; and other addresses may be arranged.

A joint dinner of the Association and the Society will be held on Wednesday evening at the Minneapolis Automobile Club. On Thursday afternoon an excursion will be made by busses or private cars to the Chisago Lakes and Taylors Falls, where time will be allowed for an inspection of the geological formations and for a short trip by motor boat through the Dalles of the Sainte Croix.

Accommodations for members of the Society and their guests will be furnished in Sanford Hall, one of the women's dormitories of the University of Minnesota. Provided the room is engaged for a minimum of three days, the

charge for a single room will be $1.50 per person per day, and $1.00 per person when the room is occupied by two. Suitable arrangements can be made for families with children. Members residing at Sanford Hall and others who desire will be served with meals at the following rates: Breakfast 50 cents, Luncheon 75 cents, Dinner 75 cents. Members who reside at Sanford Hall and who drive their automobiles to the meeting will find it convenient to store their cars at the Minnesota Garage, where a special rate of 50 cents for twenty-four hours will be granted on presentation of their registration cards. The Chairman of the Committee on Arrangements will be glad to recommend hotels to those who ask him for such information.

Requests for information concerning accommodations and opportunities for vacation in northern Wisconsin and Minnesota should be addressed to Professor Raymond W. Brink, University of Minnesota, Minneapolis, Minnesota.

Professor R. R. Fleet represented the Mathematical Association of America at the inauguration of President Robert H. Ruff of Central College, Fayette, Missouri, May 25, 1931.

At Purdue University the Departments of Mathematics and Physics sponsored a series of lectures during 1930–31 on topics of current interest in these two fields. The lectures, given by members of the staffs of the two departments, were not highly technical but were designed to be within the grasp of the intelligent layman, with opportunity for questions and discussion at the close. Attendance and interest were highly satisfactory, and it is quite probable that a similar series will be given in 1931–32.

The lectures given were:

November 17—"Integers and Related Problems" by Professor Mason.

December 2—"Johannes Kepler, 1571–1630. Three Hundred Years of Astronomy" by Professor Lark-Horovitz.

December 15—"The Mathematics of Life Insurance" by Professor Zehring.

January 12—"Einstein's Universe" by Professor Akeley.

February 2—"Mathematics and the Scheme of Things" by Professor Graves.

February 16—"What Light Tells of Atoms and Stars" by Dr. Walerstein.

March 2—"Airplane and Glider" by Professor Maier.

March 16—"Television" by Professor Abbott.

The first issue of "Physics," a new monthly publication sponsored by The American Physical Society, will appear in July, 1931. The editor will be Professor John T. Tate of the University of Minnesota. An announcement made by the American Physical Society contains the following paragraphs:

"Physics, as every science, has two aspects—an internal and an external. The one emphasizes that fascinating interplay and correlation between elementary phenomena around which the logical framework of the science is constructed. The other reveals physics in its rôle as the basic natural science, whose

discoveries have revolutionized industry and with it the material aspects of our civilization."

"The American Physical Society feels its responsibility for providing adequate publication facilities for both aspects of physics. As a natural development the 'Physical Review' has become the foremost exponent in the United States of the introspective side of physics. To care in an equally adequate manner for the more general and external aspects of physics the American Physical Society is now sponsoring the publication of a new monthly journal."

"It is intended that 'Physics' shall be devoted primarily to the rôle which physics plays as the science basic to other natural sciences and to the arts and industries. It will not compete with the more specialized Journals of the Optical Society, the Acoustical Society of America or the Society of Rheology but in conjunction with them will round out the whole field."

Professor B. A. Bernstein, of the University of California, has been granted leave of absence from July to December to complete papers on the foundations of mathematics and a book on the algebra of logic.

Dr. K. Lanczos, professor of mathematics at the University of Frankfort, will be a visiting professor of mathematics at Purdue University during the year 1931–32.

DOCTORATES IN 1930

The following eighty doctorates with mathematics or mathematical physics as major subject were conferred during 1930 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

Virgil Adkinsson, Pennsylvania, June, *Cyclically connected continuous curves whose complementary domain boundaries are homeomorphic, preserving branch points.*

R. P. Agnew, Cornell, June, minor in physics, *The behavior of bounds and oscillations of sequences of functions under regular transformations.*

Rose L. Anderson, Bryn Mawr, June, minor in physics, *A problem in the simultaneous reduction of two quadratic forms in infinitely many variables.*

C. A. Andree, Wisconsin, June, major in electrical engineering physics, minor in mathematics, *An impact strain recorder.*

J. V. Atanasoff, Wisconsin, June, major in mathematical physics, minor in mathematics, *The dielectric constant of helium.*

R. F. Bacher, Michigan, June, *Tne Zeeman effect of hyperfine structure.*

E. B. Baker, Michigan, June, *The application of the Fermi statistics to the calculation of the potential distribution of positive ions.*

C. F. Bowles, Chicago, August, *Integral surfaces of pairs of differential equations of the third order.*

M. G. Boyce, Chicago, March, *An envelope theorem and necessary conditions for a problem of Mayer with variable end-points.*

Leonard Bristow, Illinois, May, minor in physics. *Expansion theory associated with linear differential equations and their regular singular points.*

L. H. Bunyan, Wisconsin, June, *A transformation of a certain integral equation and a theorem concerning an integro-differential equation.*

J. H. Bushey, Michigan, June, *Asymptotic expressions for a certain class of definite integrals.*

Hung Chi Chang, Michigan, June, *Transformation of linear partial differential equations.*

C. M. Cleveland, Texas, June, minor in physics, *On the existence of acyclic continuous curves satisfying certain conditions with respect to a given continuous curve.*

J. B. Coleman, California, May, *Concerning the irreducibility of the characteristic equation of a ternary continued fraction.*

Elizabeth M. Cooper, Illinois, June, minor in physics, *Perspective elliptic curves.*

J. J. Corliss, Michigan, June, *On the unsymmetric top.*

A. E. Currier, Harvard, June, *The problem of the calculus of variations in m-space with end points variable on two manifolds.*

E. H. Cutler, Harvard, June, *Some properties of general subspaces of a Riemann space.*

M. N. Davis, Wisconsin, June, major in physics, minor in mathematics. *A study and partial explanation of the emission of secondary electrons from cobalt.*

E. H. Dixon, Wisconsin, June, major in physics, minor in mathematics, *The photoelectric and thermionic properties of rhodium.*

L. H. Donnell, Michigan, June, *Some examples of mechanical wave transmission.*

J. L. Dorroh, Texas, June, minor in physics, *Some metrical properties of descriptive planes.*

W. L. Duren, Jr., Chicago, August, *The development of sufficient conditions in the calculus of variations.*

W. H. Durfee, Cornell, February, minor in physics, *Summation factors which are powers of a complex variable.*

L. A. Dye, Cornell, June, minor in physics, *Involutorial transformation in S_3 of order n with an $(n-1)$-fold line.*

G. L. Edgett, Illinois, August, minor in astronomy, *Frequency distributions with given statistics which are not all moments.*

C. M. Erikson, Michigan, June, *Systems of linear difference equations with constant coefficients.*

O. J. Farrell, Harvard, June, (1). *On the expansion of harmonic functions in terms of harmonic polynomials*; (2). *On approximation to an arbitrary function of a complex variable by polynomials.*

E. J. Finan, Ohio State, August, *Determination of the domains of integrity of the complete rational matric algebra of order four.*

W. W. Flexner, Princeton, June, *On topological manifolds.*

H. L. Garabedian, Princeton, June, *On the relation between certain methods of summability.*

B. P. Gill, Columbia, July, *An analogue for analytic functions of the Thue-Siegel theorem.*

Beatrice L. Hagen, Chicago, August, *Quintuples of three dimensional varieties in a four-dimensional linear space.*

Frances Harshbarger, Illinois, May, minor in physics, *The geometric configuration defined by a special algebraic relation of genus four.*

R. N. Haskell, Rice, June, minor in mathematical physics, *The mixed boundary value problem for Laplace's equation, in the plane.*

G. A. Hedlund, Harvard, June, (1), *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*; (2) *Poincaré's rotation number and Morse's type number.*

V. A. Hoyle, Princeton, June, *Some problems in conformal geometry.*

H. K. Hughes, Michigan, June, *On the analytical extension of functions defined by factorial series.*

Aline Huke, Chicago, March, *An historical and critical study of the fundamental lemma in the calculus of variations.*

F. C. Jonah, Brown, October, *The Green's matrix and expansion problem for systems of integro-differential equations.*

T. H. Kiang, Harvard, June, *Existence of critical points of harmonic functions of three variables.*

Edna E. Kramer, Columbia, June, *On the Laguerre group and allied topics.*

C. C. Krieger, Toronto, June, minor in the theory of relativity, (1), *On the summability of trigonometrical series with localized properties*; (2) *On Fourier constants and convergence factors of double Fourier series.*

H. I. Lane, Cornell, September, *The separation of the projective plane by the lines joining six points.*

Voris Latshaw, Indiana, June, minor in physics, *The algebra of self-adjoint boundary value problems.*

D. H. Lehmer, Brown, June, *An extended theory of Lucas' functions.*

C. N. Liu, Harvard, June, *Contributions to the restricted problem of three bodies.*

Shu Ting Liu, Michigan, June, *Theory of periodic orbits for asteroids of integral types.*

W. H. McEwen, Minnesota, July, minor in physics, *Problems of closest approximation connected with the solution of linear differential equations.*

E. J. McShane, Chicago, June, *Semi-continuity in the calculus of variations, and absolute minima for isoperimetric problems.*

C. W. Mendel, Chicago, June, *Contributions to the projective differential geometry of hyperspace.*

C. A. Messick, Chicago, August, *Symmetric functions of infinitely many elements.*

E. W. Miller, Michigan, June, *Concerning subsets of a continuous curve which lie on an arc of the continuous curve.*

Ethel I. Moody, Cornell, June, minor in philosophy, *A Cremona group of order thirty-two of cubic transformations in three dimensional space.*

G. C. Munro, Michigan, June, *Systems of linear partial differential equations with constant coefficients.*

Alexander Oppenheim, Chicago, June, *The minima of indefinite quaternary quadratic forms.*

T. S. Peterson, Ohio State, June, *The invariant theory of functional forms under the group of linear functional transformations of the third kind.*

O. H. Rechard, Wisconsin, June, minor in philosophy, *The expansion problem associated with a class of ordinary differential boundary-value problems.*

D. P. Richardson, Chicago, August, *Quadratic Cremona transformations of five-dimensional space.*

D. A. F. Robinson, Chicago, June, *Fourier expansions of pseudo doubly periodic functions with applications.*

Ragner Rollefson, Wisconsin, June, major in physics, minor in mathematics, *The continuous and band spectrum of mercury.*

L. R. Salvosa, Michigan, June, *Generalizations of the normal curve of error.*

H. W. Sibert, Cincinnati, June, *Moderately thick plates with plane faces.*

R. G. Smith, Kansas, June, minor in physics, *A canonical form for the differential equations of curves in n-dimensional space.*

Elizabeth T. Stafford, Wisconsin, June, *Matrices conjugate to a given matrix with respect to its minimum equation.*

G. W. Starcher, Illinois, May, minor in physics, *On identities arising from solutions of q-difference equations and some interpretations in number theory.*

H. E. Stetson, Iowa, February, minor in physics, *Double function space.*

C. W. Strom, Illinois, June, minor in physics, *On complete systems under certain finite groups.*

Mildred E. Taylor, Illinois, August, minor in physics, *A determination of the types of planar Cremona transformations with not more than 9 F-points.*

H. P. Thielman, Ohio State, June (1), *On new integral addition theorems for Bessel functions and series of the hypergeometric type*; (2), *The application of functional operations to a class of integral equations occurring in physics.*

W. R. Thompson, Yale, June, *The possible forms of discriminants of algebraic fields.*

R. S. Underwood, Chicago, June, *On universal quadratic null forms in five variables.*

Charles Wexler, Harvard, June, *On the theory of quadratic fields.*

J. H. C. Whitehead, Princeton, June, *The representation of projective space.*

F. L. Wren, Chicago, March, *A new theory of parametric problems in the calculus of variations.*

H. M. Yarbrough, Indiana, June, minor in astronomy, *Linear cyclic groups with given fundamental region.*

R. C. Yates, Johns Hopkins, June, *The small vibrations of certain mechanical systems.*

J. Yerushalmy, Johns Hopkins, June, *The construction of equianharmonic cubics.*

L. A. Young, Michigan, June, *The Wentzler-Brioullin-Kramers approximate solution of the Schrödinger wave equation.*

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931
NUMBER 7, AUGUST-SEPTEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-eighth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at American University, Washington, D.C., on Saturday, December 13, 1930. Sessions were held in the morning and in the afternoon; Professor Clara L. Bacon, chairman of the Section, presided at both sessions.

Seventy persons attended the meeting, including the following thirty-nine members of the Association: O. S. Adams, Beatrice Aitchison, G. F. Alrich, R. N. Ashmun, Clara L. Bacon, G. A. Bingley, J. G. Burke, G. R. Clements, Abraham Cohen, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, Harry Gwinner, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, Florence M. Mears, Eugenie M. Morenus, W. K. Morrill, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, Jr., W. F. Reynolds, J. N. Rice, H. M. Robert, Jr., R. E. Root, W. F. Shenton, J. H. Taylor, John Tyler, W. J. Wallis, F. M. Weida, C. H. Wheeler, III, John Williamson, G. T. Whyburn E. W. Woolard, R. C. Yates, Oscar Zariski.

Preceding the afternoon session, a business meeting was held. A nominating committee, to report at the next meeting, was appointed, as follows: W. D. Lambert, chairman, A. Cohen and J. B. Scarborough. It was decided that the custom of having a luncheon provided for those attending the meetings of the Section should be abandoned, and that in the future each individual should personally bear the expense of his own luncheon, but that arrangements should be made for the group to have luncheon together. Appreciation of the hospitality accorded the Section by American University was expressed. An invitation from Professor C. H. Wheeler, III to hold the next meeting at the University of Richmond, Va., and to be guests of the University, was accepted.

At the morning session the following four papers were presented:

1. "An engineer's method of solving cubic and quartic equations" by Harry Gwinner, University of Maryland.

2. "Small vibrations of particles in systems analogous to certain organic compounds" by R. C. Yates, The Johns Hopkins University.

3. "Integral powers of Bazin's matrix" by J. Williamson, The Johns Hopkins University.

4. "Some naval tactics in analytic geometry" by C. H. Rawlins, Jr., Postgraduate School, U. S. Naval Academy.

An abstract of the second of these papers follows:

2. The dynamics of various systems of particles were studied with the purpose of determining the characteristic properties of the homopolar chemical bond. The equations of motion were set up and fundamental frequencies found and compared with Raman data of representative organic compounds. Certain proportionality (force) constants were calculated, and resulting wave numbers for

CO_2, CH_2Cl_2, C_2H_5OH, C_6H_6, C_3H_6, CH_3CHO, and CH_3CN were found to be in good agreement with observed data.

After luncheon the unique library of rare old mathematical books in the possession of American University was open for inspection. A special exhibit of many of the works of particular interest had been arranged by Professor Shenton, who presented much interesting information about the library.

The afternoon session was devoted to an excellent expository lecture on "The modern trend in geometry, and its connection with physical theories" by Professor F. D. Murnaghan, of The Johns Hopkins University. Professor Murnaghan traced the development of geometry from Euclid, through Riemann and Levi-Civita, to the most recent contributions of Cartan; outlined the evolution of physical theories from the ancient period of anthropomorphism and personification, through the period of mechanical models, to the present era of abstract mathematical representations; and pointed out the role of modern geometry in the construction of an absolute mathematical representation which shall be independent of the coordinate-language that is used.

EDGAR W. WOOLARD, *Secretary*

THE SEVENTEENTH ANNUAL MEETING OF THE KANSAS SECTION

The seventeenth annual meeting of the Kansas Section of the Mathematical Association of America was held in the High School Building, Topeka, on Friday, January 24, 1931. The morning session was a joint meeting with the Kansas Association of Mathematics Teachers, and Miss Mary Kelly, East High School, Wichita, President of the Kansas Association, presided. In the afternoon the Section met in separate session, Professor Emma Hyde, Kansas State Agricultural College, chairman of the Section, presiding.

There were seventy-two persons in attendance, including the following twenty-five members of the Association: C. H. Ashton, R. W. Babcock, Wealthy Babcock, Florence Black, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, A. J. Hoare, Emma Hyde, W. C. Janes, V. V. Latshaw, C. F. Lewis, U. G. Mitchell, Arthur Ollivier, O. J. Peterson, A. W. Philips, J. J. Quinn, B. L. Remick, Ethel A. Rumney, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White .

The following officers were chosen for the coming year: Chairman, Professor J. J. Wheeler, University of Kansas; Vice-chairman, Professor O. J. Peterson, Kansas State Teachers College, Emporia; Secretary-Treasurer, L. T. Dougherty, Kansas City Junior College.

At the forenoon session, the following program was presented:

1. "A geometric development of the laws governing the fundamental operations with positive and negative numbers" by R. D. Daugherty, Kansas State Agricultural College.

2. "The newer type of secondary mathematics" by John A. Swenson, Wadleigh High School, New York City.

3. "The unit in mathematics" by E. R. Breslich, University of Chicago.

At the luncheon between sessions, Professor Hyde acted as toastmaster, and Professor U. G. Mitchell, of the University of Kansas, spoke to the subject, "What we would teach."

At the session in the afternoon, the following papers were presented:

1. "A solution of the difference equation: $f(x+1)=\Gamma(x)\cdot f(x)$" by Miss Corinne Hattan, University of Kansas, by invitation.

2. "Use of hyper-space in analytic investigations" by Professor R. G. Smith, State Teachers College, Pittsburg, Kansas, by invitation.

3. "Applications of mathematics to stellar astronomy" by Professor R. W. Babcock, Kansas State Agricultural College.

Abstracts of these papers follow:

1. Miss Hattan defined a function $G(x)$ by the difference equation and the value of the function at one point. The zero points of the function were found. Weierstrass's factor theorem was used to obtain a general form of a function having these zero points. The function was then limited to a form satisfying the two conditions of the definition. This method expressed $G(x)$ as an infinite product. A simpler product was obtained when the independent variable was a positive integer. Some other forms and properties of the function were stated without proof, and a graph was drawn for real values of the independent variable.

2. The principal part of this paper was devoted to the use of the projective differential geometry of curves in hyper-space in the determination of a canonical form for ordinary linear homogeneous differential equations of order $n+1$; $n\geqq 4$.

3. The mathematics involved in the computations of stellar astronomy is very simple, and is thus in striking contrast to the work done by astronomers in locating Neptune and Pluto. The laws used are the quantitative expression of results of terrestrial laboratory experiments, or direct observations, such as the inverse square law of intensity of illumination, Kepler's third law modified by Newton, and Doppler's principle expressed in equation form. Combining such laws as these, masses of the planets, sun, and stars may be computed. From the parallax of nearby stars, trigonometrically computed, diameters of distant stars of the same class may be found, which lead to specific gravities as high as 50,000 in some cases. These extraordinary results, due to observation of spectral lines, have been checked within 5 per cent by theoretical results from the theory of relativity.

LUCY T. DOUGHERTY, *Secretary*

ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Fort Worth at Texas Christian University on January 31, 1931. During the morning session papers of a purely mathematical nature were presented, while during the afternoon session papers dealing with the problems of teaching mathematics were given and discussed. The morning session was presided over by Professor L. R. Ford, who also presided over the afternoon session until after the election of officers. Professor H. E. Bray then assumed charge of the meeting.

The Texas Section has a standing committee to consider problems connected with the teaching of mathematics in the colleges and secondary schools of the state and to make recommendations for the improvement of instruction in mathematics in the secondary schools. This committee, which is co-operating with the State Department of Education, was called together by the chairman, Professor F. W. Sparks of Texas Technological College, on the evening of January 30. Mr. H. F. Alves of the State Department of Education met with the committee. A number of questions pertaining to the mathematics curriculum of the secondary schools and to the preparation of teachers of mathematics in the secondary schools were discussed and recommendations were made. These recommendations were presented to the general meeting of the Section during the afternoon session.

Those attending the meeting were the guests of Texas Christian University at a dinner given in the University dining room on the evening of January 31. Professor Sherer acted as toastmaster and introduced Dean C. D. Hall of Texas Christian University, who in turn introduced the principal speaker of the evening, President E. O. Lovett of The Rice Institute.

Forty-five persons were at the morning session and approximately sixty were in attendance during the afternoon session, including the following thirty-one members of the Association: B. T. Adams, L. M. Blumenthal, H. E. Bray, Myrtle C. Brown, J. W. Cell, Savannah L. Cross, Alice C. Dean, Bertha K. Duncan, Nat Edmonson, Jr., H. J. Ettlinger, G. C. Evans, G. W. Evans, C. E. Ferguson, L. R. Ford, C. A. Gilley, J. W. Harrell, Deborah M. Hickey, C. M. Howard, E. O. Lovett, A. A. McSweeney, C. A. Murray, Lela Oxsheer, J. E. Redden, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer, F. W. Sparks, Elizabeth T. Stafford, Jennie Tate, R. S. Underwood.

The following officers were elected during the afternoon session: Chairman for one year, Professor H. E. Bray of The Rice Institute; Vice-chairman for one year, Professor C. R. Sherer of Texas Christian University; Secretary-Treasurer for five years, Nat Edmonson, Jr. of Texas Technological College.

The following program was presented:

I. *Morning Session*

1. "Universal quadratic null forms in five variables" by Professor R. S. Underwood, Texas Technological College.

2. "Isothermic harmonic functions" by E. F. Beckenbach of The Rice Institute, by invitation.

3. "A general Vandermondian determinant" by Dr. Elizabeth T. Stafford, Texas Technological College.

4. "Note on the double points of certain plane curves" by Dr. L. M. Blumenthal, The Rice Institute.

5. "A proof of a theorem of Popovici" by Professor H. E. Bray, The Rice Institute.

6. "An elementary treatment of the definite integral" by Professor H. J. Ettlinger, The University of Texas.

Abstracts of the papers follow:

1. In this paper the modifications of the theory of universal quadratic null forms in three variables necessary for a generalization to five variables were discussed.

2. In this paper properties of harmonic functions in three dimensions similar to those expressed by the Cauchy-Riemann equations for two dimensions were obtained.

3. Dr. Stafford gave in this paper a method of evaluating a general Vandermondian determinant which arose in her thesis for the doctorate at the University of Wisconsin.

4. In this paper theorems analogous to those obtained by Halphen (Bull. de la Société Math. de France, 1882, p. 162) for the plane sextic are deduced for all curves of order $2n$ represented by a web of curves $\rho^2+2\sigma\rho\tau+p\tau^2+2r\rho=0$ where σ, τ are of degree n and ρ of degree $2n$. The general theory is applied in detail to the case of the octavic, where it is shown that if two quadruple points and seven double points are assigned arbitrarily, the locus of other possible double points is a curve of order 13 with seven-fold points at the two assigned double points and triple points at the double points of the octavic. It is seen that the locus is, then, composite, consisting of a line and a curve of order twelve. Without the aid of Cremona transformations of which no use is made in this paper, it is shown that in the pencil of octavics with two assigned quadruple points, seven arbitrarily assigned double points, and one double point assigned on the nodal locus, there are not more than 47 octavics with a ninth double point.

5. The theorem is as follows: If $P(x)$ is a polynomial whose roots, $x_1, x_2, \cdots x_n$, are all real and non-negative; if the roots of the derivative $P'(x)$ are $y_1, y_2, \cdots y_n$; and if m is a positive integer, then $\sum_n^1 x_i^m/n \geqq \sum_1^{n-1} y_i^m/(n-1)$. The equality holds if $m=1$ or if the roots, x_i, are all equal, but in no other case. This is a natural generalization of a well known property of polynomials (corresponding to the case $m=1$), and is interesting even with the restriction to the real domain. The method of proof consists in studying the variations in the roots of $P'(x)$ when the smallest root of $P(x)$ is held fast and the other roots are increased by the same amount h. The writer is indebted to Dr. Jacques Shohat for communi-

cating to him the statement of the theorem, which was originally proposed, without proof, by M. Popovici.

6. Professor Ettlinger outlined a treatment of the definite integral designed to be within the grasp of the student taking a course in advanced calculus and at the same time to be fairly rigorous. The idea of uniform continuity is used.

II. *Afternoon Session*

1. "The place of mathematics in engineering education" by W. J. Miller, Dean of Engineering, Texas Technological College, by invitation.

2. "A modern method of teaching freshman mathematics" by Professor C. R. Sherer, Texas Christian University.

3. Report of the committee of the Texas Section.

4. "Problems in connection with the teaching of high school algebra" by H. F. Alves of the Texas State Department of Education.

Abstracts of the papers follow:

1. The value of mathematics as a tool and as a discipline for the engineering student were discussed at length. Dean Miller stressed the fact that numerous problems of design in engineering are dependent on the work of mathematicians for their solution.

2. Professor Sherer described in this paper a method of teaching freshman mathematics in use in Texas Christian University. According to this method the students are separated according to initiative and ability by means of special examinations. In the upper division each student works on his own initiative and conferences with the instructor replace formal classroom work.

3. The committee described above presented at this time its recommendations relative to the mathematics curriculum in the secondary schools and to the preparation of teachers of mathematics in secondary schools.

4. This paper was presented in connection with the work of the Committee of the Texas Section. In it Mr. Alves described the results of experiments conducted under his supervision on the teaching of high school algebra. In these experiments both the content and the method of teaching of the course in algebra were varied. Interesting and apparently favorable results were obtained.

NAT EDMONSON, JR., *Secretary*

SIXTEENTH ANNUAL MEETING OF THE OHIO SECTION

The sixteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 2, 1931, the day preceding the meetings of the Ohio College Association, with an afternoon session, dinner, and an evening session. The chairman of the Section, Professor W. G. Simon, presided at the afternoon session, which was devoted to the reading of papers. The evening session was given over to the con-

sideration of the report of a committee which is studying the mathematical situation in the high schools, and was in charge of the chairman of this committee, Professor I. A. Barnett.

Fifty-one persons registered attendance, forty-three of whom were members of the Association, namely: H. H. Alden, R. B. Allen, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, L. T. Black, M. G. Boyce, C. T. Bumer, R. S. Burington, F. E. Carr, E. H. Clarke, Rufus Crane, C. W. Dancer, D. S. Dearman, O. L. Dustheimer, T. M. Focke, B. C. Glover, Harris Hancock, R. C. Hildner, Margaret E. Jones, L. C. Knight, H. W. Kuhn, Lincoln LaPaz, F. A. Lewis, C. C. MacDuffee, Florentina Mathias, H. E. Menke, C. C. Morris, J. R. Overman, Tibor Radò, S. E. Rasor, C. E. Rhodes, Hortense Rickard, S. A. Rowland, W. G. Simon, H. E. Stelson, J. H. Weaver, R. B. Wildermuth, C. O. Williamson, B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, W. D. Cairns; Secretary-Treasurer, Rufus Crane; Member of executive committee, H. M. Beatty; Member of program committee, B. F. Yanney.

It is expected that the next meeting will be held at the Ohio State University on Thursday, April 7, 1932.

The following six papers were presented:

1. "Some doubts about the content of elementary courses in Calculus" by the chairman of the Section, Professor W. G. Simon, Western Reserve University.

2. "Linear differential equations" by Professor I. A. Barnett, University of Cincinnati.

3. "Small sample theory" by Professor C. C. Morris, Ohio State University.

4. "University publications and university presses" by Professor Harris Hancock, University of Cincinnati.

5. "Some algebraic limits and their geometric interpretation" by Professor E. H. Clarke, Hiram College.

6. "The problem of Plateau" by Professor Tibor Radò, Ohio State University.

Abstracts of these papers follow:

1. In view of the dissatisfaction with the work in mathematics done by the secondary schools, Professor Simon feels that college teachers should examine carefully the courses in college mathematics. He feels that the secondary schools have emphasized technique unduly; and after examining recent texts in calculus, he fears that students who take a first course in calculus acquire little more than a knowledge of its technique. He feels that the course should be reconstructed in such a way that more stress will be laid on fundamental principles.

2. It was shown in this paper that there exists a transformation of the Volterra kind which takes any linear homogeneous differential of the second order into one of a similar type. The formulas for the kernel of the Volterra transfor-

mation were derived in such a way that the resolvents of a certain class of kernels may readily be computed.

3. Small sample theory had its origin in a paper by Student which appeared in Vol. VI of *Biometrika* in 1908. Beginning with this paper, Professor Morris traced the development of the theory up to the present time. Solutions of typical examples served to show the difference in results obtained when formulas from the old theory and the new, were used. Special emphasis was placed upon Fisher's use of n-dimensional geometry in generalizing Student's results. The new statistical term, "likelihood," was defined and examples given of its use. Attention was called to the fact that the making of quality control charts, based upon small sample theory, marks a new epoch in industrial management.

4. For convenience Professor Hancock makes the first period of American mathematical publications begin with Nathaniel Bowditch's translation of Laplace's *Mécanique Céleste* in 1829–1839 and end with Sylvester's appearance at Johns Hopkins University in 1876. His second period extends up to the beginning of the present century. Roughly speaking, after enumerating important publications, Professor Hancock estimates that twenty-five thousand dollars would have paid for the publications of the first period which covers fifty years, and that a like sum would cover the expenses of publication of the second period. The first decade of the present century would have required at least that amount; while double that sum could have been profitably used for publications during the years 1910–1920; and twenty-five thousand dollars would be added to that amount for the decade just past. Judging from the increasing interest in mathematical studies, the rapidly growing number of universities that are offering courses in this subject, and the extraordinary number of papers that are being offered at the various meetings of mathematicians, it is conservative to say that at least two hundred thousand dollars could be profitably spent in mathematical publications for the next decade, 1930–1940. This sum is exclusive of mathematical journals. Unless something unforeseen happens this sum should be increased from decade to decade. The first question is: How should this sum be raised? Professor Hancock holds, and gives convincing argument for his contention, that the universities should contribute it, especially those universities whose buildings programs are practically completed. The next question is: How is it best to publish these mathematical works, through a few centralized presses like the Cambridge University Press of England, or through certain of our best American commercial publishing companies? Professor Hancock regards the two questions here raised as vital in the growth and promotion of American mathematics, and of such importance that they should be further considered by the mathematicians of the country.

5. Consider the ordinary plane analytic geometry problem of finding the mid-points of the sides of a triangle with permutable coefficients for the vertices. Continue the process to a second, third, etc., set of successive mid-points. The limiting point is the centroid. The formula for the resulting coordinate is in the binomial form and by equating coefficients a simple expression can be found for

the limit of the sum formed by taking the third, sixth, ninth, etc., binomial coefficient. This may readily be extended from three points to any number.

6. The object of this paper was to characterize, from the point of view of the calculus of variations, some of the classic geometrical extremal problems, such as the problem of the shortest line between two points, the problem of geodesics, and the problem of Plateau. The last problem was considered with more details, and the recent developments on this problem were discussed as sources of new type problems for the *calculus of variations.*

RUFUS CRANE, *Secretary*

ON THE ORIGIN OF THE HINDU TERMS FOR "ROOT"

By BIBHUTIBHUSAN DATTA, University of Calcutta

In a previous issue in this Monthly,[1] the writer treated of the origin and significance of two Sanskrit terms, *mûla* and *pada*, which are generally employed in Hindu mathematics to denote the "root" of a number. It is now found that this treatment requires further amplification and elucidation, especially in the light of later investigations. There is also another need of the present article. In the previous article, the writer ventured to criticise and even to contradict certain statements of Dr. Solomon Gandz hinting at an Arabic influence in the conception of the Hindu term *mûla*. He showed conclusively that while there could be no possibility of such an influence in the origin of the *mûla*, there was, on the contrary, a greater likelihood of its influencing the conception of the corresponding Arabic term *jadhr*.[2] In accepting this view of the writer in modification of his earlier one, Dr. Gandz has, in a more recent article in this Monthly,[3] made a fresh conjecture that the source and origin of the Hindu *mûla*, together with the Greek, Hebrew, and Arabic technical terms for the "root" of a number, is to be found in the corresponding Egyptian term *ḳnbt*. The literal significance of the word *ḳnbt* has been stated to be "corner" or "angle." Since "corner" or "cornerstone" implies the "lowest part," "basis," "foundation," Dr. Gandz argues that it must be equivalent to the word *mûla*, which also carries the latter significance. The modernism in this ingenious interpretation seems to be too palpable to take it as worthy of any serious consideration. Apart from that, however, it can be shown easily that such a sweep-

---

[1] Vol. 34 (1927), pp. 420–423.

[2] The oldest Arabic writer to employ the term *jadhr* was Muhammad ibn Mûsâ al-Khowârizmî who lived about the year 825, while the term *mûla* occurs in Hindu works dating from several centuries before. It is well known that the Arabs became acquainted with Hindu mathematics about the year 770, when several Hindu treatises on mathematics and astronomy were translated into Arabic. We know also from the testimony of Rabbi ben Ezra how the Arab scholars of that time made their arithmetical operations exactly according to the Hindu methods (Smith and Ginburg, *Rabbi ben Ezra and the Hindu-Arabic Problem*, this Monthly vol. 25 (1918), p. 103.

[3] *On the origin of the term "root"* (Second article), vol. 35 (1928), pp. 67–75.

ing generalisation will be wholly unjustifiable so far at least as the Hindu origin is concerned. Indeed there is not the faintest idea of "corner" or "angle" in the conception of the Hindu terms for the "root." It may also be pointed out that the metaphor of the "corner" or "cornerstone" being the "basis" or "foundation" is unknown in the Indian culture. Hence the ancient Hindus could not possibly have considered it in their search for a technical term for the "root."

It was pointed out in the writer's previous article (p. 421) that the use of the term *mûla* in connection with the theory of numbers is found in the works of Âryabhaṭa (499). Its use has recently been traced to a much more ancient work, the *Anuyogadvāra-sūtra*,[1] a canonical work written before the beginning of the Christian era. In this there occurs the terms *varga-mûla* ("the square root"), *varga-mûla-ghana* ["the cube of the square-root," meaning $(\sqrt{a})^3$]. In the same treatise there is also reference to higher roots. We next find the term in the *Tattvārthādhigama-sūtra-bhāṣya* (or "The Commentary on the Tattvārthādhigamasūtra") and the *Jambudvīpasamāsa* of Umāsvāti,[2] who lived at Kusamapura (ancient Pāṭalīputra, near modern Patna) about the middle of the second century of the Christian era, and probably much earlier.[3] Further if it be remembered that all these are religious works dealing primarily with logical, psychological, and metaphysical conceptions and the mythology of the early Jainas, the reference to the mathematical and other scientific matters being only incidental, it must be concluded that the term *mûla* was already very familiar by their time, and must have entered the theory of numbers long before. Thus it is proved conclusively that the use of the word *mûla* in the sense of the "root" of a number is much older than the corresponding use of the Hebrew word *iqqār*.[4]

Tracing the early history of the word *mûla*, we find that it was employed only once in the *Rg-veda*,[5] where it denotes the "foot" (of an *asura* or "demon"). It occurs twice in the *Vājasaneya Saṁhitā*,[6] and in each case it implies "the root of a plant." There are a several instances of its use in the *Atharvaveda*[7] in

---

[1] *Anuyogadvāra-sūtra*, Sūtra 142. For further informations see the author's article on *The Jaina School of Mathematics*, in the Bulletin of the Calcutta Mathematical Society, vol. 21, p. 137ff.

[2] *Tattvārthādhigama-sûtra* of Umāsvāti with the *Bhāsya*, edited by K. P. Mody, Calcutta, 1903, iii, 11 (*com.*). *Jambudvīpasamāsa* of Umāsvāti appears as Appendix C of this edition. The application of the term *mûla* will be found in the 4th *Āhnika* of the work.

[3] There is a great divergence of opinions about the time of Umāsvāti. According to the tradition of the *Digambara* sect of the Jainas he lived in the year 714–798 after Srī Vīra (that is,135–219). But according to the *Svetāmbara* tradition, he lived *c.* 175 B.C. Satis ChandraVidyabhushan thinks that he flourished in the 1st century A.D.

[4] The term *iqqār* occurs in the oldest Hebrew geometry, *Mishnath ha-Middoth*, which according to Shapiro and Gandz, was written *c.* 200 A.D. But Smith suspects it to be a work of much later date, posterior probably to that of Al-Khowārīzmī (825).

[5] x, 87. 10.

[6] i, 25; xxii, 28.

[7] For instance compare: "root of a plant"—ii, 7. 3; iii, 23. 6; viii, 7. 2, 12; "root of a hair"—iii, 137. 3: "basis, foundation"— xiii, 1. 56, 57; xii. 9. 4.: "root cause"—vi, 44. 3; xii, 11. 2.

relation to the vegetable kingdom as also in the abstract sense of "lowest part," "basis," "foundation," "origin," or "cause." There is copious use of the word *mûla*, in its different meanings, throughout the other Vedic literatures including the *Brāhmaṇas* and the *Śrauta-sūtras*. It is noteworthy that in the *Śulba-sūtras* of Baudhāyana, Âpastamba, and Kātyāyana, we do not find any application of the *mûla* in any sense.

The use of the word *pada* is more common than that of the word *mûla* in the ancient Sanskrit literatures of India. It will not be possible, nor is it necessary, to write here the complete history of the word *pada*. It will suffice for the purpose of the present article to state that even in the *Rg-veda*,[1] there are found numerous instances of its use in the sense of "foot," "footsteps," "places," "direction," or "source."

The earliest application of *pada* in a technical sense is found in the *Taittirīya Saṁhitā*,[2] where it denotes a unit of measure of length employed in the construction of altars. In the *Śulba-sūtras* (the earliest of which was composed before 800 B.C.), the term *pada* implies a unit of linear measure as well as a unit of surface measure. The relation between these two kinds of units has been described by Âpastamba (*c.* 500 B.C.) thus: "By means of a unit of measure (of length) is generated a unit of measure (of surface)."[3] It implies that a unit of surface measure is the area of a small square each side of which measures a unit of length. This appears clearly from the instances given by Âpastamba in illustration of the above rule. He says: "(Here and generally elsewhere) a square (is implied); other figures (are to be meant) when expressly stated. By means of two (units of length as the side of a square is produced a surface containing) 4 (units of surface measure); by means of 3 (units of length are produced) 9 (units of area)."[4] Similarly we find in the *Kātyāyana-Śulba-pariśiṣṭa*:[5] "The unit of measure (of area) is a square; it may be other figures when expressly stated. Two units (of length) generate four units (of area); three units (of length) generate nine units (of area); four units (of length) generate sixteen units (of area)." Thus here we have the most definite evidence of the application of the word *pada* in a concrete sense, as early as the time of the *Śulba-sūtras*.

The dimensions of the altars are specified sometimes by the number of *padas* (or other units of length) in their sides and sometimes by the number of *padas* (or other units of area) contained in them. Thus we have in the *Āpastamba Śulba-sūtra*,[6] "The western side is thirty *padas* or *prakramas* long, the *prâchî* or east line(i.e., the line drawn from the middle of the western side to the middle

---

[1] "Cause," "source," iii, 55. 14, 15; "foot," i, 154. 4; viii, 64. 2; "place," i, 67. 3, i, 164. 41; etc.

[2] vi, 2. 4. 5.

[3] *Āpastamba Śulva-sūtra*, iii. 4. This work has been edited by A. Bürk with a German translation and notes and comments, together with a valuable introduction, *ZDMG*, Bds. 55 and 56.

[4] *Ibid.*, iii, 5–6.

[5] iii, 5–8.

[6] v, 1; also compare G. Thibaut, "Sulvasûtras" reprinted from the *Journal of the Asiatic Society of Bengal*, 1875, p. 9.

of the eastern side of the *vedi*) is thirty-six *padas* or *prakramas* long; the eastern side, twenty-four; this is the tradition for *vedi* at the soma sacrifices." Here *pada* obviously implies the unit of length. Again we have "Twenty-eight less thousand *padas* (are in) the *Mahāvedi* . . .".[1] "The *Sautrāmanikī vedi* (contains) 324 *padas*."[2] In these two instances *pada* certainly refers to the square plots. The similar use of *pada* is found also in the *Bṛhat Saṁhitā* of Varāhamihira (505).[3] The specifications given for the *Uttara-vedi* are not quite free from ambiguity. "The *Uttara-vedi* has ten *padas*."[4] According to Dvāraknātha Yajvā, a commentator of the *Baudhāyana Śulba-sūtra* this may mean either that there are 10 *padas* on each side, hence altogether 100 *padas* in the *vedi*; or that there are only 10 *padas* in the *vedi*.[5] Now this latter interpretation is shared by Kātyāyana and the other view by Rudradatta, the commentator of the *Āpastamba Śrauta-sūtra*. Rudradatta has also quoted the authority of an ancient sage, Bharadvāja, in support of his contention. Both Âpastamba and Baudhāyana were undoubtedly of this opinion. For in the sūtra immediately following the former has referred to an alternative specification about the *Uttaravedi*;[6] "Some say that the front should be smaller" than the rear instead of there being 10 *padas* along each side. In the opinion of Baudhāyana also the shape of the *Uttaravedi* must be a square.[7] Hence in this instance we get a kind of naming a square *vedi*, as a matter of fact of any square plot, by the number of elementary squares that may be lying on each of its side and into which it is divided. This system of naming is found to have been more explicitly followed in the early Buddhist literature.

In the Buddhist *Dīgha Nikāya*[8] and *Vinaya Pitaka*[9] (c. 500 B.C.), there is reference to two varieties of chess. One was used to be played on boards of eight times eight squares and the other on boards of ten times ten squares. In the former case the board was called the *aṭṭapada* (Sanskrit *aṣṭapada*) and in the latter case the *daśapada*. In each case, the word *pada* undoubtedly denotes a "square plot," and the system of nomenclature is evidently based, as has also been pointed out by the commentator Buddhagosa, on the number of *padas* that lie on each side of the board. These games were in existence in India long before 500 B.C.[10] This system of nomenclature must consequently have been

---

[1] v, 7.

[2] v, 9.

[3] Lii, 48, 50, 56.

[4] *Āpastamba Śrauta-sūtra*, vii, 3. 10; *Āpastamba Śulba-sūtra*, vi, 8; *Baudhāyana Śulba-sūtra*, i. 98.

[5] *The Paṇḍit*, old series, vol. x, 1875, p. 49.

[6] *Āpastamba Śrauta-sūtra*, vii, 3. 11.

[7] *Baudhāyana Śulba-sūtra*, i, 80.

[8] *Dīgha Nikāya*, ed. Rhys Davids and Carpenter, vol. I, p. 6; see also Rhys Davids's English translation of the same, *Dialogues of the Buddha* I, p. 9.

[9] *Vinaya Pitaka;* ed. H. Oldenberg, vol. II, p. 10; vol. III, p. 180.

[10] Rhys Davids & Steele, *The Pali-English Dictionary* on "Daśapada."

much older. In later times, however, especially in the *Śilpa-śāstras*,[1] plots of lands similarly divided, into smaller ones, are named according to the number of plots they contain. Thus we have such names as *catuḥṣaṣṭipada* (or 64-*pada*), *ekāśītipada* (or 81-*pada*) and *śatapada* (or 100-*pada*). In the *Bṛhat Saṁhitā*[2] we get the name *aṣṭāṣṭapada* (or eight-times eight-pada) for the first, clearly indicating that there were eight rows each consisting of eight smaller squares. Bharat Muni, an ancient authority on the *Śilpa-śāstras*, quoted by Bhaṭṭotpata,[3] has given schemes for dividing plots of triangular and circular shape into 64 and 81 smaller plots but has still retained the usual old names.

We shall now proceed to discuss the source and origin of another term which is inseparably connected with the *mûla* and *pada*. It is the *varga*. It has been stated in the previous article (p. 421) that the Hindu mathematicians do not attempt a formal definition of the term *mûla* (or *pada*); they simply take it as indicative of an operation reverse to that implied by the term *varga*. For this latter term, of course, they have always a clearly formulated definition. The source and origin of the term *varga* is, therefore, bound to throw much light on the origin of the term *mûla* (or *pada*). The word *varga* literally means "rows," or "troops" (of similar things). But in the mathematical treatises of the Hindus it ordinarily denotes the square power and also the square figure. How the word *varga* came to be used in that sense has been clearly indicated by Thibaut. "The origin of the term," says he, "is clearly to be sought for in the graphical representation of a square, which was divided in as many "vargas," or troops of small squares, as the side contained units of some measure. So the square drawn with a side of five *padas'* length could be divided into five *vargas*, each containing five small squares, the side of which was one *pada* long."[4] This explanation of the origin of the term *varga* is confirmed by a passage of the *Śulba-sūtras*: "As many measures (units of some measures) a cord contains, so many troops or rows (of small squares) it produces (when a square is drawn on it)."[5] Thus it is certain that the term *varga* has a purely concrete concept in its origin. So the reverse process which means to find one out of the *vargas*, or the "basis" or "cause" of the *vargas*, must carry nothing but a similar concrete concept.

Another Sanskrit term is sometimes used to denote the square power. It is *kṛti*, literally meaning "doing," "making," or "action." This term is very expressive, for it carries with it the idea of specific preformance,—probably graphical representation. Hence the term *Kṛti-mûla* means "the cause of action." It is noteworthy that Umāsvāti has once used the term *prakṛti* for the square root.[6] The Sanskrit word *prakṛti* means "cause" or "original source."

[1] Binode Behari Dutt, *Town planning in Ancient India*, Calcutta, 1926.

[2] Lii. 55; also compare Lii, 42.

[3] Commentary on *Bṛhat Saṁihṭa*, Lii, 56.

[4] Thibaut. *Śulba-sûtras*, p. 48.

[5] *Âpastamba Śulba-sūtra*, iii, 7; *Kātyāyana Śulba-pariśista*, iii, 9; Compare—Thibaut, *Sulba-sūtra* p. 48. Also compare *Bṛhat Saṁhitā*, Lii, 49, where the word *varga* denotes a group of small squares (*padas*) in the *Ekāśīti-pada*.

[6] *Tattvārthādhigama-sūtra*, iii, 11 (*Com.*).

The oldest Hindu term for the square root of a number was *karaṇī*. This term is still retained in the Hindu mathematics, but its application has been restricted. It is now used in connection with the square roots of those numbers, the root of which cannot be obtained exactly. Or, in other words, it now indicates a surd. But in the beginning it had a wider scope and was used to denote the square root of any number.[1] Umāsvāti (*c.* 150) treated the terms *mûla* and *karaṇī* as synonymous.[2] Even in the *Āpastamba Śulba-sūtra*,[3] we find the use of term *catuṣkaraṇī*, meaning $\sqrt{4}$. The origin of the term *karaṇī* is interesting. Literally it means the "making one" or "producing one." It seems to have been originally employed to denote the cord used for the measuring of a square.[4] It then meant the side of any square,[5] and is so called because it makes a square (*caturasra-karaṇī*).[6] Thus we see that the term *karaṇī* which in later mathematics has only the highly abstract idea of a surd number, was in the beginning a concrete concept. Hence the original term for the square root was a concrete concept. We also learn from it why when in later times there arose the need of a new term for the square root the Hindus chose the word *mûla* ("cause," "basis," or "foundation") to replace *karaṇī*.

From what has been stated above it will be sufficiently clear that there is not the slightest trace of the idea of the "corner" or "angle" in the origin and source of the Hindu terms for the "root" of a number. Even the word *mûla* or the word *pada* is not known to have ever been used in Sanskrit in that sense.[7] Hence the conjecture of Dr. Gandz, about the influence of the Egyptian term *knbt* (meaning literally "corner" or "angle") on the conception of the Hindu term *mūla* is wrong.

---

[1] Thibaut rightly apprehended this to be the original meaning of *karaṇī* but he was somewhat doubtful. "*Karani* meant at first the side of any square, after that possibly the square root of any number. Possibly I say, for in reality the mathematical meaning of *karani* was restricted. It was not used to denote the square roots of those numbers, of which the root can be exactly obtained, but only of those the root of which does not come out exact . . ." (*Sulba-sūtra*, p. 48). As will appear from the instances cited below, his doubt was due to insufficient knowledge in this matter. It is also noteworthy that we do not find in the *Śulba-sūtras*, any other technical term for the root of a number, though the operation of extraction of roots was not uncommon in them.

[2] Compare *Tattvārthādhigama-sūtra bhāsya* (iii, 11)—"viskambha krterdaśagunasya *mūlam* vrttapariksepa" and *Jambudrīpasamāsa* (iv) "viskambhavargadasaguna*karanī* vrttapariksepa." There is also another like passage which contains the word *karanī*,—"vikkambhavagga dahaguna *karanī* vaṭṭa pariraya hoi," I cannot exactly locate it, but it must be from one of the mūla-sūtras of the Jainas and hence very ancient.

[3] ii, 6.

[4] Thibaut, *Śulba-sūtras.*

[5] *Āpastamba Śulva-sūtra*, ii, 4, 5; xii, 1, 5, 9; xiii, 1, 11; xv, 10; xviii, 5. *Baudhāyana Śulba-sūtra*, i, 53–56, 57, 60. Thibaut, *Śulbat-sūtra*, pp. 78, 49.

[6] *Baudhāyana Śulba-sūtra*, i, 60. In the *Āpastamba Sulba-sūtra* occurs the term *pakṣakaraṇī* which literally means "The maker of the wing."

[7] In the *Rgveda*, we find though in rare cases, the application of the compounds *nava-pada* (i, 164. 41; viii, 76. 12) and *nava-srakti* (viii, 76. 12) in the same sense, meaning the "nine directions" of the heaven. Hence in these cases the words *pada* and *srakti* have equivalent significance. Though

CONSTRUCTION OF A RATIONAL CANONICAL FORM FOR A LINEAR TRANSFORMATION

By ALBERT A. BENNETT, Brown University

The existence and properties of a rational canonical form for a linear transformation have proved of interest to various writers[1] and have significance for applications to difference and differential equations. A characterization of a common canonical form is provided by each of these writers here mentioned. The linear transformation in each case involves only a finite number n of linearly independent variables, but there are an infinite number (∞^n) of possible "chains" as the term is used by Dickson. Even in this problem one may distinguish between an existence proof and a construction. Both Kowalewski and Dickson call upon the reader to take a chain of maximum possible length. It is true that no chain can have a length greater than the given n. On the other hand for a given linear transformation the actual maximum attainable length may be less than n. So far as existence is concerned there is no doubt that among the ∞^n chains there is at least one of maximal length. It would seem desirable however (and to the school of Brouwer and Weyl, necessary) to provide the reader with some actual method of recognizing when such a chain is obtained and of obtaining such a chain by a finite sequence of steps. This constructive feature is missing in their discussions. Lattès adopts a somewhat different procedure. Here the reader is to determine the polynomials with rational coefficients which are factors of a given polynomial. This suffers from the same constructive failure. While the problem is rational in the sense that the desired factors are rational the operation of determining such factors is at least not elementary. The reader may be unable to ascertain whether a given polynomial is or is not reducible in the domain of polynomials with rational coefficients.

It seems of interest that by altering the sequence of steps, involving otherwise slight modifications of Dickson's discussion and by a corresponding change in definition, an effective construction may be provided by which in a finite number of steps, each strictly rational (not involving factorization or the choice of an unknown element to satisfy given conditions), the same canonical form may be secured. It is this construction that is here provided. The notation employed will be essentially that of Dickson, and a portion of the language here used at the start will be quoted verbatim from this source.

---

in Sanskrit the word *srakti* commonly denotes the "corner" or "angle," those instances can hardly be relied upon in support of the theory of Dr. Gandz. They are mentioned here for what they are worth.

[1] S. Lattès, Faculté des Sciences de l'Université de Toulouse Annales, (3) vol. 3 (1914), pp. 1–84 (with references to Burnside, Laguerre, Frobenius, etc.). Lattès considers only the case of a non-vanishing determinant.

G. Kowalewski, Leipzig, Berichte, vol. 69 (1917), pp. 325–335.

L. E. Dickson, *Modern Algebraic Theories* (1926), pp. 89–98.

Let L be any linear transformation

$$L\colon T_i = \sum_{j=1}^{n} a_{ij}t_j \qquad (i = 1, \cdots, n)$$

with coefficients in any field F.

Let x_i be any homogeneous linear function of $t_1, \cdots, t_n$, whose coefficients belong to F and are not all zero. Let X_1 denote the corresponding function of $T_1, \cdots, T_n$. From x_1 we may generate an *absolute chain*,

$$(1) \qquad X_1 = x_2,\ X_2 = x_3, \cdots X_{a-1} = x_a,\ X_a = [x_1, \cdots, x_a],$$

where the bracket denotes a homogeneous linear function of $x_1, x_2, \cdots, x_a$ with coefficients in F. We shall call a the length of the chain, or say that the chain has a links, and shall refer to x_1 as the *leader* of the chain. If $X_1 = mx_1$ where m is in F, (but may be zero), the chain has one link only and $a = 1$. If $X_1 \neq mx_1$, we write x_2 for X_1. Then X_2 is formed, being the same function of $T_1, \cdots, T_n$, that x_2 is of $t_1, \cdots, t_n$. If X_2 is a linear function of x_1 and x_2 (perhaps identically zero), we have (1) for $a = 2$; in the contrary case we write x_3 for X_2 and have three linearly independent functions x_1, x_2, x_3. Proceeding in this manner each x_1 gives rise through L to a uniquely determined absolute chain of a linearly independent functions $x_1, x_2, \cdots, x_a$, with coefficients in F. It may be noted that the process of testing linear dependence involves only the evaluation of a numerical determinant and is strictly rational.

For the absolute chain (1) it will be convenient to indicate more explicitly the expression $X_a = [x_1, \cdots, x_a]$. The following notation may be used:

$$(2) \qquad X_a = [x_1, \cdots, x_a] = \sum_{i=1}^{a} a_i x_i.$$

Within the absolute chain no subscript greater than a is required. However one may define arbitrarily x_q for every natural number q, by the recursion relation,

$$(3) \qquad x_{q+1}(t_1, \cdots, t_n) = x_q(T_1, \cdots, T_n).$$

It may happen that for every q greater than a certain number, x_q vanishes identically.

The equation called the *characteristic condition* for the chain of x_1,

$$(4) \qquad s_{a+1} = \sum_{i=1}^{a} a_i s_i,$$

is satisfied not only for $s_1 = x_1$, but also (for each $q \geqq 1$) by $s_1 = x_q$, since the substitution L is linear. For example,

$$x_{a+2} = \sum_{i=1}^{a} a_i x_{i+1},$$

since this may be written as

$$L(X_a) = \sum_{i=1}^{a} a_i X_i,$$

which is a consequence of (2).

For $n>a$, let y_1 be any homogeneous linear function of the initial variables, t_i, with coefficients in F, which is linearly independent of $x_1, \cdots, x_a$. Let Y_1 denote the corresponding function of the variables T_i. We may then generate from Y_1 a *relative chain* of b links, relative to the absolute chain of x_1, expressed by

$$(5) \qquad Y_1 = y_2,\ Y_2 = y_3, \cdots Y_{b-1} = y_b,\ Y_b = [x_1, \cdots, x_a, y_1, \cdots, y_b],$$

where $x_1, \cdots, x_a, y_1, \cdots, y_b$ are linearly independent. Indeed if $Y_1 = [x_1, \cdots, x_a, y_1]$, we have (5) for the case $b=1$. In the contrary case, write y_2 for Y_1. Then if $Y_2 = [x_1, \cdots, x_a, y_1, y_2]$ we have (5) for $b=2$. In the contrary case write y_3 for Y_2, and so forth.

The term "relative chain" may be extended to apply to a chain relative to a given number of suitably related independent absolute chains as will be discussed presently.

An absolute chain,

$$(6) \quad Y_1 = y_2,\ Y_2 = y_3, \cdots, Y_{b-1} = y_b,\ Y_b = [y_1, \cdots, y_b] = \sum_{i=1}^{b} b_i y_i,$$

is said to be *canonically consequent* to an absolute chain,

$$(1) \qquad X_1 = x_2,\ X_2 = x_3, \cdots, X_{a-1} = x_a,\ X_a = [x_1, \cdots, x_a] = \sum_{i=1}^{a} a_i x_i$$

if and only if

(i) the elements of the set $x_1, x_2, \cdots, x_a; y_1, y_2, \cdots, y_b$ are linearly independent, and also

(ii) y_1 satisfies the characteristic condition of the chain of x_1.

The final canonical form satisfies the following theorem.[1]

Theorem: *By the introduction of new variables which are linearly independent homogeneous linear functions of the initial variables t_i with coefficients in the field F, any linear transformation L with coefficients in F may be reduced to a canonical form defined by a sequence of absolute chains*

---

[1] In the other discussions the length of chain is made the defining feature.

$$
(7)\qquad \left\{\begin{array}{l}
X_1 = x_2,\ X_2 = x_3, \cdots, X_{a-1} = x_a,\ X_a = \sum_{i=1}^{a} a_i x_i,\\
\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\\
Y_1 = y_2,\ Y_2 = y_3, \cdots, Y_{b-1} = y_b,\ Y_b = \sum_{c=1}^{b} b_i y_i,\\
\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\\
Z_1 = z_2,\ Z_2 = z_3, \cdots, Z_{c-1} = z_c,\ Z_c = \sum_{i=1}^{c} c_i z_i,\\
\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot
\end{array}\right.
$$

where $x_1, x_2, \cdots, x_a; \cdots; y_1, y_2, \cdots, y_b; \cdots; z_1, z_2, \cdots, z_c; \cdots;$ *are linearly independent, and where each given chain of the sequence is canonically consequent to every preceding chain in the sequence.*

The method of procedure is a ruthless one in which if at any stage an absolute chain is obtained of length greater than some previously tentatively accepted chain, this previously obtained chain and all subsequent to it in the tentative system are rejected. Each step in the construction adds a new chain or replaces an old one by a longer chain. The total sum of the lengths of all the chains in a canonical system is at most n, the original number of arguments. The process must therefore terminate.

The process of choosing canonical variables in place of such original variables as are not in the final canonical system is one by which explicit linear combinations of variables, t_i, with coefficients in F are introduced. As each new linear combination is introduced, it takes the place of an original variable t_i, which is no longer needed in the system. The original variable when once dropped need never be taken up again, whether or not the new variable is eventually retained. At each stage we have n linearly independent arguments.

Suppose for the purpose of induction, we have at a given stage the following n linearly independent linear functions of t_i, with coefficients in F,

$$x_1, x_2, \cdots, x_a; \cdots; y_1, y_2, \cdots, y_b; \cdots; z_1, z_2, \cdots z_c; u, \cdots, w,$$

where $a+\cdots+b+\cdots+c=p$, and where the number of variables in the set, $u, \cdots, w$, is $n-p$. Suppose the first of these constitute the temporary tentative partial canonical system,

$$
(8)\qquad \left\{\begin{array}{l}
X_1 = x_2, \cdots, X_{a-1} = x_a,\ X_a = \sum_{i=1}^{a} a_i x_i,\\
\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\\
Y_1 = y_2, \cdots, Y_{b-1} = y_b,\ Y_b = \sum_{i=1}^{b} b_i y_i,\\
\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\\
Z_1 = z_2, \cdots, Z_{c-1} = z_c,\ Z_c = \sum_{i=1}^{c} c_i z_i,
\end{array}\right.
$$

and suppose also that each of the chains in this temporary partial sequence is canonically consequent to every preceding chain in the sequence. At the initial

stage one may merely assume that $p=0$, and that $u, \cdots, w$, are respectively $t_1, \cdots, t_n$. The p variables $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; \cdots; z_1, \cdots, z_c$, will be called the *classified*, and the $n-p$ variables, $u, \cdots, w$, the *unclassified* variables.

Take u and form its absolute chain. If this absolute chain exceeds in length that of any one of the tentative canonical chains, reject this shorter chain from the tentative canonical system, and relegate its variables to the unclassified set. Hence we may assume that no classified chain is shorter than the absolute chain of $u=u_1$. Form the relative chain of u_1 relative to all the classified variables. We then have

$$U_1 = u_2, \cdots, U_{h-1} = u_h, U_h = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; u_1, \cdots, u_h].$$

Three cases may be distinguished: (1) This relative chain may be an absolute chain, that is the brackets may not include any of the variables $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_b; \cdots; z_1, \cdots, z_c$ explicitly. (2) This relative chain may not be an absolute chain, (and hence will be necessarily of length less than that of the absolute chain of u_1) and also h may be such that $h<e\leqq b$, where y_e is the last element of the classified system to appear explicitly in the brackets. (3) This relative chain may not be an absolute chain but it may happen that $h\geqq e$.

In the first case, as already stated, there will be no absolute chain retained in the classified system whose length exceeds h links. In the second case, write $e=h+g$, $g>0$. Write

$$U_h = u_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + my_e + [u, \cdots, u_h], (m \neq 0).$$

Form the linear combination

$$v_1 = u_1 - my_g \quad (m \neq 0).$$

Since L is linear, we may form the chain of v_1, in place of that of u_1 and obtain for every q,

$$v_{q+1} = L(v_q) = u_{q+1} - my_{g+q}.$$

By hypothesis $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_e; u_1, \cdots, u_h$, are linearly independent. Hence also, $x_1, \cdots, x_a; \cdots; y_1 \cdots, y_e; v_1, \cdots, v_h$, are linearly independent, since each v_i introduces a new u_i explicitly. Forming v_{h+1}, we have

$$v_{h+1} = u_{h+1} - my_e = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + [v_1 - my_g, v_2 - my_{g+1}, \cdots, v_h - my_{e-1}],$$

or, combining,

$$v_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + [v_1, \cdots, v_h].$$

Now drop the unclassified variable u_1 in favor of the variable v_1 which is again an explicit linear combination of the original variables t_i, with coefficients in F. But for v_1 the last explicitly appearing bracketed argument precedes y_e. By repetitions of this process we are led either to Case 1, or shall succeed in obtaining

an unclassified variable that may be called again, u_1, for which the previous final explicitly appearing set has been entirely removed from the brackets. If this process by repeated application does not provide a new absolute chain it can only be by the occurrence of Case 3.

For the third case let $h \geqq b$, and write

$$U_h = u_{h+1} = [x_1, \cdots, x_a; \cdots; y_1, \cdots, y_{e-1}] + my_e + [u_1, \cdots, u_h] \quad (m \neq 0, e \leqq h).$$

In this case we shall show that the absolute chain of u_1 necessarily exceeds in length that of y, so that this case also may be eliminated. Since by hypothesis $x_1, \cdots, x_a; \cdots; y_1, \cdots, y_e; u_1, \cdots, u_h$ are linearly independent, the following are certainly linearly independent, namely, $u_1, u_2, \cdots, u_h, u_{h+1}, \cdots, u_{b+h-e+1}$. Indeed u_{h+1} introduces y_e for the first time and with non-vanishing coefficient m. Similarly u_{h+2} introduces y_{e+1} for the first time and with the same non-vanishing coefficient. Finally $u_{b+h-e+1}$ introduces y_b for the first time and with this same non-vanishing coefficient. Since $e \leqq h$, the subscript $b+h-e+1$ at least exceeds b, so that as desired the absolute chain of u_1 in this case exceeds in length the absolute chain of y_1. In accordance with the general directions, the chain of y_1 is then relegated to the unclassified set.

In every case therefore we are led to a new absolute chain. The previous temporary classified system is retained unless the new absolute chain exceeds in length some classified chain.

It remains only to show that the new absolute chain is canonically consequent to all the classified chains still retained or gives rise to an explicitly determined still longer chain. The process is not interrupted even if an absolute chain is obtained which requires the rejection of the entire set of hitherto tentatively adopted classified chains, for this only occurs when an absolute chain arises longer than any in the temporary classified system. A single chain and all other variables in the unclassified set is always a possible temporary classified system.

Suppose then for the sake of induction that u_1 generates an absolute chain of length not exceeding that of the chains generated by each of the classified variables, $x_1, \cdots, y_1, \cdots, z_1$, and suppose that the characteristic condition for z_1 is satisfied but that the latest of the characteristic conditions not satisfied by u_1 is that of y_1. Then by hypothesis,

$$u_{b+1} \neq \sum_{i=1}^{b} b_i u_i.$$

Form now the new linear combination $v_1 = y_1 + u_1$, and form the absolute chain of v_1. Since in particular $y_1, y_2, \cdots, y_b, u_1, \cdots, u_h$ are linearly independent and each subsequent u_q is a linear combination of u's alone, we conclude that $v_1, v_2, \cdots, v_b$, are surely linearly independent. If v_{b+1} is a linear combination of $v_1, \cdots, v_b$, say,

$$v_{b+1} = \sum_{i=1}^{b} m_i v_i.$$

we have

$$v_{b+1} = y_{b+1} + u_{b+1} = \sum_{i=1}^{b} b_i y_i + [u_1, \cdots, \mu_h] = \sum_{i=1}^{b} m_i(y_i + u_i).$$

But this can only occur if $m_i = b_i$ $(i = 1, 2, \cdots, b)$ since the y's and u's are linearly independent. But this relation is impossible by the hypothesis that

$$u_{b+1} \neq \sum_{i=1}^{b} b_i u_i.$$

Hence the absolute chain of v_1 exceeds in length that of y_1 and is to be retained while relegating y_1 to the unclassified set.

By these successive reductions a canonical system at length survives. That the system obtained is indeed the invariant system used by Dickson, follows from the fact that an arbitrary linear combination of these canonical variables, must satisfy the first characteristic condition, since each individual term satisfies this homogeneous relation. Thus the absolute chain $x_1, x_2, \cdots, x_a$ eventually obtained as leading chain of the canonical system is of maximum possible length for any linear combination of these canonical variables. But the canonical variables constitute n linearly independent linear combinations of the original n variables, t_i. Hence this chain is of the maximum attainable length for any possible linear combination of the original variables, t_i, with coefficients in F. Similar remarks hold for any linear combination of the variables $y_1, y_2, \cdots, y_b; \cdots; z_1, \cdots, z_c; \cdots$. The construction has become possible by using the characteristic conditions rather than the maximal lengths of chains, as the determining conditions.

ON THE EQUILATERAL HYPERBOLA

By J. R. MUSSELMAN, Western Reserve University

Some time ago Professor P. Appell[1] proposed the following problem: *Let* A_1, B_1, C_1, D_1 *be four points on an equilateral hyperbola, the orthocenters of* $B_1 C_1 D_1$, $C_1 D_1 A_1$, $D_1 A_1 B_1$, $A_1 B_1 C_1$ *lie also on the curve. Thus from four points* A_1, B_1, C_1, D_1 *we derive four points* A_2, B_2, C_2, D_2. *This process can be continued indefinitely; under what conditions will* A_n, B_n, C_n, D_n *coincide entirely, or in part, with* A_1, B_1, C_1, D_1?

Appell suggested a method of attack which was used by Emch.[2] The problem was solved in other ways by Goormaghtigh[3] and Ser.[4] The purpose in bring-

[1] Nouvelles Annales de Mathématiques, vol. 18 (1918), pp. 41–42.
[2] The American Mathematical Monthly, vol. 26 (1919), pp. 63–65.
[3] Nouvelles Annales de Mathématiques, vol. 18 (1918), pp. 445–448.
[4] Nouvelles Annales de Mathématiques, vol. 19 (1919), pp. 220–228

ing up the problem again is to discuss the case in which the points A_2, B_2, C_2, D_2 fall in other orders on A_1, B_1, C_1, D_1; and also to point out an interesting case in which some of the points are not real.

If we take the equation of the equilateral hyperbola $xy=c^2$ in a parametric form as

$$x = ct, \quad y = c/t \tag{1}$$

and let the parameters of A_1, B_1, C_1, D_1 be t_1, t_2, t_3, t_4 respectively, then the parameters of A_2, B_2, C_2, D_2 are $-t_1\sigma^{-1}$, $-t_2\sigma^{-1}$, $-t_3\sigma^{-1}$, $-t_4\sigma^{-1}$ respectively where $\sigma=t_1t_2t_3t_4$. The parameters of A_3, B_3, C_3, D_3 are $t_1\sigma^2$, $t_2\sigma^2$, $t_3\sigma^2$, $t_4\sigma^2$ and in general the parameters of A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} are

$$(-1)^n t_1\sigma^k, \quad (-1)^n t_2\sigma^k, \quad (-1)^n t_3\sigma^k, \quad (-1)^n t_4\sigma^k, \tag{2}$$

respectively, where $k=[(-3)^n-1]/4$. Hence if A_{n+1} should coincide with A_1

$$\begin{aligned} (-1)^n t_1\sigma^k &= t_1 \\ \text{or } (-1)^n\sigma^k = 1; \quad k &= [(-3)^n - 1]/4. \end{aligned} \tag{3}$$

For real solutions we have the well-known cases: $n=1$, $\sigma=-1$, which states that the given four points A_1, B_1, C_1, D_1 must themselves be orthocentric; $n=2$, $\sigma=1$, which arises when the four points lie also on a circle.

Can A_2, B_2, C_2, D_2 coincide with B_1, A_1, D_1, C_1? This would require

$$-t_1\sigma^{-1} = t_2, \quad -t_2\sigma^{-1} = t_1, \quad -t_3\sigma^{-1} = t_4, \quad -t_4\sigma^{-1} = t_3, \tag{4}$$

which equations have the solution $t_2=-t_1$, $t_4=-t_3$ and $\sigma=1$. Hence the given four points must lie on a circle with A_1 and B_1, C_1 and D_1 at opposite ends of two diameters of the hyperbola. The center of the circle is at the origin and the square of its radius is $c^2(t_1^2+t_3^2)$. The four given points form a rectangle whose area is $2c^2(t_1^2-t_3^2)$. Now the tangents at A_1 and B_1 meet in a point whose coordinates[1] are

$$2ct_1t_2/(t_1+t_2), \quad 2c/(t_1+t_2). \tag{5}$$

For the above particular choice of parameters for A_1, B_1, C_1, D_1, the four tangents at these points become two sets of parallel lines which form a rhombus whose area is $16\,c^2/(t_1^2-t_3^2)$. The product of the areas of the rectangle and rhombus is $32c^4$. Hence we have the theorem: *If A_2, B_2, C_2, D_2 coincide with B_1, A_1, D_1, C_1 the given four points lie on a circle with A_1 and B_1, C_1 and D_1 at opposite ends of two diameters of the hyperbola. Hence the product of the area of the rectangle which they form, and the area of the rhombus formed by the tangents to the hyperbola at these points, is a constant.*

Naturally the four points A_1, B_1, C_1, D_1 can be so chosen that A_2, B_2, C_2, D_2 will coincide with C_1, D_1, A_1, B_1, or with D_1, C_1, B_1, A_1. No other arrangements are possible for real solutions.

---

[1] Loney, *Coordinate Geometry*, p. 300.

For the next theorem it is necessary to use the fact that if A^1 be any point on an equilateral hyperbola, there are four circles through A^1 and the center of the hyperbola O which touch the hyperbola at points M_1, M_2, M_3, M_4 respectively.[1] These last four points lie on a circle through O, with center at A, the point on the hyperbola diametrically opposite to A^1.

If M_1^1 be the point diametrically opposite to M_1, then A^1, B^1, M_1, M_1^1 are an orthic set of points, B^1 is the further intersection of the circle, through A^1 and O, and tangent at M_1, with the equilateral hyperbola. Hence the line A^1B^1 is perpendicular to $M_1M_1^1$ and cuts OM_1^1 at K which is the midpoint of OM_1^1. Hence it is easy to construct a circle tangent at any point of an equilateral hyperbola. To construct a circle tangent at M, produce OM to M^1, erect a perpendicular bisector of OM^1 which cuts the hyperbola in two points. The circle through these two points and the center O will be tangent to the equilateral hyperbola at the point M.

Let us look further at these points M_1, M_2, M_3, M_4. If we call their parameters t_1, t_2, t_3, t_4 respectively; then A the center of the circle passing through them will have for its parameter $\sigma_1/2$, where $\sigma_1 = t_1+t_2+t_3+t_4$. The parameter of A^1 will be $-\sigma_1/2$. Since there are four circles through A^1 tangent to the hyperbola at the points M_1, M_2, M_3, M_4 respectively, there will be four further points of intersection B_i^1 $(i=1, 2, 3, 4)$ whose parameters are $-2/t_i^2\sigma_1$ respectively. Now the coordinates[2] of the center of a circle passing through any four points of a hyperbola with parameters t_1, t_2, t_3, t_4 are $c\sigma_1/2$, $c\sigma_3/2$ where $\sigma_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4$. But since A lies on the hyperbola, these coordinates must satisfy the equation of the hyperbola, hence

$$\sigma_1\sigma_3 = 4. \tag{6}$$

Since the circle also passes through O, the center of the hyperbola, if

$$\begin{aligned} \sigma_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4, \\ \sigma_2 &= 0. \end{aligned} \tag{7}$$

Hence the quartic equation, whose roots are the parameters of the four points M_i, can be written as

$$t^4 - \sigma_1 t^3 - 4\sigma_1^{-1}t + 1 = 0. \tag{8}$$

The quartic equation whose roots are the parameters of the four points B_i^1 is

$$\sigma_1^4t^4 + 32\sigma_1t^3 - 24\sigma_1^2t^2 + 8\sigma_1^3t + 16 = 0. \tag{9}$$

Now this quartic (9) is nothing else than the Hessian of (8) as can be verified. Further if we ask for those two quartics of the pencil $(8)+\lambda(9)=0$ for which the invariant I_2 vanishes, we find they are precisely (8) and (9) themselves. Hence the four points M_i are an equianharmonic set; the same is true for the four points

[1] Lemaire, Nouvelles Annales de Mathématiques, (6), vol. 11 (1927), p. 63.

[2] Loney, *Coordinate Geometry*, p. 301.

B_i^1. So we have the theorem: *If we draw through any point of an equilateral hyperbola the four circles which touch the hyperbola, these four points of contact form an equianharmonic set, the four further points of intersection form a second equianharmonic set, which set is also the Hessian of the first set.*

In general the four points B_i^1 do not lie on a circle. But if $\sigma_1=2$, the points B_i^1 will lie on the circle whose equation is

$$(x + 2c)^2 + (y + 2c)^2 = 14c^2. \tag{10}$$

The equation of the circle on the points M_i, if $\sigma_1=2$ will be

$$(x - c)^2 + (y - c)^2 = 2c^2. \tag{11}$$

The points A and A^1 are now the vertices of the equilateral hyperbola, so the whole figure can easily be constructed. If we look at the six Jacobian points given by the Jacobian of the quartic (8) when $\sigma_1=2$, we find only two are real, namely the vertices; the other four lie on the circle whose equation is

$$(x - c)^2 + (y - c)^2 + 4c^2 = 0. \tag{12}$$

In the general case there are likewise four circles through A and O which are tangent to the curve at four points M_i^1 ($i=1, 2, 3, 4$), which points lie on a circle passing through O with center at A^1. There will likewise be four points B_i, the further intersections of these circles and the equilateral hyperbola. But these points are the diametrically opposite points of the sets M_i and B_i^1, so they present nothing new. However the four points M_i furnish an illustration of a set of four points whose orthocenters are M_i^1; and the orthocenters of M_i^1 again, are the original points M_i.

A NOTE ON THE CHARACTERISTIC DETERMINANT OF A MATRIX

By ALFRED K. MITCHELL, Yale University

Several authors[1] have pointed out that when a determinant is defined by means of the generalized Kronecker delta many of the properties of determinants follow almost immediately. It is the purpose of this note to show how the properties of the generalized Kronecker delta lend themselves to the derivation of the characteristic determinant of a matrix as a polynomial in λ.

The generalized Kronecker delta has k superscripts and k subscripts and is alternating both in superscripts and subscripts. It may be denoted by

$$\delta^{r_1 r_2 \cdots r_k}_{s_1 s_2 \cdots s_k}.$$

If the superscripts are distinct from each other and the subscripts are the same

---

[1] See F. D. Murnaghan, in this Monthly, Vol. 32 (1925), p. 233; and O. Veblen, *Cambridge Tracts*, No. 24.

set of numbers as the superscripts, the value of the symbol is $+1$ or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; in all other cases its value is zero. F. D. Murnaghan[2] has shown that the generalized Kronecker delta has the properties of a mixed tensor, but we are not concerned here with its tensor properties.

We shall use, however, the following relations[3] satisfied by the generalized Kronecker delta. These relations are easily verified by counting the number of terms which occur in the indicated summations.[4]

$$\delta^{\alpha r}_{\alpha s} = (n-1)\delta^r_s, \tag{1}$$

where $\delta^r_s = 1$ if $r = s$; $\delta^r_s = 0$ if $r \neq s$.

$$\delta^{r_1\cdots r_k\alpha_{k+1}\cdots\alpha_m}_{s_1\cdots s_k\alpha_{k+1}\cdots\alpha_m} = \{(n-k)!/(n-m)!\}\delta^{r_1\cdots r_k}_{s_1\cdots s_k} \tag{2}$$

$$\delta^{\alpha_1\cdots\alpha_k}_{\alpha_1\cdots\alpha_k} = \frac{n!}{(n-k)!}\,. \tag{3}$$

Remembering that the repeated indices are summation labels we see that the expression[5]

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}E^{\beta_1}_{\alpha_1}\cdots E^{\beta_n}_{\alpha_n}$$

will be by definition the n rowed determinant $|E^r_s|$. Furthermore it is readily seen that

$$I_1 = \delta^\alpha_\beta E^\beta_\alpha = E^1_1 + E^2_2 + E^3_3 + \cdots$$

is the sum of the elements of the principle diagonal of the determinant $|E^r_s|$;

$$I_2 = \frac{1}{2!}\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}$$

is the sum of the two rowed principle minors;

$$I_3 = \frac{1}{3!}\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}E^{\beta_3}_{\alpha_3}$$

is the sum of the three rowed principle minors; etc.

Let us observe that when we have an expression of the type

$$(\delta^{\alpha_1}_{\beta_1}E^{\beta_1}_{\alpha_1} + \delta^{\alpha_2}_{\beta_2}E^{\beta_2}_{\alpha_2} + \cdots \delta^{\alpha_r}_{\beta_r}E^{\beta_r}_{\alpha_r})$$

---

[2] Bulletin of the American Mathematical Society, Vol. 31 (1925), p. 323.

[3] See O. Veblen, *Cambridge Tracts*, No. 24, p. 9.

[4] Repeated Greek letter indices will be used for summation labels, the summation being from 1 to n.

[5] See F. D. Murnaghan, in this Monthly, vol. 32 (1925), p. 233.

where, as above, the repeated indices are summation or dummy labels then obviously

$$(\delta^{\alpha_1}_{\beta_1}E^{\beta_1}_{\alpha_1} + \delta^{\alpha_2}_{\beta_2}E^{\beta_2}_{\alpha_2} + \cdots \delta^{\alpha_r}_{\beta_r}E^{\beta_r}_{\alpha_r}) = r\delta^{\alpha}_{\beta}E^{\beta}_{\alpha}. \tag{4}$$

Now by the characteristic determinant of a square matrix $\|E^r_s\|$ is meant the determinant of the matrix $\|\delta^r_s\lambda - E^r_s\|$. Thus the characteristic determinant of the matrix $\|E^r_s\|$ will be

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}(\delta^{\beta_1}_{\alpha_1}\lambda - E^{\beta_1}_{\alpha_1})(\delta^{\beta_2}_{\alpha_2}\lambda - E^{\beta_2}_{\alpha_2})(\delta^{\beta_3}_{\alpha_3}\lambda - E^{\beta_3}_{\alpha_3}) \cdots (\delta^{\beta_n}_{\alpha_n}\lambda - E^{\beta_n}_{\alpha_n}).$$

From this we obtain, on multiplying the factors in parenthesis and collecting terms, using (4) above,

$$\frac{1}{n!}\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\Big\{\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\cdots\delta^{\beta_n}_{\alpha_n}\lambda^n - n\delta^{\beta_2}_{\alpha_2}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}\lambda^{n-1} + \frac{n(n-1)}{2!}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}E^{\beta_2}_{\alpha_2}\lambda^{n-2}$$

$$+ \cdots (-1)^s\frac{n!}{s!(n-s)!}\delta^{\beta_{n-s}}_{\alpha_{n-s}}\cdots\delta^{\beta_n}_{\alpha_n}E^{\beta_1}_{\alpha_1}\cdots E^{\beta_{n-s}}_{\alpha_{n-s}}\lambda^{n-s} + \cdots (-1)^n E^{\beta_1}_{\alpha_1}\cdots E^{\beta_n}_{\alpha_n}\Big\}.$$

Remembering that $\delta^r_s = 1$ if $r = s$; $\delta^r_s = 0$ if $r \neq s$, and multiplying through by the factor $(1/n!)\ \delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ and making use of (1), (2) and (3) and the definitions of I_1, I_2, etc., we obtain

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} - I_3\lambda^{n-3} + \cdots (-1)^s I_s\lambda^{n-s} + \cdots (-1)^n I_n,$$

the characteristic polynomial of the matrix $\|E^r_s\|$.

THE HESSIAN CONFIGURATION AND ITS RELATION TO THE GROUP OF ORDER 216

By H. C. SHAUB, Washington and Jefferson College,

and

HAZEL E. SCHOONMAKER, Cornell University

The group of order 216 was first discussed synthetically by Herman Wiener,[1] who made models of cubic surfaces. Unfortunately, however, this discussion appeared only in pamphlet form and is not generally available. For this reason the following treatment,[2] believed to be new, may be of interest.[3]

---

[1] H. Wiener, *Die Einteilung der ebenen Kurven und Kegel dritter Ordnung*, Martin Schilling (1901).

[2] Suggested by Professor Virgil Snyder in a recent course in cubic curves at Cornell University.

[3] It was not until the present paper was in galley form that our attention was called to Professor R. M. Winger's article, *The Ternary Hesse Group and Its Invariants*, Univ. of Washington Publications in Mathematics, vol. 1, pp. 60–80. An excellent bibliography accompanies this article.

Using homogeneous coordinates, the equation of any non-singular plane cubic curve can be put in the Hesse form

(a) $$f \equiv x_1^3 + x_2^3 + x_3^3 + 6m x_1 x_2 x_3 = 0.$$

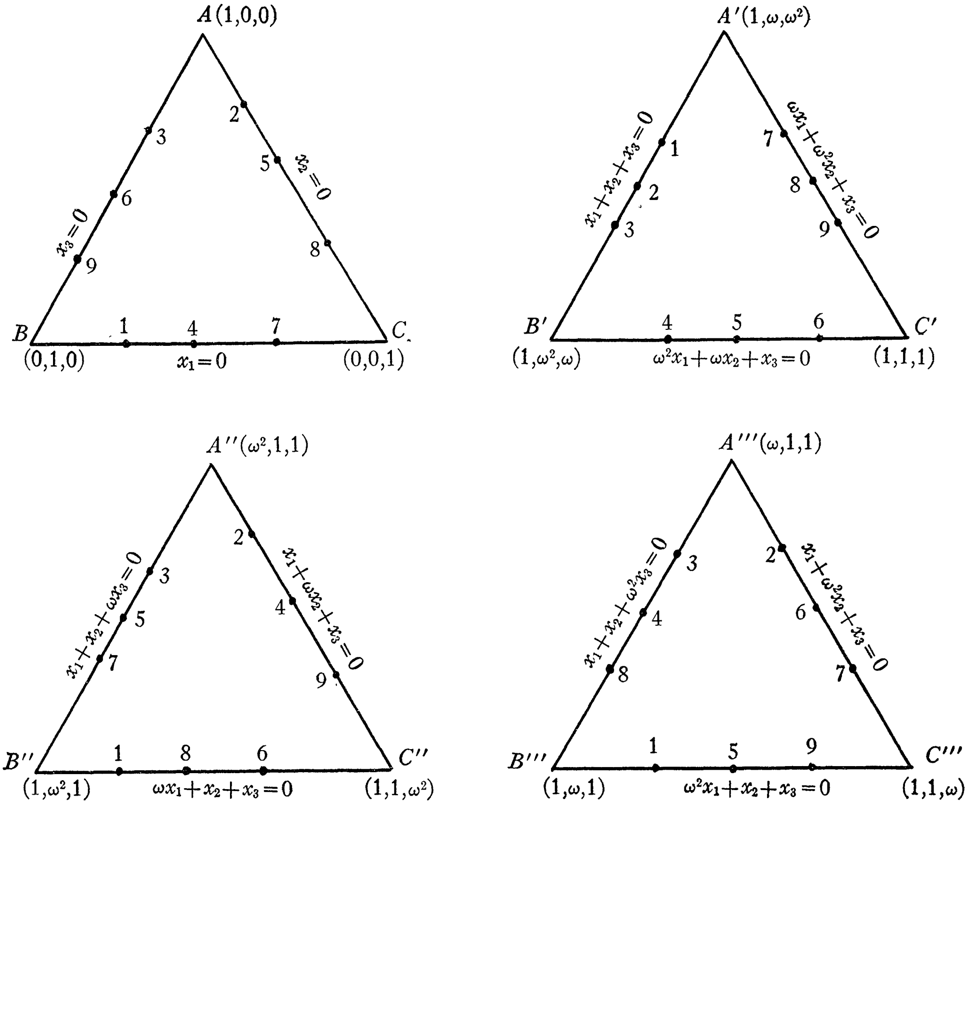
This curve has nine points of inflexion whose coordinates are

$$\begin{array}{lll} (0,\ 1,\ -\ 1), & (1,\ 0,\ -\ 1), & (-\ 1,\ 1,\ 0), \\ (0,\ \omega^2,\ -\ 1), & (\omega,\ 0,\ -\ 1), & (-\ 1,\ \omega,\ 0), \\ (0,\ \omega,\ -\ 1), & (\omega^2,\ 0,\ -\ 1), & (-\ 1,\ \omega^2,\ 0), \end{array}$$

where ω and ω^2 are the imaginary cube roots of unity.

Equation (a) contains one parameter m, and hence represents a pencil of cubics. Since the coordinates of the points of inflexion are independent of m, these points are points of inflexion for any curve of the pencil, called by Cayley a *syzygetic* pencil. For brevity we shall number these points and arrange them in the following square array:

(b) $$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$



It is easy to show that there are twelve lines, each of which contains three of these points of inflexion. These lines, called inflexional lines, are the lines containing the points in the horizontal, vertical, and diagonal rows of the above array. Each group forms a triangle, called an inflexional triangle. Any two of these triangles are in six-fold perspective; and any three are perspective from some vertex of the fourth. These triangles, with the coordinates of the vertices, the equations of the sides, and the points of inflexion which they contain are shown schematically in the accompanying figure. The notation is due to Clebsch[1] and the scheme is called the *Hessian Configuration.* This is the most famous configuration in the plane, with the possible exception of that of Desargues.

The polar conic of a point of inflexion is composite; it consists of the tangent at that point and a line called the harmonic polar of that point. Each harmonic polar contains a vertex from each of the triangles, but does not contain any of the points of inflexion. Denote by H_1 the homology which has (1) as center and its harmonic polar as axis, and by a similar notation the homologies which have the other points of inflexion as centers. Under H_1 the nine points of inflexion are permuted among themselves but in such a way that three points which are collinear remain collinear. Using the notation of groups we have, for example,

$$H_1 = (1)(23)(47)(59)(68),$$
$$H_2 = (13)(2)(49)(58)(67).$$

There are nine of these and it is obvious that H_i^2 is the identity.

Considering the product of any two of these transformations we find, for instance,

$$H_1H_2 = (132)(465)(798).$$

It is seen that the points group themselves by three's according to the rows of the original matrix (b). It is easy to verify that $(H_1H_2)^3$ is the identity. By constructing the multiplication table for the H's we find that the product of any two of them is equivalent to H_1H_i $(i=1, \cdots 9)$. Thus we have eight new operations of this type.

Next consider the product of three transformations, as

$$H_1H_2H_3 = (13)(2)(49)(58)(67).$$

But this is H_2. This illustrates the fact that if three centers are collinear, the product of the three corresponding homologies is the middle one. On the other hand the product $H_1H_2H_1$ is found to be H_3. That is, the product of three H's involving only two centers is the homology whose center is the third point collinear with the other two. If the centers are not collinear we have for example

$$H_1H_2H_4 = H_1H_1H_6 = H_1^2H_6 = H_6.$$

Indeed a product of the form $H_iH_jH_k$ can always be reduced to a single homology H_x.

[1] Clebsch-Lindemann, *Vorlesungen über Geometrie*, Bd. I, S. 507 (1876).

From the geometrical meaning of our transformation it is obvious that multiplication is associative, for instance,

$$(H_1H_2)H_3 = H_1(H_2H_3) = H_1H_2H_3,$$

and that the inverse is always the operation itself, two points being *interchanged* at each step.

Of the operations, H_i, H_iH_j, $(i=1 \cdots 9; j=1 \cdots 9)$ only 18 are distinct. These form a G_{18} called the Hessian Group.

The polar conic of the point $(0, 1, -1)$ with respect to the cubic (a) reduces to

$$(x_2 - x_3)(x_2 + x_3 - 2mx_1) = 0,$$

where $x_2 - x_3 = 0$ is the harmonic polar of $(0, 1, -1)$. Since under an homology lines through the center go into themselves and points on the axis are invariant, we can easily get the equations of H_1 by taking two lines through the center, and the new axis as the negative of the old, thus

$$x_1' = x_1;\ x_2' + x_3' = x_2 + x_3;\ x_2' - x_3' = x_3 - x_2.$$

Solving these equations gives

$$H_1;\ x_1' = x_1,\ x_2' = x_3,\ x_3' = x_2.$$

The equations for the other eight homologies can be found in a similar manner.

An example of one of the products is

$$H_1H_2;\ x_1' = x_2,\ x_2' = x_3,\ x_3' = x_1.$$

The transformations, I, H_1, H_2, H_3, H_1H_2, and H_2H_1 will be found to form a group. This is the symmetric group of permutations on three letters. These transformations are real. It follows that the Hessian Group has a sub-group of order six; these are the only transformations of the group that are real.

By applying any of these transformations it is found that $f=0$ is invariant. An ordinary point of $f=0$ is associated with 17 others, no three of the 17 being collinear. A point will not be invariant unless it lies on the harmonic polar of an inflexional point. A point of inflexion is transformed into a point of inflexion, and hence is associated with eight others. The behavior of the inflexional triangles will be considered later.

Now consider a transformation of the form

$$T_1:\ x_1' = \omega x_1,\ x_2' = x_2,\ x_3' = x_3,$$

of which there are three. There are nine combinations of these forming a group. The operations are commutative. Of these nine however, three are not new for $T_1^2T_2 = H_1H_4$, $T_1^2T_3 = H_1H_7$, and $T_1T_2T_3$ is the identity. These transformations are of period three. The six new ones represent homologies. For example T_1 and its inverse T_1^2 are homologies which have A for center and $x_1 = 0$ for axis.

The equations for these and further transformations are easily set up and will not be given.

If we multiply $H_1 \cdots H_9$ by T_1, T_2, T_3, we get transformations of period six. Multiplication in general is not commutative, and duplications occur. There are nine new transformations of this type. If we multiply the transformations H_iH_j by T_1, T_2, T_3, we get six new transformations of period three, making 15 in all. The products of these 15 by T_1^2, T_2^2, T_3^2 give 15 other new transformations which are the inverses of the original 15. We now have 54 transformations. These form a G_{54}, under which the triangle ABC is invariant. The other three triangles are permuted in various ways.

Turning now to the triangle $A'B'C'$, we find a similar situation with six new transformations T_i and T_i^2 $(i=4, 5, 6)$. These combine with the elements of the original G_{18}, to give 30 other new transformations, making in all a G_{54} of transformations under which this triangle is invariant as in the preceding case. A similar group exists for each of the other two triangles.

We now have 24 transformations of the form T_i or T_i^2. If we multiply any of these by any of the nine H_i we get transformations which are of period six. There are 72 of these. If we multiply T_i by H_jH_k we get 24 transformations of period three of which the products of T_i^2 by H_jH_k are the inverses. Thus we have 48 transformations of this type. Multiplication is not commutative.

If we combine the T_i's among themselves we get nothing new. However, products of the form $T_i^2T_j$ $(i=1, 2, 3; \cdot j=4, 5 \cdots 12)$ do give something new. These and their inverses give 54 transformations of period four. The general product of type T_iT_j $(i=1, 2 \cdots 12; j=1, 2 \cdots 12)$ gives nothing further.

It is easy to show that the square of any of the products $T_i^2T_j (i=1, 2, 3; \cdot j=4, 5 \cdots 12)$ is one of the harmonic homologies and the product of any one of them by any of the G_{18} gives one of the 54. Hence G_{18} and these 54 form a group of order 72 of which G_{18} is an invariant sub-group.

We have now found 216 transformations which keep the points of inflexion of a cubic curve invariant, and these transformations form a group. Besides the identity there are six types:

(1) Nine harmonic homologies with the nine points of inflexion as centers. These are of period two. Each triangle is invariant, one side being fixed. On this side one point of inflexion is fixed and the other two points are interchanged.

(2) Eight transformations, each a product of two of type (1). Their period is three. Under any of these the sides of one triangle are invariant but the points of inflexion are permuted. The sides of the other triangles are permuted. The line joining the two centers is fixed.

The identity together with the transformations (1) and (2) form a G_{18}, the Hessian group.

(3) The 24 homologies having a vertex of an inflexional triangle for center and the opposite side for axis. They are of period three. In this case one triangle is invariant and the other three are permuted. The invariant triangle has one side invariant point by point, and the other two sides invariant as a whole.

(4) The 72 transformations formed by the product of (1) and (3). These are of period six. These homologies keep one triangle fixed and permute the others. The triangle that is invariant has one side invariant and the other two interchanged. This invariant side has one point fixed and the other two interchanged.

(5) The 48 transformations formed by the product of (3) and (2). These are of period three. One triangle is fixed with its sides permuted and the other three triangles are permuted.

(6) The 54 transformations formed from the products of the elements of (3). These are of period four, and leave no triangle invariant. On the other hand the triangles are permuted in pairs.

This G_{216} sends the syzygetic pencil $f=0$ into itself, each curve being sent into one of twelve (including itself). Each of these is invariant under a G_{18}. The images are as follows:

$$x_1^3 + x_2^3 + x_3^3 + Kx_1x_2x_3 = 0,$$

where K has the values

$$6m,\ 6m\omega,\ 6m\omega^2,\ \frac{6-6m}{1+2m},\ \frac{6\omega-6m\omega}{1+2m},\ \frac{6\omega^2-6m\omega^2}{1+2m},$$

$$\frac{6-6m\omega}{1+2m\omega},\ \frac{6\omega-6m\omega^2}{1+2m\omega},\ \frac{6\omega^2-6m}{1+2m\omega},\ \frac{6-6m\omega^2}{1+2m\omega^2},\ \frac{6\omega-6m}{1+2m\omega^2},\ \frac{6\omega^2-6m\omega}{1+2m\omega^2}.$$

We omit the details indicating which value of K corresponds to the separate transformations.

If $m=0$ the cubic is called *equianharmonic*. Under the G_{216} it goes into itself or one of the form

$$x_1^3 + x_2^3 + x_3^3 + 2ax_1x_2x_3 = 0,$$

where $a=1$, ω, or ω^2, and hence is invariant under a group of order 54. This curve has received more attention from the standpoint of group theory than any other cubic curve. The tangents at three collinear points of inflexion pass through the opposite vertex of the inflexional triangle. This is the only cubic curve whose inflexional triangles meet in a point.

The only other cubic curve which is invariant under a larger group than the Hessian is the *harmonic* cubic. The cross ratio of the four tangents to a cubic from an arbitrary point on the curve is constant and is called the characteristic of the curve. If this characteristic is -1 the curve is called harmonic. Such a curve is invariant under a cyclic, non-perspective transformation of period four whose square is the harmonic homology. The products of this transformation by those of G_{18} give 18 new ones, forming a group of order 36 of which the G_{18} is a sub-group. This curve then goes into one of six under the G_{216}.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

ON A TOTALLY DISCONTINUOUS FUNCTION

By ROBERT E. MORITZ, University of Washington

The following example, suggested by the Weierstrass function representing continuous curves which have no derivative at any point, seems to the writer unusually suggestive of the caution that must be employed in dealing with limits.

Consider the function

(1) $$y(x, n) = f(x) + b^n \sin(a^n x\pi),\ 0 < b < 1,$$

together with its derivative

(2) $$y'(x, n) = f'(x) + (ab)^n\pi \cos(a^n x\pi),$$

for increasing values of n, in any interval of x for which $f(x)$ is continuous.

We note that in such an interval both y and y' are continuous.

The graph of $y(x, n)$ results from the composition (addition of ordinates) for the two graphs of $y_1 = f(x)$ and $y_2 = b^n \sin(a^n x\pi)$ respectively, and consists, therefore, of a wave-curve superimposed on the curve $y = f(x)$. Both the wave-length and the amplitude of y_2 decrease indefinitely with increasing values of n. As n approaches ∞, $y(n, x)$ approaches $y = f(x)$.

Let us consider the nature of this approach more closely. While the wave-length $2/a^n$, and the amplitude b^n, of y_2 both decrease, its maximal slope (slope at a node) $(ab)^n\pi$ increases or decreases according as ab is greater or less than unity.

In the case $ab < 1$, the undulations of the curve y_2 not only grow shorter and lower with increasing values of n but they also become smoother and, in the limit, flatten out completely. In this case the tangent to $y(x, n)$ at any point in the interval assumes a definite direction as n approaches ∞. (2) shows that this direction is the direction of the tangent to $y = f(x)$.

In the case $ab > 1$, the undulations of the curve y_2 also grow shorter and lower with increasing values of n, but instead of becoming smoother, they become rougher, assuming a saw-tooth like shape; the teeth, though diminishing in size, grow sharper and steeper as n increases. In this case the tangent to $y(x, n)$ does not assume a definite direction as n approaches ∞. In fact, since $a^n x\pi$ changes by 2π while x changes by $2/a^n$, $(ab)^n\pi\cos(a^n x\pi)$ must take on all values from $-(ab)^n\pi$ to $+(ab)^n\pi$ while x changes by $2/a^n$, an interval which can be made less than any assignable quantity by taking n sufficiently large.

When $ab = 1$, the variations of the tangent to $y(x, n)$ for every interval of x however small are between $-\pi$ and $+\pi$.

This shows that $\lim y'(x, n)$ is totally discontinuous when $ab \geqq 1$, while when $ab < 1$, it is equal to the derivative of $\lim y$.

On Bell's Functional Equations

By Aaron Herschfeld, College of the City of New York

In the November, 1930 issue of this Monthly (vol. 37, p. 484) E. T. Bell requested the solution of the following two functional equations,

$$3.1 \qquad f(x, n_1)f(x, n_2) = f(x, n_1 + n_2 + c),$$

$$3.2 \qquad f(x, n_1)f(x, n_2) = f(x, cn_1n_2),$$

where n_1 and n_2 are integers $\geqq 0$ and c is a constant integer $\geqq 0$. A solution follows.

I. From (3.1) we have

$$f(x, n)f(x, 0) = f(x, n - 1)f(x, 1) = f(x, n + c), \qquad n > 0.$$

$$\therefore f(x, n) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^n f(x, 0), \quad n \geqq 0, \quad f(x, 0) \neq 0,$$

while if $f(x, 0) = 0$ then by (3.1), $f(x, n) = 0$, $n \geqq 0$. Substituting the value of $f(x, n)$ in (3.1), we get, provided[1] $f(x, 1) \neq 0$,

$$f(x, 0) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^c,$$

so that

$$f(x, n) = \left\{\frac{f(x,1)}{f(x,0)}\right\}^{n+c} = \{F(x)\}^{n+c},$$

$F(x)$ being an arbitrary function. This proves the generality of the solution given by Bell.

---

[1] For $f(x, 1) = 0$, we have the single solution $c = 0$, $f(x, 0) = 1$, $f(x, n) = 0$, $n > 0$. Professor Franklin omitted this case in his solution of the first of the two functional equations. On page 154 of the March issue of this "Monthly" Franklin writes:

"$\cdots$; and also that if $f(x, 0) \neq 0$, no $f(x, n)$ can vanish."

The truth is that if $f(x, 0) \neq 0$, no $f(x, n)$ can vanish if $c > 0$, but if $c = 0$ and $f(x, 0) \neq 0$, then $f(x, n) = 0$, $n > 0$ and $f(x, 0) = 1$ is a solution. The following is a proof:

Case I. $c > 0$, $f(x, 0) \neq 0$. Using Professor Franklin's equations,

$$f(x, n-1)\, f(x, n+1) = f(x, 2n+c) = [f(x, n)]^2, \quad n = 1, 2, \cdots,$$

we see that should $f(x, n_1) = 0$ for some $n_1 > 0$, then $f(x, n) = 0$ for every $n > 0$. In that case $f(x, c) = 0$. But $[f(x, 0)]^2 = f(x, c)$, so that $f(x, 0) = 0$, which contradicts the hypothesis.

Case II. $c = 0$, $f(x, 0) \neq 0$. As before if any $f(x, n_1) = 0$, $n_1 > 0$, then $f(x, n) = 0$, $n > 0$. But now $[f(x, 0)]^2 = f(x, 0) \neq 0$. Hence $f(x, 0) = 1$. It is easily verified that the function $f(x, 0) = 1$; $f(x, n) = 0$, $n > 0$, actually satisfies Bell's equation.

II. From (3.2) we get

$$f(x, n_1)f(x, n_2) = f(x, 1)f(x, n_1n_2) = f(x, cn_1n_2). \tag{1}$$

Define the new function $\phi(x, n)$ by means of the relation, valid if $f(x, 1)\neq 0$,

$$\frac{f(x, n)}{f(x, 1)} = \phi(x, n).$$

Then

$$\phi(x, n_1)\phi(x, n_2) = \phi(x, n_1n_2). \tag{2}$$

For the case $f(x, 1)=0$ we have, by (1), $f(x, n)=0$, $n\geqq 0$.

Excluding the two trivial cases

$$\phi(x, n)=1 \text{ for all } (n\geqq 0), \text{ which must hold if } \phi(x, 0)\neq 0,$$
$$\phi(x, n)=0 \text{ for all } (n\geqq 0), \text{ which must hold if } \phi(x, 1)\neq 1,$$

the general solution of (2) is: $\phi(x, 0)=0$; $\phi(x, 1)=1$; for prime values of n, $\phi(x, 2)$, $\phi(x, 3), \cdots, \phi(x, p), \cdots$ are arbitrary functions of x; for any $n=p^aq^b\cdots$, where $p, q, \cdots$ are distinct primes,

$$\phi(x, n) = \{\phi(x, p)\}^a\{\phi(x, q)\}^b\cdots.$$

Hence, substituting $f(x, n)=f(x, 1)\phi(x, n)$ in (3.2), we get $f(x, 1)=\phi(x, c)$, so that $f(x, n)=\phi(x, nc)$. The particular choice $\phi(x, p)=p^{F(x)}$, yields the special solution $f(x, n)=(cn)^{F(x)}$, given by Bell.

A Note by the Editor

A solution of the same problem was given by Philip Franklin, in this Monthly, vol. 38 (1931), page 154. The present paper, without the footnote, was received while Franklin's article was in the press. R.E.G.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Numerical Mathematical Analysis. By James B. Scarborough. The Johns Hopkins Press, Baltimore, 1930. 416 pages. $5.50.

That there has been a remarkable development in the interest in numerical mathematics is shown by the numerous publications of this nature such as the valuable series of *Tracts for Computers* and other publications of the Biometric Laboratory headed by Karl Pearson, the publications of the Mathematical Laboratory of the University of Edinburgh, Running's *Empirical Formulas*,

the *Handbook of Mathematical Statistics*, edited by H. L. Rietz, Whittaker & Robinson's *Calculus of Observations*, Steffensen's *Interpolation*, to mention only a few of them. The subject of this review is a worthy addition to the group.

The author states in the preface that he has purposely refrained from employing symbolic methods and divided differences in deriving the standard formulas of interpolation. This is to be regretted, in my opinion, as not only is the derivation of these formulas much more direct, but all the common formulas are derived by a single method and these can be modified to suit the requirements of the special problem, where the ordinary formulas are inapplicable.

In Chapter I there is a valuable discussion on the accuracy of approximate calculations covering 37 pages. A valuable feature of the book is the excellent collection of examples at the end of each chapter. This feature would be improved if the answers were given, excepting in cases where the answer would spoil the problem. In Art. 10 is a discussion of the accuracy of series approximations with Taylor's formula (two forms) and Maclaurin's formula, with two forms for the remainder term for each series, one of them being in the form of an integral. This latter form is applied to an example: Find the remainder after n terms in the expansion of $\log_e(x+h)$. The result is that $R_n<(h/x)^n/n$ which is the first term of the series neglected—a well-known property of alternating series. It is possible, however, to obtain by elementary algebra a closer limit than this, viz.

$$R_n = h^n/[(x+h)n - h]x^{n-1}$$

As an example of the closeness of the approximation, let $h=1$, $x=1$; then $R_n=1/(2n-1)$, with an error $<1/4n^3$. Taking $n=10$, the error in log 2 is .00014. To get as close an approximation by taking the error as the first term neglected would require nearly 7000 terms of the series. In the example given by the author, $x=1$, $h=0.01$ and $n=4$, the formula just stated gives the remainder with an error only one-tenth that by the author's formula. On page 32 is given the logarithmic series

$$\log_e (m+1) = \log m + 2[1/(2m+1) + 1/3(2m+1)^3 + \cdots].$$

The remainder term given is not as close as the following,

$$R_n = 1/[(4m^2+4m)n + 2m^2 + 2m + 1](2m+1)^{2n-1}.$$

If we put $m=4$ and $n=5$, the series with the author's remainder term gives the value of log 5 to ten decimals. With the same number of terms of the series and the remainder term just given, the value of log 5 is correct to 14 decimals. Incidentally, there is a misprint at the top of page 32. In the first line, 5 should be 9. In the second line, 7 should be 5, and in the third the word "seven" should be replaced by "five."

Chapter II: *Interpolation. Differences. Newton's formulas for forward and backward interpolation.* The author gives the ordinary notation for differences

when arranged diagonally and another when arranged horizontally. It does not seem to the reviewer that the advantages in the new notation are great enough to compensate for the disadvantage of having a new notation to learn and use. If divided differences were used it would not be necessary to have two formulas, one for forward and the other for backward interpolation.

Chapter III: *Interpolation. Central-difference formulas.* The statement is made that Newton's interpolation formulas do not in general converge as rapidly as the central-difference formulas. It is doubtful if there is any appreciable difference in the rapidity of convergence in most cases. Stirling's and Bessel's interpolation formulas are derived by a long, involved method. Here would have been a good place to give Everett's central difference interpolation formula, which has many advantages over other formulas, one of them being that only half as many differences are required as with other formulas. It was for this reason that this formula was adopted in the *Logarithmetica Britannica*, the twenty-place table now being computed by Alexander J. Thompson and being issued by the Biometric Laboratory. (Four of the ten parts have been published.)

Chapter IV: *Lagrange's formula. Inverse interpolation.* Since the formulas derived in the preceding sections are applicable only when the values of the independent variable are given at equidistant intervals, it is important to have a formula which may be used when the values are given for any values of the independent variable. Such a formula is Lagrange's formula, which, however, was discovered by Euler originally. The author states that Lagrange's formula is tedious to apply and involves a great deal of computation. It has another disadvantage that if not enough data are used and it is desired to use more data, the whole computation must be gone over from the beginning. The author does not give any other method. Newton's *general interpolation* formula is the equivalent of Lagrange's formula and does not have the disadvantages of the latter. Although this formula is generally stated in terms of divided differences, it can be stated without them; see Boole's *Finite Differences*, 2nd edition, Chapter 3, Exercises 6 and 19. The problem of inverse interpolation may be solved by Lagrange's formula, but owing to the objections just stated, it is not commonly used. An alternative method is the method of successive approximations, or iteration, which is easily applied and has the advantage that any method of iteration possesses, which is that any errors are eliminated in the next trial, so that the result proves itself. Another method given is that of reversion of series. The coefficients in the reverted series are given as far as the 5th power of the variable and the method is applied to Newton's, Stirling's, and Bessel's formulas. The method is long and involves a great deal of computation with a good chance of errors. Newton's general interpolation formula could be used, of course.

Chapter V: *The accuracy of interpolation formulas.* The formulas for the remainder terms in the different interpolation formulas are given.

Chapter VI: *Interpolation with two independent variables. Trigonometric interpolation.*

Chapter VII: *Numerical differentiation and integration.* To find the derivatives of a function, it may be represented by an interpolation formula and this formula differentiated as many times as desired.

Numerical integration or mechanical quadrature is the process of computing the value of a definite integral from a set of numerical values of the integrand. By representing the function by Newton's interpolation formula, integrating, and substituting for the differences their values in terms of the ordinates, the formulas known as the Newton-Cotes formulas are obtained. Let the number of intervals be n. When $n=1$, we have the trapezoidal rule; when $n=2$, Simpson's one-third rule; when $n=3$, Simpson's three-eighths rule; when $n=6$, by modifying the coefficients slightly, we have Weddle's rule. The author observes that Simpson's three-eighths rule is less accurate than Simpson's one-third rule, although the reason is not stated. The reason is that if $n=2r$ the coefficient of the $(2r+1)$th difference vanishes so that the approximation is as good as or better than from the formula obtained by putting $n=2r+1$. This is the reason why the formulas obtained by putting $n=2, 4, 6, \cdots$ are more useful than those in which n is odd. Then the author's statement on page 120, line 5 that the geometric significance of Simpson's one-third rule is that we replace the graph of the given function by $n/2$ arcs of second-degree parabolas should be changed to "third-degree parabolas."

The remarks near the end of page 123 as to the relative accuracy of the different quadrature formulas admit of exceptions. For example in the integral

$$\int_{-1}^{1} dx/(e^x + 1)$$

the trapezoidal rule, Simpson's one-third rule, Simpson's three-eighths rule, and Weddle's rule all give the same value 1, which is the exact value. Merrifield, in his *Report on the Present State of Knowledge of the Application of Quadratures and Interpolation to Actual Data* (British Association Report, 1880), points out that in the rectification of the quarter-ellipse, quadrature formulas of high order do not give as good an approximation as those of lower order. If the semi-axes of the ellipse are 1 and $1/\sqrt{2}$, we have $y=\frac{1}{2}(3+\cos 2x)^{1/2}$, the limits for x being 0 and $\pi/2$. The correct value according to Legendre is 1.35064 38820. With 3 ordinates,

by the trapezoidal rule. . . . . . . 1.35055 with an error of 0.00009 0;

by Simpson's one-third rule. . . 1.35381 8 with an error of 0.00317 4.

With 5 ordinates,

by the trapezoidal rule. . . . . . . 1.35064 3855 with an error of 0.00000 0027;

by Simpson's one-third rule. . . 1.35088 3396 with an error of 0.000239;

by Cotes's formula ($n=4$) . . . . 1.35051 9660 with an error of 0.000124.

A good elementary presentation of Gauss's quadrature formulae is given. On page 135, the values of the u's and R's should be given in the surd form as well as in decimal form, as the former are frequently easier to substitute in a given problem. In Example 1, page 140, the value of an integral is found and the result is true to the 8th figure. However, if the calculations had been carried to more decimals, the value would have been found to be true to the 11th figure. Another important quadrature formula—not mentioned by the author—is Sheppard's Rule XI. (See the *Handbook of Mathematical Statistics*, edited by H. L. Rietz, page 18.)

In a great many of the computations, the author has not carried the work far enough to show the correct value of the error. An illustration of this is given in Example 1, page 121. By taking the logarithms to eleven places, the value given by Weddle's formula is 1.8278 47407 and the true value is 1.8278 47408. This gives the error by Weddle's formula of 1×10^{-9} instead of 3×10^{-8} as given by the author.

On page 142 the article "Caution in the use of quadrature formulas" the author warns against the use of quadrature formulae without ascertaining the nature and behavior of the integrand over the interval and illustrates with two examples. These are good illustrations but the author seems to have missed the reason for the lack of a good approximation to the value of the integral. The reason is stated by several authors, e.g. Merrifield (loc. cit.) and W. F. Sheppard in article "*Mensuration*" (Encyclopedia Britannica), §88, "Cases of Failure." Since the derivative in both examples becomes infinite at both limits, the integrand cannot be represented by a polynomial with integral exponents only. Several writers on the subject have derived special formulas for this case. However, a simple method which improves the accuracy of the formula is to consider a small portion of the curve adjoining the point where the derivative becomes infinite as a portion of a parabola with axis on the x-axis and vertex at the point mentioned. The area of such a segment bounded by a parabola is known to be two-thirds of the circumscribed rectangle.

In Example 1, page 142 there are some misprints. When $x=0.80$ the value of y is 0.038468. On page 144, line 4 from the bottom, instead of 0.000035, read 0.0000031. In line 2 from the bottom, instead of 0.023548, read 0.0206514. The same two corrections should be made in the last line. The result is given as 0.0235 and the inference is that all the figures are correct. However, if the number of ordinates is increased, the value is found to be 0.0238.

In Example 2, page 145, the author finds for the value of the elliptic integral for $h=0.1$ by Simpson's one-third rule 2.1934, the correct value being 2.2033. The error is 0.0099 or 0.45%. However, if we find the value of the integral between the limits -0.9 and 0.9 by Simpson's rule, we get 2.1226140. The area between -1 and -0.9 by the formula for the parabola, as suggested above, is $\frac{2}{3}(.1)0.7422\ 94=0.0494\ 863$. In the same way the area between $x=0.9$ and 1 is $\frac{2}{3}(.1)0.4571\ 65=0.0304\ 777$. Adding these three, gives for the value of the integral 2.2025780 with an error of only 0.0007 or 0.032% or less than one-tenth the error by using Simpson's rule alone.

A brief treatment of mechanical cubature or finding the approximate numerical value of a definite double integral is given with two examples worked out.

Chapter VIII is devoted to the accuracy of quadrature formulas. The relative accuracy of the different quadrature rules with formulas for the errors is discussed with illustrative examples.

Chapter IX is devoted to the solution of numerical algebraic and transcendental equations. Since Horner's method is explained in all college algebras, it is not discussed. The first method given is the method of false position. This is equivalent to using interpolation with first differences only, although it is not so stated. The Newton-Raphson method is next explained with a discussion of the inherent error. The method of iteration which is next given has many advantages which were mentioned in discussing the method applied to inverse interpolation. It requires only a slight knowledge of methematics and the errors are eliminated automatically. The last two methods are applied to simultaneous equations in several unknowns.

In Chapter X is a detailed explanation, covering 20 pages, of Graefe's root-squaring method for solving algebraic equations. There are a number of other methods of solving numerical equations which deserve at least a brief mention. In Workman's *Memoranda Mathematica* (Oxford, 1912) pp. 45–55 are given twelve methods. The graphical method is especially valuable when applied to cubic equations, as the roots may be found from the intersection of the curve $y=x^3$ and a straight line. This curve can be obtained in celluloid, making the graphic solution very rapid. Workman suggests that in Graeffe's method the series $f^3, f^9, f^{27}, \cdots$ is to be preferred to Graeffe's series for equations of degree not higher than 5. Another method for solving cubic equations is by trigonometry. (See Hobson's *Plane Trigonometry*.) Carl Runge in his *Praxis der Gleichungen* gives a method of calculating the root of any trinomial equation by addition and subtraction logarithms. In Thiele's *Interpolationsrechnung* is a solution of equations by divided differences, also interpolation of functions of two arguments.

Chapter XI treats the numerical solution of differential equations by the method of successive approximations; Chapter XII, the convergence and accuracy of the iteration process; and Chapter XIII, other methods for the numerical solution of differential equations, including the methods of J. C. Adams, Runge-Kutta, and W. E. Milne.

Chapter XIV: The normal law of errors and the principle of least squares.

Chapter XV: The precision of measurements.

Chapter XVI: Empirical formulas. The problem is to fit an equation to a set of numerical data. The methods used are: (1) The graphic method or method of selected points; (2) The method of averages; (3) The method of least squares.

The book has many admirable features. The explanations and derivations of formulae are given in detail and there are numerous examples at the end of each chapter. The author has avoided introducing new and complicated nota-

tions which, although they may conduce to brevity, are a serious stumbling block to the reader. The typography and paper are excellent and misprints are few.

The bibliographical references are very few. The book would be much improved by a good bibliography, so that those who wished a more extensive treatment of any topic would know where to go.

E. B. ESCOTT

A Manual of Greek Mathematics. By Sir Thomas L. Heath. Oxford University Press, 1931. xvi+552 pages. Price $5.00.

Although the author states in his Preface that "the story told in these pages is in substance the same as that unfolded in somewhat greater detail" in his *History of Greek Mathematics* (Clarendon Press, 2 vols., 1921), it is something much more than that. The main body of facts is naturally the same, but the work differs widely in purpose, somewhat in style, and naturally in various important details. In purpose it seeks to meet the needs and tastes of the general reader instead of the classical scholar; in style it cultivates somewhat more that which a recent essayist has called "a fetich of our time," namely, genuine simplicity,—the simplicity that appeals to the educated reader who is discouraged by any uncalled-for excess of erudition; and in details of knowledge it includes several discoveries which have been made since the History was published.

In the opening sentence the author seems to draw up a comfortable chair for his reader and to speak to him as an old friend,—"Why should we study Greek mathematics?" he asks, and this unassuming style is evident throughout the book. It is the style which the best teachers always seek, and it always appeals to the seeker after truth. And when the author answers the question he does so with a common-sense frankness that must appeal to even our most vociferous educational revolutionists, especially when he considers mathematics in the education of the Greek youth. For the Athenian boy of Plato's time, brought up under the latter's influence, was in great measure, a pupil of John Dewey's some two thousand years ahead of his time. Under the shadow of the Acropolis was born the project method, and here it was seen at its best. It is probable that our generation has often seen it at its worst. Pythagoras met and solved, for his day, the problems of "arrested development," "individual differences," the "I.Q.," and the College Entrance Examination Board,—but he thought and talked of something else.

These are, however, mere educational *obiter dicta* in this interesting summary of the story (for it is not a dry-as-dust "history") of the greatest intellectual development of ancient times,—probably of all time. In twenty-eight chapters the author runs the scale of Greek mathematics from the numbers of commerce and of calculation, through the remarkable development of geometry, to the algebra of Diophantus and the commentaries of the minor writers. This he does with a view to the needs of "the general reader who has not lost his interest in the studies of his youth and would wish to know how it came about that a Greek

but it also tends to confirm the common belief that Thales (c. 600) had access to similar records. (See also p. 89.)

The reference to the commentary of Pappus on Euclid X as "partly extant in Arabic" (p. 241) will, of course, need changing in view of the elaborate work of Junge and Thomson (*The Commentary of Pappus on Book X of Euclid's Elements*, Arabic text and translation, Harvard University Press, 1930), which appeared while this work was in press.

Such discoveries and studies give hope that definite information may sometime be forthcoming with respect to the first use of the *myriad* (Greek *myrias*, 10,000), the counting by 10,000's being oriental as well as Greek. Was there a common source in the Sumerian civilization? In speaking of Greek numeration some readers may regret that the work should give the impression that the numerals in the classical period were always represented by small letters instead of capitals. This often gives to beginners in the study of the history of notation a false idea of the facts. Another source of misunderstanding occurs on p. 25, where a medieval writer is quoted as saying that "the 'mensa Pythagorica' was so called in honor of the Master who taught its use." Since the quotation seems, at first sight at least, to be approved, these same beginners feel that it is worthy of credence. It is something like the confusion arising from the names "Naperian logarithms," logarithms which Napier never used, and "Arabic numerals" with which the Arabs were almost wholly unacquainted.

Altogether, the work is by far the best condensed presentation of the history of Greek mathematics that we have in any language.

DAVID EUGENE SMITH

College Algebra (*Alternate Edition*). By William L. Hart. D. C. Heath & Co., New York, 1931. 380 pages.

It would be interesting to know how many new editions of mathematical text-books are due to a burning desire on the part of the authors to improve the originals, and how many are inspired by the publishers' wish to eliminate used copies and teachers' suspicions that home-made keys to the problems have become too numerous. The title of this book implies that it belongs in the second category and, if this inference is correct, mathematical teachers should hope that the frankness of the author and publisher will be imitated by others, and that in the future no good book will be marred by attempts to make the new edition appear due to the first rather than to the second motive. This is certainly not true of the book under review, whatever the motives of the author; the old edition was a good book and the new one is better.

As to the difference between the two editions it is stated in the preface that "the original edition was rewritten in detail, without alteration of its main features." While this statement is literally true, nevertheless there is so little change in the text that a reader could shift from one to the other without noticing the difference. The principal alterations are a change in the order of four elementary chapters and a combination of two later chapters into one. The most

noticeable other changes are the omission of the numbering of steps in demonstrations, which does not seem to be an improvement, and the rewording, or rearrangement of the wording, of many sections in the interest of greater clarity. In contrast to the text there is an almost entirely new set of problems and their number has been increased from about 3,000 to 4,500.

For the benefit of those not acquainted with the old edition, it may be stated that Chapters I–VIII (pp. 1–111) cover algebra up to quadratics; Chapters IX–XVII (pp. 112–199) contain quadratics, ratio and proportion, variation, progressions, mathematical induction, the binomial theorem, and inequalities; and Chapters XVIII–XXVI (pp. 200–368) contain complex numbers, the theory of equations, logarithms, permutations and combinations, probability, determinants, partial fractions, the mathematics of investment, and infinite series. This is followed by seven pages of tables, an index, and answers to the odd-numbered problems. The object in using over half the book for high-school algebra is to provide a suitable starting place for the freshman with any degree of lack of preparation, an additional aid being the insertion of collections of review problems at the ends of Chapters IV, VI, and VIII.

The strong features of the book are the wealth of material, the clearness of presentation, and the attractive typography and arrangement; less desirable features are the lack of really difficult problems like the "stickers" in Hall and Knight and the encouragement to teacher and pupil to omit demonstrations, which raise a doubt that the student will gain enough power to comprehend the one really difficult chapter, that on determinants. However, such a criticism loses force when it is remembered that the book is for freshmen and therefore the emphasis is primarily not on rigorous thinking, but on "doing" algebra. In conclusion, the reviewer feels safe in predicting a successful career for this book in spite of his belief that the problem of more algebra for college students is better solved by letting the student get his algebra review in trigonometry, analytic geometry, and calculus, and by presenting the advanced topics to those who wish them is a more mature course later.

W. A. WILSON

Grundlehren der neueren Zahlentheorie, von Professor Dr. Paul Bachmann, Dritte, neu durchgesehene Auflage, herausgegeben von Professor Dr. Robert Haussner. Walter de Gruyter & Co., Berlin and Leipzig, 1931. xvi+252 pages. RM 9.50.

This third edition of Bachmann's well known introduction to the elementary theory of numbers from one fairly modern point of view, first published in 1907, differs from previous editions only in minor changes and in the following respects. Haussner has prefixed a short biography of his friend Bachmann, (1837–1920); Bachmann has added Frobenius' presentation of Zeller's proof of the law of quadratic reciprocity and a table classifying the 56 proofs of the law given up to 1914. Of the 56, 32 are by the Gauss lemma or variations of it, 8 by cyclotomy, 2 by quadratic fields.

The book can be warmly recommended to beginners with a good reading knowledge of German. The exposition is unusually detailed, and Bachmann spares no pains to make clear his reasons for attacking various arithmetical problems in the way he does. Opinions may differ whether a beginner should read some such book as this as his initiation to the theory of numbers, or whether he should start with something closer to the classical tradition. One who stops with the theory of integers and ideals for a quadratic field, as this book does, may get the too optimistic impression that the theory of numbers is rather an easy subject, and that all of its interesting problems yield their secrets to the methods of the theory of algebraic numbers. Such optimism should be corrected by further reading.

The first 143 pages cover the elementary theories of divisibility, congruence, quadratic residues, binary linear forms, and binary quadratic forms. Those who like geometrical pictures of algebraic or arithmetical facts will find several in this presentation of classical material. The second part of the book, pp. 144–252, is devoted to the quadratic field, and gives a detailed discussion of quadratic integers, modules and ideals, units of a quadratic field, divisibility in a quadratic field, ideals and lattice numbers. In this part much of the matter of the first part is re-presented in the language of quadratic algebraic number fields. Any inquiring reader who is tempted to ask why the first part was presented at all, in view of the greater power and elegance of the second, may be invited to try his hand on the theory of binary forms of degree 4, by any method that occurs to him.

E. T. BELL

Logarithmetica Britannica. Being a Standard Table of Logarithms to Twenty Decimal Places. By Alexander John Thompson. Part V, Numbers 50000 to 60000 Issued by the Biometric Laboratory, University of London, to Commemorate the Tercentenary of Henry Briggs' Publication of the *Arithmetica Logarithmica*, 1624. Subscription Issue. Cambridge, The University Press, 1931.

This is the fifth part (the fourth not yet published) of this tremendous undertaking. It consists of twenty-place logarithms of numbers of five digits, accompanied by values of second and fourth differences. The project speaks for itself; it is sufficient to say that the result is all that is to be expected of any product of the Cambridge Press.

R. A. J.

Standard Four-Figure Mathematical Tables. By L. M. Milne-Thomson and L. J. Comrie. London, Macmillan, 1931. xvi+246 pages, $8\times 10\frac{1}{2}$ inches. $4.50.

This monumental work is issued in two editions, the one with negative characteristics of logarithms where they occur, the other with all such logarithms increased by 10. All the usual tables of logarithms and trigonometric functions, and logarithms of the latter, are given at length. There are also hyperbolic and

exponential functions and their inverses, natural logarithms, Gudermannians, etc. There is a single page for the gamma function, and one for the probability integral, represented by the symbol *Erf* x. There is a collection or formulae; including, for example, 139 integrals, and 53 series.

For convenience in interpolation a table of proportional parts is published on a card approximately the size of the page, so that it can be used in connection with any part of the book.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3503. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The apex angles of two cones of revolution are α_1 and α_2; the cones intersect in an ellipse whose semi-axes are a and b; show that the angle β which the axes of the cones make with each other is given by the equation.

$$\cos\beta = \left[-\sin^2\tfrac{1}{2}\alpha_1 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_1\right]^{1/2}\left[\sin^2\tfrac{1}{2}\alpha_2 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_2\right]^{1/2} - \left[1-(b/a)^2\right]\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2.$$

3504. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Two cones of revolution whose apex angles are α_1 and α_2 intersect in an ellipse whose semi-major and semi-minor axes are a and b; show that

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2$$

in which γ_1 and γ_2 are the angles which the axes of the cones make with the plane of the ellipse.

3505. *Proposed by J. Rosenbaum, Milford, Conn.*

Under what condition will the lines joining the vertices of a tetrahedron with the points of contact of the opposite faces and the insphere be concurrent?

3506. *Proposed by E. B. Escott, Oak Park, Illinois.*

Solve these n simultaneous equations with n unknowns:

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ x_n & x_1 & \cdots & x_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ x_3 & x_4 & \cdots & x_1 \end{vmatrix} = a_1; \quad \begin{vmatrix} x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ x_4 & x_5 & \cdots & x_2 \end{vmatrix} = a_2; \cdots$$

$$\begin{vmatrix} x_n & x_1 & \cdots & x_{n-2} \\ x_{n-1} & x_n & \cdots & x_{n-3} \\ \cdot & \cdot & \cdot & \cdot \\ x_2 & x_3 & \cdots & x_n \end{vmatrix} = a_n.$$

3507. *Proposed by Charles K. Robbins, Purdue University.*

Find a solution of

$$\frac{\partial v}{\partial t} = A\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{dv}{\partial r}\right)$$

subject to these conditions:

(1) $\partial v/\partial t = B - C(\partial v/\partial r)$ if $r = R$ and $0 < t < \infty$;

(2) $v = B$ if $t = 0$ and $0 \leqq r \leqq R$;

(3) $v = 0$ if $t = 0$ and $R < r < \infty$;

where A, B, C, and R are positive real constants.

3508. *Proposed by the late Artemas Martin, Washington, D. C.*

The sides of a plane triangle are a, b, c. It is required to determine the radius of the circle circumscribing the escribed circles of this triangle.

See the *Annals of Mathematics*, March, 1894.

SOLUTIONS

196. (192) [1913, 223; 1919, 214]. *Proposed by Charles Macaulay, Chicago, Illinois.*

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all the b's in the same column; all the c's in the same column, etc.

Solution by H. L. Olson, Michigan State College

It is easily seen that if there are more than two letters, it is not possible to arrange *all* the combinations containing a given even number of letters in this way; for if a and b are two letters which are to be placed in the same column, evidently the combination ab can not be used.

Assume that there is, in all, an even number, $2n$, of letters; we first separate

them into two groups, each containing n letters, which can be done in $(2n)!/(n!)^2$ ways if we understand that interchanging the groups gives a different separation. If each combination is to contain $2k$ letters, we combine each combination of k letters from the first group with each combination, in turn, of k letters from the second group. Thus the total number of ways in which the letters can be combined according to the conditions of the problem is ${}_nC_k^2$, or $(n!)^2(k!)^{-2}[(n-k)!]^{-2}$ for each separation into two groups.

291 [1914, 122; 1931, 171]. *Proposed by Emma M. Gibson, Springfield, Missouri.*

The time of descent, down a rough inclined plane, of a spherical shell which contains a smooth solid sphere of the same material as itself is t_1. The time of descent, down the same plane, of a solid sphere of the same material and radius as the shell is t_2. Determine the thickness of the shell.

From Loudon's *Elementary Theory of Rigid Dynamics*, p. 188.

Solution by Ralph P. Agnew, Princeton University.

Let the spheres be supposed to roll a distance l down a plane making an angle α with the horizontal. Let R be the radius of the spheres, and let r (unknown) be the radius of the inner sphere of the first sphere. Let the spheres have density d and mass M. Let I_1 denote the moment of inertia of the spherical shell, and let I_2 denote the moment of inertia of the second sphere. Let $s=s(t)$ be the distance rolled at time t, and let $\theta=\theta(t)$ be the angular displacement at time t; then $s=R\theta$.

For each sphere, the force parallel to the inclined plane is $Mg\sin\alpha$; and for each sphere this force may be regarded as the sum of two forces F_1 and F_2, the first of which produces the linear acceleration of the sphere and the second of which produces the rotational acceleration. By fundamental formulae, we have

$$F_1 = M(d^2s/dt^2),\ F_2R = I_n(d^2\theta/dt^2),\ n = 1, 2,$$

where $n=1$ refers to the first sphere and $n=2$ to the second. Hence, since $F_1+F_2=Mg\sin\alpha$ and $s=R\theta$,

$$(M + I_nR^{-2})(d^2s/dt^2) = Mg\sin\alpha,\ n = 1, 2,$$

and after integrating twice and using the facts that $s=(ds/dt)=0$ when $t=0$ and $s=l$ when $t=t_n$, we obtain

$$(M + I_nR^{-2})l = \tfrac{1}{2}Mgt_n^2 \sin\alpha,\ n = 1, 2.$$

Eliminating l from these equations, we obtain

$$(MR^2 + I_1)t_2^2 = (MR^2 + I_2)t_1^2.$$

Since $M=(4/3)d\pi R^3$, $I_2=(8/15)d\pi R^5$, and $I_1=(8/15)d\pi(R^5-r^5)$, the preceding relation reduces to

$$r^5 = 7R^5(t_2^2 - t_1^2)/(2t_2^2).$$

The thickness x of the spherical shell is therefore the following linear function of R:

$$x = R - r = \{1 - [7(t_2^2 - t_1^2)/(2t_2^2)]^{1/5}\}R.$$

Note by the Editors: It is implicitly assumed in the solution that, in the first case, the radius of the inner sphere is equal to the inner radius of the shell.

309 [1915, 162; 1931, 228]. *Proposed by Jos. B. Reynolds.*

The tangent at one cusp of a vertical three-arched hypocycloid is horizontal and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

Solution by the Proposer

Let the Y-axis be vertical and the coordinates of a point P on the curve be

$$x = 2a\cos\theta + a\cos 2\theta,\quad y = 2a\sin\theta - a\sin 2\theta.$$

Let the curve cut the X-axis at the cusp A and let B be the highest cusp. For these points $\theta=0$ and $2\pi/3$ respectively. The normal at P to the arc $AP=s$ makes a clockwise angle $\frac{1}{2}\theta$ with the vertical. The radius of curvature is $R=8a$ $\sin(3\theta/2)$ and $ds=4a\sin(3\theta/2)\,d\theta$. Let w be the weight of the particle and k the coefficient of friction. Then resolving forces along the tangent and normal respectively at P we have

$$(1)\qquad w\sin\tfrac{1}{2}\theta - kS = -\,wg^{-1}v\,dv/ds,$$

and

$$(2)\qquad S - w\cos\tfrac{1}{2}\theta = wg^{-1}v^2/R;$$

in which v is the velocity of the particle and S the reaction of the curve.

Elimination of S from these equations gives,

$$(3)\qquad g(\sin\tfrac{1}{2}\theta - k\cos\tfrac{1}{2}\theta) = kv^2/R - v\,dv/ds.$$

Substitution of $8a\sin(3\theta/2)$ for R and $4a\sin(3\theta/2)\,d\theta$ for ds gives the equation

$$(4)\qquad 8\,ag\,(\sin\tfrac{1}{2}\theta - k\cos\tfrac{1}{2}\theta)\sin(3\theta/2)\,d\theta = kv^2\,d\theta - 2v\,dv.$$

Integration of this equation leads to

$$(5)\qquad 4ag\left[\frac{(2-k^2)\sin 2\theta - 3k\cos 2\theta}{k^2+4} - \sin\theta\right] = v^2 + Ce^{k\theta},$$

in which C is the constant of integration. Now $v=0$ for $\theta=0$ and also for $\theta=2\pi/3$. These two pairs of values in (5) lead to two equations from which C may be eliminated giving the required equation in k or

$$2\sqrt{3} - k = 2ke^{2\pi k/3}.$$

Also solved by W. H. Erskine.

313 [1915, 202; 1931, 228]. *Proposed by Clifford N. Mills.*

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1+ekgw/2)$.

Solution by L. M. Hoskins, Palo Alto, Calif.

The stress per unit area upon a cross-section distant x below the point of support is $kg(c-x)$, and the elongation per unit length at this point is $kg(c-x)/e$. The total elongation is

$$\frac{kg}{e}\int_0^c (c - x)dx = kgc^2/2e.$$

The length of the stretched wire is therefore $c(1+kgc/2e)$ instead of the value given in the statement of the problem.

Also solved by Paul Capron.

315 [1915, 309; 1931, 229]. *Proposed by H. S. Uhler.*

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

Solution by L. M. Hoskins, Palo Alto, Calif.

Since in each case gravity is the only force doing work upon the cylinder, its kinetic energy *in any given position* during the descent would have the same value in the two cases. For the sliding body the energy is wholly translational, while for the rolling body it is in part rotational. The energy of translation is $\frac{1}{2}mv^2$ and that of rotation $\frac{1}{2}I\omega^2$, m being the mass, I the moment of inertia about the geometrical axis, v the velocity of the mass-center, and ω the angular velocity. If r is the radius, $I=\frac{1}{2}mr^2$, and for the case of rolling $\omega=v/r$, so that $\frac{1}{2}I\omega^2=\frac{1}{4}mv^2$. The total kinetic energy of the rolling cylinder is therefore $\frac{3}{4}mv^2$.

If v_1 and v_2 denote values of v *for the same position of the body* in the two cases of sliding and rolling

$$\tfrac{1}{2}mv_1^2 = \tfrac{3}{4}mv_2^2, \text{ or } v_1/v_2 = \sqrt{(3/2)}.$$

Since this fixed ratio of velocities holds for every position, its reciprocal gives the ratio of the times of descent. That is,

$$\frac{\text{time of descent of sliding body}}{\text{time of descent of rolling body}} = \sqrt{(2/3)}.$$

This result is independent of the radii of the cylinder and track.

For a sphere, since $I=2mr^2/5$, the rotational energy in the case of rolling is $\frac{1}{5}mv^2$ and the total energy $7mv^2/10$, leading to $\sqrt{(5/7)}$ instead of $\sqrt{(2/3)}$ as the ratio of the times of descent for the cases of sliding and rolling.

3464 [1930, 551]. *Proposed by I. Maizlish, Centenary College of Louisiana.*

A telegraph pole is in the form of a frustum of a right circular cone of altitude h, diameter of lower base D, and diameter of upper base d. A rope, of radius r, is wound spirally around this pole from the bottom to the top, covering the whole pole. A device is constructed which unwinds the rope—beginning at the top. If the angular velocity, w, with which the rope is being unwound is a function of the length of the rope already unwound, find the time it will take the device to unwind the whole rope.

Solution by H. R. Cooley, New York University

Let s be the length of the rope already unwound after it has turned through the angle θ. Then the time consumed by the complete unwinding is

$$t = \int_0^{\theta_1} \frac{d\theta}{f(s)}, \tag{1}$$

where $f(s)$ is the angular velocity and θ_1 is determined below in (2). Set $\frac{1}{2}(D-d)=a$, then the slant height of the frustum is $l=(a^2+h^2)^{1/2}$. If n is the number of turns of the rope necessary to cover the pole, then

$$2nr = l,\quad \theta_1 = 2\pi n = \pi l/r. \tag{2}$$

Consider a rectangular system of axes with the origin at the center of the upper base, the x-axis passing through the upper end of the rope, and with the z-axis downward along the axis of the frustum. Let ρ be the distance from the z-axis of a point of contact of the rope with the frustum. Then the equations of the curve of contact are

$$x = \rho\cos\theta,\quad y = \rho\sin\theta,\quad z = k\theta, \tag{3}$$

where k is a constant such that for $\theta=\theta_1$, $z=h$. From (2) $k=rh/l\pi$. From similar right triangles we have

$$\frac{\rho-\frac{1}{2}d}{a} = \frac{k\theta}{h},\quad \text{or}\quad \rho = \frac{1}{2}d + \frac{ak\theta}{h}\cdot \tag{4}$$

Hence from (3) and (4) we have

$$s = \int_0^{\theta} [(ak\theta/h + \tfrac{1}{2}d)^2 + r^2/\pi^2]^{1/2} d\theta;$$

and the integral on the right is easily evaluated by the formula for such integrals. With this value of s the time t is determined by (1).

A Note by the Editors: After reading the proof sheet of this solution, the solver has remarked that the expressions for z and l are not strictly correct, but for practical purposes, as in the case of an actual telegraph pole, the assumptions seem to be sufficiently accurate.

3467 [1930, 552]. *Proposed by J. Rosenbaum, Melford, Conn.*

Solve in positive integers: $2x^2+2x+1=y^2$.

Solution by H. C. Bradley, Massachusett Institute of Technology

This equation may be written: $x^2+(x+1)^2=y^2$. In this form, we see that the solution depends on that series of Pythagorean triangles whose perpendicular sides differ by unity, x being the value of the smaller side, and y the hypothenuse.

These triangles can be derived from the series of convergent fractions which result from extracting the square root of 2 by continued fractions. This series is

$$1/1,\ 3/2,\ 7/5,\ 17/12,\ 41/29,\ 99/70,\ 239/169,\ \cdots .$$

Each alternate fraction, beginning with the first, gives a solution, the numerator being $2x+1$, and the denominator y.

Simple as this is, it can be made still simpler. Write at once the fractions x/y in a series, thus:

$$0/1,\ 3/5,\ 20/29,\ 119/169,\ \cdots .$$

Then, let a/b and c/d be two consecutive fractions in this series; we note that $c=3a+2b+1$; $d=c+a+b+1$. By this law, the series may be continued as far as we please; the next two terms are 696/985 and 4059/5741.

Also solved by W. E. Buker, Mannis Charosh, Raymond Garver, Theodore Lindquist, R. E. Moritz, A. Pelletier, A. W. Randall, Wallace Smith, F. Underwood, E. E. Whitford, and the Proposer.

A Note by Otto Dunkel: The process above gives all the solutions; for the general theory of continued fractions for quadratic surds shows that any solution in positive integers of $2y^2-z^2=1$ must furnish a convergent z/y of the continued fraction development of $2^{1/2}$. If p_n/q_n is the nth convergent, it is also shown that

$$2q_n^2 - p_n^2 = (-1)^{n+1}, \tag{1}$$

and therefore the solution must give an odd convergent. Conversely an odd convergent furnishes a solution. From this last equation it is clear that p_n must be an odd integer, and hence it is always possible to find positive integral solutions for x_i, where $z=p_{2i+1}=2x_i+1$, $y_i=q_{2i+1}$. Then x_i, y_i is a solution of the equation of the problem, with $x_1=0$, $x_2=3$, $y_1=1$, $y_2=5$.

From the relations $p_{n+2}=2p_{n+1}+p_n$, $q_{n+2}=2q_{n+1}+q_n$, we easily derive $p_{n+2}=6p_n-p_{n-2}$ with a similar relation for the q's. From these result the equations

$$x_{i+2} = 6x_{i+1} - x_i + 2, \quad y_{i+2} = 6y_{i+1} - y_i, \tag{2}$$

which enable us to calculate a solution from the two preceeding solutions. Other relations may be obtained from

$$p_n + 2^{1/2}q_n = \alpha^j(p_{n-j} + 2^{1/2}q_{n-j}), \quad \alpha = 1 + 2^{1/2}, \tag{3}$$

by equating the rational and irrational parts of each side. Thus if $j=2$, $\alpha^2=3+2.2^{1/2}$, and

$$
\begin{aligned}
(4) \qquad p_n &= 3p_{n-2}+4q_{n-2}, \qquad q_n = 2p_{n-2}+3q_{n-2};\\
x_{i+1} &= 3x_i+2y_i+1, \quad y_{i+1} = 4x_i+3y_i+2.
\end{aligned}
$$

This last pair of equations is essentially the same as the pair in the last part of the solution above. By means of (3) x_i and y_i may be expressed directly in terms of i, but the formulae thus obtained would not be as convenient for computation as the process given in the above solution.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS
1930–1931

Gamma of Pennsylvania, Lehigh University.

The officers for 1930–1931 were: Professor Tomlinson Fort, Director; John C. Mertz, Vice Director; John E. Freehafer, Second Vice Director; Harry C. Kelly, Secretary; Stephen L. Gregg, Treasurer; George J. Schaumburg, Librarian. The officers are elected by a majority vote near the end of the spring semester.

The meetings and programs were as follows:

October 24, 1930: "Foundations of geometry" by Professor Tomlinson Fort; "An introduction to non-Euclidean geometry" by Harry C. Kelly.

November 19, 1930: "A general introduction to the topic of projective geometry" by Professor Tomlinson Fort; "The principle of duality" by A. N. Rogers; "Cross ratio" by D. L. MacAdam.

December 15, 1930. "Second order configurations" by W. J. Tomlinson; "Complete quadrangle" by R. H. Raring.

February 16, 1931: "Theorems of Pascal and Brianchon" by W. C. Elmore; "Quadric surfaces" by P. G. Reynolds.

March 18, 1931: "Philosophical aspects of probability" by Benj. Rabinowitz; "Elementary theory of probability" by C. H. Krott; "Examples of probability using calculus methods" by J. B. Hartman.

April 22, 1931: "Humanism in mathematics and science" by Professor C. J. Keyser, Columbia University (This lecture was open to the public).

It is the policy of this chapter to concentrate on one subject of study, developing the topic in greater detail at successive meetings. Each spring there is a lecture open to the public.

This chapter has 45 active members. Twenty new members were initiated. The new members are tapped at a regular chapel exercise sometime in April and initiated in May. A banquet is held in connection with the initiation.

H. C. KELLY, *Secretary*

Alpha of California, University of California at Los Angeles.

California Alpha of Pi Mu Epsilon extends greetings and best wishes to its fraternal chapters. A very successful year has been enjoyed by the chapter under the direction of the following officers: Sibyl Rock, Director; Virginia Woods, Vice Director; Alta Blackford, Secretary; John Hill, Treasurer; Dr. Whyburn, Librarian; Dr. Daus, Faculty Adviser.

The meetings and programs were as follows:

October 8, 1930: Election to membership of Marjorie Easterly; Reed Lawlor; Abram Loshokoff; Jean Robb.

November 5, 1930: "Brocard points" by Annie Peterson.

December 3, 1930: "Inversion" by Francis Herrmenn.

December 6, 1930: Initiation of the newly elected members at the home of Alta Blackford.

December 28, 1930: Christmas party in conjunction with the Engineers Club and the Mathematics Club.

February 11, 1931: Dr. Slaught of the University of Chicago was made an honorary member of our chapter. At this time he gave a very interesting talk on "The growth and development of mathematical organizations." Dr. Slaught is our first honorary member.

March 9, 1931: Election to membership of Margaret Barney; Goldie Ivener; Chester Lagenbeck; John Montgomery.

March 25, 1931: "Subjects of tangents" by Marjorie Easterly; "Bohr theory" by Ernest Von Seggern.

April 18, 1931: Initiation of the newly elected members at the Mona Lisa cafe.

April 20, 1931: Special business meeting for the revision of rules and regulations concerning elegibility for membership.

May 1, 1931: Beach party at the Deauville Beach Club in conjunction with the Mathematics Club.

May 6, 1931: "Annual calculus examination." The award of ten dollars was given to Russell Doescher. Honorable mention was given to Donald Hyers.

May 13, 1931: "Definition of algebra" by Sibyl Rock. (Election of officers.)

ALTA BLACKFORD, *Secretary*

Pi Mu Epsilon of Syracuse University.

The officers for 1930–1931 were: Mr. David MacAlpine (Instructor in the College of Applied Science), Director; Sara Fister, Vice Director; Helen E. Kelley, Secretary; W. E. Moulton, Treasurer.

This chapter has a Scholarship Committee the members of which are: Professor I. S. Carroll, Chairman; Professors Taylor and Keenan; Sara Fister; Ruth Kohman; Malcolm Adler; Tilroe Hedden.

The meetings and programs were as follows:

October 7, 1930: Picnic. Major students in mathematics were the guests of the Chapter.

October 24, 1930: "Puzzles" by Dr. Hurwitz of Cornell University.

December 3, 1930: "Conic sections" by Mr. John Randolph. Twenty two students were elected to membership at this meeting. Of these, eight were graduate students, fourteen were undergraduates.

January 15, 1931: Initiation.

February 19, 1931: "Fourth dimension" by Dr. A. D. Campbell.

March 26, 1931: "The mathematics of aerial photography" by Professor Earl Church.

April 22, 1931: Business meeting.

May 6, 1931: Picnic. Major students in mathematics were the guests of the Chapter.

May 13, 1931: Initiation and Banquet. Forty-five members were present. Sara Fister was chairman of the banquet committee. Election of officers took place at the business meeting after the banquet.

MAY SPERRY, *Corresponding Secretary*

Pi Mu Epsilon of the University of Illinois.

The officers for 1930–1931 were: Mr. Marion T. Bird, Director; Mr. Donald M. Brown, Vice Director; Mr. Harry E. Crull, Corresponding Secretary; Miss Elsie Zelle, Recording Secretary; Mr. Julian K. Knipp, Treasurer.

Mr. Brown was elected October 14, 1930 to fill a vacancy. The other officers were elected at the regular spring election, May 20, 1930. All officers were elected by vote of the members present at these meetings.

The University of Illinois Chapter has 55 active members, of whom 30 were members at the beginning of the academic year, 14 were initiated December 16, 1930 and 11 were initiated May 4, 1931.

The meetings and programs were as follows:

October 28, 1930: "Analysis situs" by Professor Brahana.

January 13, 1931: "Space of many dimensions" by Dr. Wilson.

February 24, 1931. "An original theorem concerning the separation of the zeros of a function and its derivative" by Mr. F. S. Wood.

March 10, 1931: "Reverberation equation applied to a room" by Professor Watson. (This meeting was an open meeting.)

March 24, 1931: "Functions similar to the trigonometric functions" by Dr. O. K. Bower.

March 31, 1931: "A method of numerical integration of a first order differential equation" by Mr. Crull.

April 7, 1931: "Some remarks about the circle" by Professor Levy.

April 21, 1931: "Inversive invariants of three points" by Dr. Peters. (This meeting was an open meeting.)

On December 16, 1930 and May 4, 1931 we had our initiation banquets at the Inman Hotel in Champaign and on February 13, 1931, we had a theatre party at the Rialto Theatre in Champaign.

HARRY E. CRULL, *Corresponding Secretary*

Alpha of Alabama, University of Alabama.

The officers for 1930–1931 were: Charles E. Watkins, Director; Joseph S. Gelders, Vice Director; William F. Adams, Secretary; Brent G. Clark, Treasurer; Edith Gregory, Librarian.

We have 36 active members. An initiation was held on December 2, 1930 at which time 16 were initiated.

The meetings and programs were as follows:

September 30, 1930: "Exploring the house of arithmetic" by William Sell.

October 28, 1930: "Problem involving the determination of the orbit of one binary star around another having given the law of force" by Sara E. Haughton.

December 2, 1930: "To find the envelope of all the circles that cut two circles at a fixed angle" by Brent G. Clark.

January 27, 1931: "The Lorentz transformation" by Dr. B. A. Wooten.

February 24, 1931: "Two theorems of characteristic equations of square matrices" by H. S. Thurston.

April 7, 1931. "The foundations of mathematics" by Dr. W. P. Ott.

Alpha of Alabama has enjoyed two social meetings.

WILLIAM F. ADAMS, *Secretary*

Pi Mu Epsilon of the University of Pennsylvania.

This chapter of Pi Mu Epsilon has had a most successful year under the direction of the following officers. Dr. Maurice J. Babb, Director; Dr. George Gailey Chambers, Vice Director; Mr. Paul A. Knedler, Treasurer; Miss Margaret E. Koons, Secretary; Mr. Weinstein, Librarian.

The meetings and programs were as follows:

October 17, 1930: "Rational right triangles of equal areas" by Mr. W. A. Bristol.

November 21, 1930: "Mayan mathematics" by Miss Helen O'Boyle.

January 16, 1931: "The counting of molecules" by Dr. Horace C. Richards. (Department of Physics.)

February 20, 1931: "The cat's cradle and string figures" by Dr. H. M. Lufkin. (Department of Physics.)

March 20, 1931: "Infinitely continued exponents" by Mr. R. P. Bailey.

April 17, 1931: "Applications of mathematics to chemistry" by Mr. J. C. Miller. (Department of Chemistry.) At this meeting, 10 new members were initiated.

Prizes have been offered for the solution of interesting problems in mathematics and so far two excellent problems have been suggested and successfully solved by undergraduate students.

The May meeting is an open meeting. At this meeting the address is given by an outside speaker. We also have our annual picnic in May and this event is always well attended.

MARGARET E. KOONS, *Secretary*

Beta of Ohio, Ohio Wesleyan University.

The officers for the year 1930–1931 were: Raymond F. Felts, Director; Dean Friedly, Vice Director; Adelene Offinger, Secretary; John F. Foster, Treasurer.

The meetings and programs were as follows:

October 8, 1930: "Polygons and polyhedrons" by Professor Rufus Crane.

October 29, 1930: Business meeting. Election of new members.

November 11, 1930: Initiation. The club was entertained at the home of Professor and Mrs. Rowland.

December 8, 1930: "Education conference" by Miss Marie Gugle, Assistant Superintendent of Schools, Columbus, Ohio.

January 14, 1931: "Pythagoras and his theorem" by Mr. Paul Mathews.

February 11, 1931: "History and development of logarithms" by Mr. Wilbur Robinson.

March 9, 1931: "Problems of the student teacher" by five members. (Banquet.)

April 15, 1931: "Determinants" by Evelyn Coates.

May 13, 1931: "Statistical approach to the business depression" by Adelene Offinger.

ADELENE OFFINGER, *Secretary*

Delta of Pennsylvania, Pennsylvania State College.

The Pennsylvania Delta Chapter of Pi Mu Epsilon at present has 52 active members. Seven are charter members, 12 were elected at the beginning of the fall semester, 24 during that term and 9 during the current semester.

The officers for the first semester elected in October at the first meeting of the Chapter were: Dr. Haskell B. Curry, Director; Dr. Orrin Frink, Vice Director; Miss Gladys Quigg, Treasurer; Dr. Aline Huke, Secretary; Mr. H. L. Van Velzer, Librarian.

The officers for the second semester were: Mr. Evan Johnson, Director; Mr. George Fisanick, Vice Director; Miss Gladys Quigg, Treasurer; Mr. Eugene M. Fry, Secretary; Dr. Haskell B. Curry, Librarian.

Meetings were held with reasonable regularity during the year. The speakers were for the most part members of the Graduate Mathematics Faculty. Two meetings, however, were conducted by students and one by Dr. Altar, Instructor in Physics.

The meetings and programs were as follows:

October 29, 1930: "The fundamental laws of algebra" by Dr. Curry.

December 4, 1930: "Eclipses" by Dr. Rupp.

January 14, 1931: "Indeterminate equations" by Dr. Frink.

February 12, 1931: "Chinese mathematics" by Mr. Herple.

March 13, 1931: "A topic in descriptive astronomy" by Mr. Fry.

March 27, 1931: "The structure of physical theories" by Dr. Altar.
May 7, 1931: "The solutions of equations by successive approximations" by Dr. Sheffer.
May 20, 1931: "The foundations of geometry" by Dr. Owens.

Since the organization has now become more firmly established at Penn State it is hoped that next year we will be able to broaden our program and bring in visiting lecturers from other institutions. It is believed that the year's activities have justified the establishment of the chapter and we hope that, with the carrying through of the new program, the fraternity may come to be a real asset to the campus.

EVAN JOHNSON, *Director*

B.

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the George Washington University.

The officers for the year 1930–1931 were: Professor F. E. Johnston, President; Albert Wertheimer, Secretary. The officers are elected annually at the first meeting of the academic year.

The aim of the club is to stimulate a creative interest in mathematics. Membership is open to all persons having a genuine interest in the subject. This year we had 25 members.

The meetings and programs were as follows:

October 13, 1930: "Measurements of relation" by Dr. Frank M. Weida.
October 27, 1930: "The controversy over the foundations of mathematics" by Captain E. E. Hagler, Jr.
November 10, 1930: "Curve tracing" by Dr. F. E. Johnston.
November 24. 1930: "The history of non-Euclidean geometry" by Dr. Tobias Dantzig.
December 8, 1930: "Magic squares" by Abraham Sinkov.
January 12, 1931: "Ovals" by P. J. Federico.
February 25, 1931: "Maximum polyhedra" by Michael Goldberg.
March 11, 1931: "Applications of least squares" by Dr. W. E. Deming.
March 25, 1931: "Isometric projection in analytic geometry" by Dr. Walter F. Shenton.
April 15, 1931: "Hypercomplex variables" by Dr. Edgar W. Woolard.
April 29, 1931: "The reality of the imaginary" by Katherine G. Hawley.
May 16, 1931: "Some problems on the foundations of geometry" by Dr. Wilhelm Blaschke of Hamburg, Germany.

ALBERT WERTHEIMER, *Secretary*

The Undergraduate Mathematics Club of the University of Iowa.

The officers for the year 1930–1931 were: Carl H. Fischer, President; Thelma T. Coate, Secretary-Treasurer; Professor L. E. Ward, Faculty Advisor.

The meetings and programs were as follows:

October 29, 1930: "A recent problem in the American Mathematical Monthly" by Professor Roscoe Woods.
November 20, 1930: "The Galois theory of equations" by Mr. Allen T. Craig.
February 19, 1931: "The positive integers" by Mr. Deane Montgomery.
March 12, 1931: "An elementary method of computing logarithms" by Professor L. E. Ward.
April 16, 1931: "Contemporary American Mathematicians of the last thirty years" by Professor H. L. Rietz. At this meeting the officers for the year 1931–1932 were elected.

THELMA COATE, *Secretary*

The Newtonian Society of Lehigh University.

The officers for the year 1930–1931 were: Richard Lindabury, President; John B. Hancock, Vice President; David G. Wright, Secretary-Treasurer. The officers are elected immediately after the initiation of new members in February, the third meeting of the second semester.

The purpose of the society is to promote interest in mathematics among members of the

freshman class, to give its members opportunity for intellectual activity outside the classroom, and to promote friendship between its members and the members of the faculty.

Membership is an honor conferred only upon students of high standing in mathematics. Members are chosen from the freshman class at the end of each semester. The membership is limited to 40. The candidates for membership are nominated by their teachers and elected by a majority vote of the members present at a regular meeting. At present, we have 33 active members.

The meetings and programs were as follows:

February 23, 1931: "Life of Sir Isaac Newton" by J. Ricards.

March 9, 1931: "An old Hindu solution of the general quadratic" by R. Byers; "The slide rule and its inherent errors" by B. Beach.

March 23, 1931: "Paradoxes" by B. Fortman; "Outline of the history of mathematics" by K. Honeyman.

April 13, 1931: "Proof of the functions of the sum of two angles" by F. Geiger.

April 27, 1931: "The intersection of two parabolas" by C. MacDonald; "Derivation of logarithms" by P. Loughran; "Determination of three numbers, the sum of squares of two of which are equal to square of third" by Dr. E. H. Cutler.

May 11, 1931: "Universal utility of mathematical training" by Professor John Stocker. After this meeting the society held a social hour and smoker.

DAVID G. WRIGHT, *Secretary*

The Mathematics Club of Brown University.

The Mathematics Club at Brown University is run upon as informal a basis as possible. Beyond the necessary membership in the committees for the program and arranging the details of the meetings, there are no student officers. All undergraduates beyond the freshman year who are taking courses in mathematics are eligible for membership without other formality than the payment of $1.00 annual dues. The dues are used to cover the cost of refreshments at meetings. The student speakers are thoroughly drilled by members of the faculty, and the programs have been of sufficient interest to attract from 80 to 90 persons to most of the meetings. The program is printed and distributed in October, and no instance has occurred of incomplete preparation or absence of the announced speaker. Following each program there is a brief social hour at which on a few occasions there has been informal dancing. The students from Pembroke College share equally with the undergraduate men from Brown University.

The meetings and programs were as follows:

November 4, 1930, Wilson Hall, Edward M. Read, 3rd, '31, presiding: "Diophantus and Fermat" by Ruth Barden Eddy, '32; "Rational triangles" by Delbert S. Wicks, Jr., '32.

December 2, 1930, Pembroke Hall, Enis E. DeMagistris, '31, presiding: "The meaning of the mean" by James B. Brown, '31; "Prisms and pyramids" by Howard W. Memmott, '33.

January 13, 1931, Wilson Hall, Professor Currier, presiding "How Mother Earth keeps tab on Father Time" by Charles H. Smiley, Assistant Professor of Mathematics, Brown University.

February 24, 1931, Wilson Hall, Donald L. Fowler, Jr., '31, presiding: "Consequences of cutting a cone" by Tina Codianni, '33; "Seeing curves in the dark" by Raymond C. Archibald, Professor of Mathematics, Brown University.

March 24, 1931, Pembroke Hall, Ethel McKechnie, Gr., presiding: "Chow's tail" by William Solomon Wilson, '31; "Endless decimals and fractions" by Marjorie H. Smith, '32. (The club picture was taken at this meeting.)

April 28, 1931, Wilson Hall, Professor Archibald, presiding: "Symmetrical space forms," Illustrated by models, by Albert Harry Wheeler, North High School, Worcester, Massachusetts.

Committee on Program: Professor Bennett; Professor Oakley; Enis Eva DeMagistris, '31; Marjorie Helen Smith, '32; William Solomon Wilson, '31.

Committee on Arrangements: Mr. Jonah; Emma Mae Breyer, '31; Elizabeth Alma Partridge, '33; Austin Hazen, Jr., '32; Delbert Swan Wicks, Jr., '32.

ALBERT A. BENNETT, *Faculty Supervisor of the Program Committee*

The Junior Mathematics Club of the University of Chicago.

The Junior Mathematics Club of the University of Chicago is conducted by the graduate students of the department of mathematics. The officers of the Junior Club are elected by vote of the Club at the end of each academic year. The officers for 1930–1931 were Mr. Arnold E. Ross, President; Mr. Saunders MacLane, Secretary-Treasurer; Miss Ruth Mason, Social Committee.

The subjects presented at the meetings of the Club are usually of general mathematical interest. That is, they are historical sketches, surveys of certain domains of mathematics, or of applied mathematics, or discussions of famous classical problems. Any student in mathematics is eligible to membership. There are about fifty members.

In the current academic year the President and the Secretary of the Club edited the "Bulletin of the Junior Mathematics Club." The Bulletin consisted of a few typewritten pages, and contained an abstract of a forthcoming lecture, report of happenings in mathematics and among mathematicians, and other topics of interest to the members of the club. A new issue was placed on the bulletin board on alternate Mondays preceding the meetings.

The meetings and programs were as follows

October 8, 1930: "Historical Sketch of the Junior Mathematics Club" by Professor H. E. Slaught; "The Mathematical Association of America and the American Mathematical Society" by Professor M. I. Logsdon.

October 22, 1930: "An historical sketch of the theories of the calculus of variations" by Professor G. A. Bliss.

November 5, 1930: "Newton polygons in factorization of polynomials" by Mr. Saunders MacLane

November 19, 1930: "The three pile match problem" by Professor W. D. MacMillan.

December 4, 1930: Trip to Adler Planetarium.

January 7, 1931: "Probabilities" by Professor Walter Bartky.

January 21, 1931: "The isoperimetric property of the circle" by Mr. Max Coral.

February 4, 1931: "Mathematical theories of life assurance" by Miss Frances Wiancko.

February 18, 1931: "Approximating plane curves by Newton polygons" by Rev. J. E. Case, S. J.

March 4, 1931: "Graphical methods for the determination of the elements of orbits of double stars" by Professor Kurt Laves.

April 10, 1931: "A revision of the theory of curvature" by Professor J. A. Schouten of Delft, Holland.

April 22, 1931: "An analogue of continued fractions" by Mr. M. R. Hestenes.

May 6, 1931: "A method for constructing cipher messages" by Dr. Mina Rees.

May 20, 1931: "Problems in mathematical economics" by Professor Henry Schultz.

June 3, 1931: "Life of Galois" by Miss Ruth Mason.

The Junior Club largely sponsors the social activities of the Department. A social half hour is held in the Common Room of Eckhart Hall before each Club meeting. Tea is served. During this year the Club has sponsored two dances and several bridge parties. One outstanding social evening in which all members of the Mathematics Department are included is always held some time during the year. The activities of the Junior Mathematics Club are quite distinct from and in addition to the work of the graduate research club which holds bi-weekly meetings.

ARNOLD ROSS, *President*

The University of Colorado Mathematics Club.

The object of this club shall be to stimulate among the members a spirit of inquiring interest in mathematics which will reveal to them more perfectly the cultural value of that science, and to promote a spirit of co-operation and friendship.

Any student enrolled in the University of Colorado who has completed successfully in a University or College of reputable standing three hours of mathematics is eligible to active membership. This year we had 21 active members.

The officers for 1930–1931 were: Mr. Robert A. Merrill, President; Mr. Marvin Halldorson, Vice President; Miss Janet Hall, Secretary-Treasurer; Professor A. J. Kempner, Faculty Advisor.

The meetings and programs were as follows:

October 8, 1930: Reading and ratification of the constitution.
October 16, 1930: "Fallacies in elementary mathematics" by Professor A. J. Kempner.
October 29, 1930: "Mathematical problems and their solutions" by Mr. Richard Furr.
November 13, 1930: "Fourier series" by Professor C. A. Hutchinson.
December 2, 1930: "Relations of time and space in the theory of relativity" by Mr. H. T. James.
January 8, 1931: "Hyperbolic functions and their use in engineering" by Mr. W. A. Wildhack.
January 22, 1931: "Crazy functions" by Mr. Jack Britton.
February 19, 1931: "Quadrature of the circle" by Miss Doris Hyddleston.
March 5, 1931: "Two theorems in geometry" by Miss Elizabeth Cole.
April 23, 1931: "Probability" by Mr. Robert A. Merrill.
May 14, 1931: "The history of calculus" by Miss Letha Lyon.
May 26, 1931: Math. Fry.
May 28, 1931: "Majoring in mathematics" by Professor G. H. Light.

ROBERT A. MERRILL, *President*

The Bryn Mawr Mathematics Club.

The officers for 1930–1931 were: Gretchen Mueller, '32, President; Ruth Unangst, '31, Vice President and Treasurer; Pauline Huger, '32, Secretary.

The officers were elected at the last meeting of the club last year on May 13, 1930. The club has eight active undergraduate members, three faculty members, and five honorary graduate members. The primary aim of the Mathematics Club is to stimulate interest in mathematics, and to give the students, who are taking advanced courses, a chance to hear interesting topics reported on and discussed. Only those students who are taking at least the major course are eligible for membership.

The meetings and programs were as follows:

December 2, 1930: "Some aspects of number theory" by Dr. Widder.
January 12, 1931: "Non-Euclidean geometry" by Pauline Huger.
February 2, 1931: "Geometry of compasses" by Dr. Hedlund.
April 29, 1931: "The Waring problem" by Gretchen Mueller.

There will be another meeting on May 23, to which all those students are invited who are planning to take the major course in mathematics. Dr. Lehr will speak, and after her talk the club plans to go on a picnic, where baseball and singing will most probably be indulged in.

PAULINE HUGER, *Secretary*

The Mathematics Klub of Adelphi College.

The officers for 1930–1931 were: Dr. Joseph Bowden, Honorary President; Lois Corbe, '31, President; Kate Sandowsky, '31, Vice President; Theresa Cartereau, '32, Secretary; Sarah Gordon, '32, Treasurer.

The officers are elected in May of the preceding semester by a two-thirds vote by ballot of the members present.

The object of the Mathematics Klub is to promote interest in the study of mathematics. Any person may be elected to membership in the Klub who is interested in the purpose of the organization. We have twenty-five active members.

The meetings and programs were as follows:

September 30, 1930: Annual party to welcome new members. Mathematical problems were presented by Kate Sandowsky, '31; Sarah Gordon, '32; Marie Garlichs, '31; Lois Corbe, '31.
October 8, 1930: "Congruences" by Dr. Joseph Bowden.
November 6, 1930: "The distance to the moon" and "Saturn's rings" by the President of the Astronomy Club. Observations were held in the College Observatory. Joint supper meeting of Mathematics and Astronomy Clubs.

November 24, 1930: "Trisection of an angle" by Adele Shrage, '32; "Gottfield von Leibnitz" by Kate Sandowsky, '31; "Harriot" by Sarah Gordon, '32.

Some mathematical problems were presented by various members of the klub.

December 2, 1930: "Scientific Calvinism" or "Predetermination" by Professor John A. David.

December 16, 1930: "Rhythm" by Professor William A. Thayer.

February 16, 1931: Professor Joseph Bowden and Lois Corbe, '31, gave proofs of interesting problems.

March 3, 1931: "Relation of mathematics to thoughts of God" by Dr. Ernest N. Henderson.

March 12, 1931: Supper meeting. A social meeting preceded the supper.

April 21, 1931: "Historical notes on the theory of the slide rule" by Theresa Cartereau, '32; "Demonstration of the slide rule" by Lois Corbe, '31.

May 5, 1931: Supper meeting. A play "Discord in Mathematics Land" was presented by the members. "Suggestions for future teachers" by Miss Ada Ostrander, Head of Mathematics Department, Sewanhaka High School, Floral Park, N. Y.

May 12, 1931: Nomination of officers for 1931–1932.

May 18, 1931: Election of officers for 1931–1932.

May 19, 1931: Installation of officers for 1931–1932.

THERESA CARTEREAU, '32, *Secretary*

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Eugenio Rignano Prize

The Review "Scientia," in agreement with the family of its late illustrious Director Eugenio Rignano, who died in Milan on February 9th., 1930, has founded a "Rignano Prize," of the value of 10,000 Lire, to be awarded by competition to an author of a study on "The Evolution of the Notion of Time." The Review believes that in taking this step it is paying the best tribute to the memory of its late Director by giving a new impulse to the scientific and philosophical studies to which he devoted his whole life, and by developing at the same time that union and spiritual emulation between the scientists of all countries, which was his highest aspiration.

The competition is completely international and open to all. Anyone can take part in it, without regard to school, tendency, or faculty.

Conditions of the Competition:

1.—The aspirants for the prize shall make known their intention of competing by sending their works to the Editor of "Scientia" not later than the 31st. of December, 1932.

2.—The works submitted must either be unpublished, or published since the year 1930, and must be written in one of the following languages: Italian, French, English, German, or Spanish. Works submitted must be typewritten or printed.

3.—In all cases, each work must be accompanied by a typewritten summary of not more than 10 pages (4000 words) in length, capable of being published as an article in the Review.

4.—The examination of the works submitted will be entrusted to a Committee nominated by the Directorate of the Review.

For all further information apply to the Editor of "Scientia," 12, Via A. De Togni, Milano (116), Italy.

Professor Henri Fehr, Head of the Department of Mathematics at the University of Geneva, has been appointed Recteur of the University. He is well-known to Americans and to the mathematical and educational world at large as the editor of l'Enseignement Mathématique, and as Secretary of the International Commission on the Teaching of Mathematics, not only the present one, but the one that was founded in 1908. He was a student at the Sorbonne and at the Collège de France. He has held a professorship in the University of Geneva since 1900, and was twice Dean of the Faculty of Science. From 1928 to 1930, he was Vice-Recteur of the University, and has frequently represented it at various ceremonies, at Oslo, Paris, Brussels and elsewhere. He was one of the founders and was president of the Société Mathématique Suisse. He is also a member of the Légion d'Honneur, besides being corresponding member of various mathematical societies.

Through the courtesy of Professor W. F. Shenton of the George Washington University, the Library of the Mathematical Association now possesses a complete set of the *Mathematical Visitor* published by Artemas Martin during the years 1877–1894, and of the *Mathematical Magazine* published by him during the years 1882–1910. This includes a copy of the very rare first edition of Volume 1, Number 1 of the *Mathematical Visitor*.

Todhunter's *History of the Theory of Probabilities* is being reprinted by G. E. Stechert & Co. This book has been so rare that it has been valued in the second hand market at from \$40 to \$50. But even at that price no copy could be found from which to make the reprint. It was necessary to borrow a copy from a university library. The price of the photographic print, equal to the original, will be \$7.50.

Jenaro Moreno Garcia-Conde, Professor of Mathematics in the School of Arts and Crafts, School of Military Engineers and in the Military Staff College, Santiago, Chile, has been granted a fellowship by the John Simon Guggenheim Memorial Foundation for studies, in the United States, of the theory of functions of real variables and of complex variables, and of the calculus of variations of simple and multiple integrals.

M. Albert Linton was elected in June 1931 to the presidency of the Provident Mutual Life Insurance Company of Philadelphia. He entered the service of the Provident Mutual in 1909, became mathematician in 1913, associate actuary in 1915, and vice-president in 1916. He is a charter member of the Mathematical Association, fellow and vice-president of the Actuarial Society of America, fellow of the Institute of Actuaries of London, and chairman of the Life Insurance Sales Research Bureau of Hartford.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 8, OCTOBER

PART I

The October Number is issued in two parts of which Part II is the Library Catalog. Price 50¢ to non-members.

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28. 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE FIFTEENTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The fifteenth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colo., on Friday and Saturday, April 17 and 18, 1931. There were three sessions, Professor Claribel Kendall presiding at each.

The attendance was forty-eight, including the following twenty-three members of the Association: C. F. Barr, Jack Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, G. W. Gorrell, Sidney Hacker, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, O. C. Lester, A. J. Lewis, G. H. Light, S. L. Macdonald, A. S. McMaster, J. Q. McNatt, W. K. Nelson, Greta Neubauer, E. J. Purcell, E. D. Rainville, O. H. Rechard, Mary S. Sabin, C. H. Sisam.

At the business session the following officers were elected for the next year: Chairman, Professor O. H. Rechard, University of Wyoming; Vice-chairman, Professor A. G. Clark, Colorado Agricultural College. Plans were made to meet at the University of Wyoming in the spring of 1932.

The following papers were presented:

1. "On a biological application of the Poisson series" by Professor A. G. Clark, Colorado Agricultural College.
2. "A theorem on foci" by Mr. E. J. Purcell, University of Colorado.
3. "Projective geometry and some of its relations to other courses in mathematics and mathematical physics" by Professor J. R. Everett, Colorado School of Mines.
4. "Solution of a problem in dynamics" by Professor D. F. Gunder, Colorado Agricultural College, by invitation.
5. "On Riccati equations" by Professor C. A. Hutchinson, University of Colorado.
6. "Geometry as an avocation" by Professor A. J. Kempner, University of Colorado.
7. "The practical experiences of engineering in the mountain districts of Colorado" by Mr. J. Q. McNatt, Colorado Fuel and Iron Company.
8. "Linear systems of curves on an algebraic surface" by Professor C. H. Sisam, Colorado College.
9. "On the application of equations to the solution of congruences with prime moduli" by Mr. E. D. Rainville, University of Colorado.

Abstracts of some of the papers follow, the numbers corresponding to the numbers of the titles:

1. Professor Clark showed how the Poisson distribution could be used to simplify the work of the seed analyst in handling the problem of noxious weed seeds in certification work.

2. Mr. Purcell presented and proved the following theorem on foci, which he believes to be new: If a real algebraic curve of class M has one or more real

axes of symmetry, then in general exactly M foci lie on each axis of symmetry. Any such set of M collinear foci completely determines the remaining foci.

4. The problem of a particle flying off a fixed curve while sliding under the force of gravity is a familiar one to students of elementary mechanics. The corresponding problem when the force of friction is considered had been solved for various particular curves. Professor Gunder gave the solution of the problem for any curve considering friction and the particle sliding from any given position. The solution in the case of the circle was given as an illustrative example.

5. Professor Hutchinson's paper gave an exposition of the elementary theory of the Riccati equation; discussed a few points of contact with the general theory of differential equations; and exhibited graphically the continuity of the integral curves with respect to the constant of integration.

7. Mr. McNatt discussed the unusual situations met in surveying in mountain districts. He showed the application of mathematics to the solution of various problems met in such work.

8. This expository paper deals with the definition of a birational transformation of an algebraic surface; the invariant properties, under birational transformations, of linear systems of curves on an algebraic surface; the significance and invariance of the genera of the surface; the simple and double integrals of the first kind on the surface, and the meaning of irregularity and its connection with non-linear systems of curves on the surface.

9. Mr. Rainville's paper was an expository account of some of the results obtained by extending the congruence notation of the theory of numbers. Some simplifications in solution of certain congruences is effected. A few of the results are to be found in Gauss's works. The most important contribution is probably Poinsot's "Sur l'application de l'algèbre à la théorie des nombres."

A. J. LEWIS, *Secretary*

THIRTEENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The thirteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at the Bradley Polytechnic Institute, Peoria. Illinois, on May 1 and 2, 1931. The chairman, Professor H. B. Curtis, presided at all the sessions.

Forty-three persons registered attendance. Among these were the following twenty-nine members of the Association: Beulah Armstrong, Edith I. Atkin, H. W. Bailey, R. W. Barnard, Walter Bartky, C. E. Comstock, H. B. Curtis, Edna M. Feltges, Elinor B. Flagg, A. E. Gault, L. M. Graves, Martha Hildebrandt, Mildred Hunt, J. M. Kinney, W. C. Krathwohl, Luise Lange, Mayme I. Logsdon, W. D. MacMillan, C. N. Mills, G. E. Moore, E. J. Moulton, Mary W. Newson, W. A. Richards, Mina S. Rees, Mary B. Rumsey, E. W. Schreiber, H. E. Slaught, W. A. Spencer, C. A. VanVelzer, F. E. Wood.

The following officers were elected for the coming year: Chairman, R. W.

Barnard; Vice-chairman, W. C. Krathwohl; Secretary-Treasurer, C. N. Mills; and Associate Secretary-Treasurer, Edith I. Atkin. The meeting for the year 1932 will be at the University of Illinois.

The following seven papers were presented:

1. "Reorganization of material for freshman mathematics" by Professor Mayme I. Logsdon, University of Chicago.

2. "Real branches of algebraic curves" by Professor H. R. Brahana, University of Illinois, by invitation.

3. "On some points relating to the restricted theory of relativity" by Dr. Luise Lange, Crane Junior College.

4. "Stellar systems" (Evening lecture) by Professor W. D. MacMillan, University of Chicago.

5. "Analysis Situs" by Dr. N. E. Rutt, Northwestern University, by invitation.

6. "An experimental study of the effects of sectioning in college mathematics" by Miss Helen Taylor, University of Illinois, by invitation.

7. "Some recent work on the application of Padé approximants to divergent series" by Dr. H. S. Wall, Northwestern University, by invitation.

Abstracts of six of these papers follow:

1. A reorganization of the material of freshman mathematics is to be desired because of certain weaknesses in the present almost universal practice of allowing three (or four) semester hours or five quarter hours to each of the pre-calculus subjects. These weaknesses are, (1) the inequitable distribution of time to the three subjects, trigonometry, college algebra, and analytical geometry; (2) the ineffectiveness of teaching an unrelated succession of topics in algebra; (3) the loss in time and in understanding by presenting without the notions of the calculus topics which should properly be developed from that point of view; (4) the unavoidable duplication due to the fact that certain topics and certain methods belong to more than one of the subjects.

The arrangement described by Professor Logsdon is believed by her to be free from the above objections, to present the material in a logical manner, and to secure and hold the interest of the students because they can see from the beginning that they are building a structure in which each part has a definite relation to the whole.

2. Mr. Brahana gives an exposition of the representation of an algebraic curve by means of a closed surface in a Euclidean three-space and discusses the location of the real part of the curve on this surface. Some curves of orders three and four are considered in detail.

3. The purpose of the paper is to point out how, in an introductory presentation of the theory of relativity, greater emphasis on certain basic ideas and greater consideration for the reader's preconceived notions may be helpful in achieving better understanding and preventing misconceptions. Two points are chosen for discussion: (a) on time and time determination; (b) on the unique role played by the velocity of light in this theory.

(a). The distinction is drawn between psychological and physical time; principles of measurement for either are discussed; these introduce subclassification of physical time as that of "nearby" and "distant" events. The relevancy of these distinctions is maintained against arguments to the contrary. A new classification is then obtained: "local" time, which is psychological time plus the time of nearby events, and "distant" time. It is further pointed out that, in applying the Lorentz transformation, judgment has to be used because in the formula the above distinctions are not apparent; and that some erroneous results have been derived in such manner, e.g. that an observer *sees* the lengths in a moving system contracted, and the derolement of time retarded. Correctly applied the Lorentz transformation yields the result that an observer sees an approaching object stretched out, and a receding one contracted; an approaching clock run faster, and a receding one run slower, in the ratio $[(c+v)/(c-v)]^{1/2}$: 1. It is further shown that these results followed already from the classical theory, only with the respective ratios $(1-v^2c^{-2})^{-1/2}$ times greater; and that in the Jupiter system such a clock, alternatingly approaching and receding from the earth, is available to observation. The difference between the effects calculated from the one and other theory is too small to afford an experimental test, being of the order $8 \cdot 10^{-4}$ seconds only.

(b). (1) the idea of a "transcendental simultaneity" is discussed and evaluated; (2) the idea of establishing distant simultaneities by carrying synchronized clocks from place to place is shown to be intrinsically inapplicable. (3) the unique role played by light as a signal is shown to spring from the concurrence of two facts of immediate or indirect experience: that it is the greatest signal velocity known, and that it is said to be constant relative to systems in uniform relative motion. In conclusion it is pointed out that the postulate of the constancy of the velocity of light can and should likewise be treated with sufficient care so as to prevent the impression of its being contrary to sound reason.

5. This is an expository paper describing briefly the theory of dimensionality the separation of spaces by means of sums of their connected subsets, and the notion of a curve.

6. The mathematics department at the University of Illinois conducted a four-year experiment in sectioning students in analytic geometry on the basis of their grades in college algebra and trigonometry. Sixty-six sections including sixteen hundred students and fifty-five different instructors were involved in the experiment.

When three sections met at one hour students were divided into high, middle, and low sections. When two sections met at an hour students were put into high and low sections. Control sections were maintained in which no sectioning was attempted. Students from the experimental and control sections who made equal scores on the Iowa placement tests were considered to be of approximately equal ability and were then compared at the end of the semester on the basis of a new type test in analytics and an old type test in analytics. Through the four years, the students sectioned into three ability groups averaged 7% higher scores

on the old type test and 5% higher on the new type test on the basis of a perfect score. In terms of the average score for the 1600 students the gain is $12\frac{1}{2}\%$. Students sectioned into two groups at an hour show a 4% gain on the old type test, but less than 1% on the new type test. The experiment affords strong evidence for the value of sectioning when three groups meet at an hour and some evidence for sectioning two at an hour. The method of sectioning has been adopted as a policy in the department at the University of Illinois for classes in analytics.

7. Dr. Wall shows that the summability of a power series $\Sigma c_i z^i = P(z)$ afforded by the Padé table is a species of the following kind of summability. Let

$$y_s = \sum_{i=0}^{s} c_i z^i + z^{s+1} \sum_{i=0}^{\infty} c_i^{(s)} z^i,$$

where

$$\sum_{i=0}^{s} \alpha_{i,p}^{(s)} c_i + \sum_{i=0}^{p} \alpha_{s+i+1,p}^{(s)} c_i^{(s)} = 0, \; p = 0, 1, 2, \cdots .$$

Now, if, over a region R of the z-plane, $\mathrm{Lim}_s y_s = y$, then y is a generalized sum (over R) of $P(z)$. For a proper choice of the $\alpha_i^s{}_{,\,p}$, y_s is a Padé approximant of $P(z)$, and is a rational fraction. This "Padé summability" is discussed from various angles, e.g. regularity, applicability, connection with other summabilities, etc. The writer's notion (Bulletin of the American Mathematical Society, vol. 36, p. 646) of a straight line of functional equivalents of $P(z)$ is discussed and amplified.

C. N. MILLS, *Secretary*

THE EIGHTH ANNUAL MEETING OF THE INDIANA SECTION

The eighth annual meeting of the Indiana Section of the Mathematical Association of America was held at Ball State Teachers College, Muncie, Indiana, on Friday and Saturday, May 1 and 2, 1931.

There were one hundred fifty present at the meeting, including the following twenty-five members of the Association: W. C. Arnold, E. R. Bowersox, G. E. Carscallen, P. T. Copp, H. T. Davis, W. E. Edington, P. D. Edwards, T. C. Fry, E. D. Grant, H. E. H. Greenleaf, F. H. Hodge, H. K. Hughes, Florence Long, Juna M. Lutz, H. R. Mathias, T. E. Mason, H. A. Meyer, T. W. Moore, Mary S. Paxton, J. A. Reising, C. K. Robbins, L. S. Shively, W. O. Shriner, R. O. Virts, K. P. Williams.

On Friday evening at 6:30 a banquet was held at Lucina Hall on the campus which was attended by 58 members and guests of the Association. Professor L. H. Whitcraft of the mathematics department of Ball State Teachers College presided. Following two musical selections, the keys of the College were turned over to the visitors in a felicitous address by President L. A. Pittenger. The welcome

was further emphasized in a short address by Dean Ralph Noyer who spoke of the great debt owed to mathematicians by the world as exemplified in the work of Clerk Maxwell and others who have connected pure mathematics with the world of our experience.

At eight o'clock a public lecture was given in Science Hall by Dr. Thornton C. Fry, of the Bell Telephone Laboratories of New York City, who spoke on the subject, "Mathematics comes into its own." Dr. Fry began by painting a picture of the history of mathematical physics from its origin in the atomic theory of Democritus. He showed in epitome the long warfare between proponents of the corpuscular theory of nature and advocates of the theory of continuity in the underlying stratum of things. Bringing the story to the modern era he showed the great perplexity of physics today in its attempt to rationalize the experiments of the quantum theory. The speaker affirmed that we have had three major syntheses of physical experience: (1) Newton's theory of universal gravitation; (2) Maxwell's theory of electricity and magnetism; (3) Einstein's geometrization of nature. He predicted that the fourth synthesis will appear in the mysteries of the quantum theory and stated that great progress toward this end appears in the work of Schrödinger, Born, de Broglie, Heisenberg, Dirac and other mathematical physicists. He emphasized the fact that the first three syntheses were of a definitely mathematical character and that there were indications that the fourth would be of the same nature. The speaker asserted that mathematics only fully comes into its own in so far as it comes in intimate contact with more objective sciences such as physics, quoting in this connection from the preface to the second edition of Newton's *Principia:* "Those who fetch from hypotheses the foundations on which they build their speculations, may form, indeed, an ingenious romance; but a romance it will still be." He concluded with a plea for more attention in America to this border-line work which has heretofore been left almost exclusively to Europeans.

The session on Saturday morning in Science Hall was presided over by Professor P. D. Edwards, Ball Teachers' College, chairman. The following officers were elected: Chairman, Professor G. E. Carscallen, Wabash College; Vice-chairman, Professor T. E. Mason, Purdue University; Secretary-Treasurer, Professor H. T. Davis, Indiana University.

A chairman's address was made by Professor Edwards on "Reorganization of secondary mathematics." Professor Edwards discussed the advantages that would result from a reorganization of secondary mathematics on the plan recommended by the National Committee. It was pointed out that at present 405 of the 841 high schools in Indiana are organized on the 6–6 plan. Mathematics is a required subject through grade nine, but is an elective in grades 10–12. Comparison was made with the mathematics curriculum in Europe where algebra and intuitive geometry are begun as early as grade six. The advantages of the longer period of instruction in algebra were emphasized. The reorganization results in advantages to the student who does not continue his mathematical studies beyond grade nine as well as to the student who continues his study in the senior

high school and college. Emphasis was placed on the rapid increase in recent years of the need for knowledge of statistics, graphs, equations, compound interest and annuities. For the person continuing in mathematics it was pointed out that the three-year interval for the acquisition of the essentials of algebra would result in better fixation of habits of thinking in terms of algebraic symbols. Suggestions were made concerning possible reorganization of material in the senior high school. It was shown that in the decade since the National Committee made its report there has been practically no change in the teaching of secondary mathematics in Indiana. A plea was made that the members of the Mathematical Association take the initiative in bringing about such a reorganization as will best serve the interests of the state.

Mr. Russell Sullivan of Indianapolis presented by invitation an illustrated lecture on "The evolution of the stars." Mr. Sullivan devoted special attention to modern interpretations of the various kinds of nebulas. He sketched the step-ladder evolutionary theory which puts the red giant stars at one foot of the ladder (youth), the blue stars at the top, and the red dwarfs at the other foot (old age). He concluded with an exposition of the interpretation of the high recessive velocities of the spiral nebulas as evidence of the curvature of space-time.

The remainder of the program consisted of the following papers, the third one being read by title:

1. "The early history of Kepler's equation" by Professor K. P. Williams, Indiana University.

2. "Class size, past, present and future" by Mr. C. E. Trueblood, Arsenal Technical Schools, Indianapolis, by invitation.

3. "Synthetic projective geometry as an aid to high school teachers" by Professor W. C. Arnold, De Pauw University.

4. "Some applications of the calculus of residues to the theory of functions" by Professor H. K. Hughes, Purdue University.

5. "Notes on generatrix functions with an application" by Mr. Fred Robertson, Iowa State College, Ames, Iowa, by invitation.

6. "A problem in grade distribution" by Professor C. K. Robbins, Purdue University.

7. "Some recent results in the theory of elimination" by Professor T. W. Moore, Indiana University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper described Kepler's formulation of his famous problem and his treatment of it as given in the *Astronomia Nova* (1609), the *Epitome Astronomiae Copernicae* (1618), and the *Tabulae Rudolphinae* (1627). The tabular solution in the latter disposed of all the actual astronomical necessities of the time, but the comments in the *Astronomia Nova*, which despaired of the possibility of a solution, and the method of approximations in the *Epitome*, usually overlooked by commentators, showed Kepler's desire for an adequate treatment. The later his-

tory of the problem was traced to the time of Lagrange. It was pointed out how the problem could be made an instructive one in a course on the history of mathematics that drew on actual sources.

2. In 1924–25 the speaker established large classes with the idea of developing a suitable technique for handling large numbers in mathematics, in view of the conclusion reached from educational studies made since 1896 that size of class has little influence upon student achievement. His classes ranged in size from 80 to 120. The technique which he established in his first four or five classes has been used with minor improvements ever since. He is now teaching his seventeenth class of 100 in mathematics. His conclusions are: (1) In spite of the success of large class technique there will always be a place for the small class; (2) one teacher will be able to teach three classes of 100 students as efficiently as six classes of 25; (3) one serious fault of present teaching technique is that highly trained teachers are required to perform too many minor details. This can be eliminated by improved methods.

3. In this paper a plea is made for the wider study of synthetic projective geometry among high school teachers. It is pointed out that this mathematical discipline is self-contained and hence is admirably adapted to the needs of one who might wish to extend his knowledge without class-room instruction. Synthetic projective geometry is also more closely connected with the geometry in which the high school teacher is giving instruction than with either analytics or calculus, and its range of beautiful theorems has an esthetic appeal that is not exceeded by other mathematical subjects of similar difficulty.

4. In this paper Professor Hughes considered functions defined by certain types of infinite series, the series themselves being regarded as given. The particular problem was to extend analytically the function defined by the given series into regions exterior to the region of convergence of the series. As consequences of the results obtained, certain further results pertaining to the asymptotic development of the functions in question were discussed. Reference was made to results already established by Barnes, Ford, and others regarding functions defined by power series. The speaker considered functions defined by factorial series of the first and second kinds, and by Dirichlet series. The methods of the calculus of residues was employed to solve the problem of analytic extension. Some asymptotic properties were also obtained.

5. If z^{-n}, where $z=d/dx$, is an n-fold integral operator, it can be shown that this operator is equivalent to another, i.e., $x^nQ_n(\mu)$, $\mu=xz$, which has a Taylor's expansion about $\mu=0$. This operator is called a *generatrix function*. Both μ^{-n} and $Q_n(\mu)$ may be shown to satisfy the differential equation: $\mu Q''(\mu)+(n+\mu+1)Q'(\mu)+nQ(\mu)=0$. The object of the present paper is to show that the solution of the equation of heat conduction, expressed in spherical coordinates, r, θ, ϕ, is given by a function of the form: $V=F(r,\ t)\Theta(\theta)\Phi(\phi)Q_n(\nu)$, where $\nu=4r^2kt$.

6. Professor Robbins presented a solution of the following problem: The grades used in a certain institution are A, B, C, and D. Suppose that the distri-

bution of grades for a certain period of time was A', B', C', D', where A' is the number of A grades etc. The grades are redefined in such a manner that it is estimated that the distribution would have been A'', B'', C'', D'', if the new definitions had been in effect during the above period of time. Suppose that the distribution of an individual for this same period was a', b', c', d'. What would the distribution of this individual necessarily have been under the new definition of grades?

7. This paper was a brief resumé of the results contained in two papers published recently in the Annals of Mathematics under the titles: "Extended results in elimination," (vol. 30, pp. 92–100), and "On the resultant of two binary forms," (vol. 31, pp. 185–189). They are concerned with the problem of representing the eliminant of a definite number of the forms, where the number depends upon the dimension of the domain of definition, as a single determinant free of extraneous factors. In the first paper the question of forms in more than one set of variables was considered, and in the second, new forms of the resultant determinant of two binary forms were exhibited.

At the afternoon session of the Section a resolution was adopted expressing the sorrow of the members at the news of the sudden death of Professor W. A. Zehring of Purdue University, "who has been a faithful attendant of the Section meetings, and a zealous and enthusiastic teacher of mathematics for the past quarter of a century." A second resolution was adopted expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by President Pittenger and the members of the mathematics department of Ball State Teachers College.

In close connection with the meetings of the Section a conference on the teaching of high school mathematics was held. The conference joined with the Section for the morning program, but met separately in the afternoon.

H. T. DAVIS, *Secretary*

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section of the Association was held at Lincoln on May 8, 1931, jointly with the mathematics section of the Nebraska Academy of Sciences. Thirty persons were in attendance, including the following members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, M. M. Flood, M. G. Gaba, A. L. Hill, J. M. Howie, R. M. McDill, T. A. Pierce, Lulu L. Runge. Mr. M. M. Flood, chairman of the Section, presided.

The following officers were elected for the ensuing year: Chairman, Prof. A. K. Bettinger, Creighton University, Omaha; Secretary, Prof. A. L. Hill, Peru State Teachers College; Treasurer, Prof. J. M. Howie, Nebraska Wesleyan University.

The program at the joint meeting was as follows:

1. "On generalized differentiation" by R. M. Ely.
2. "Study of some new orthogonal polynomials" by Mrs. Madeline Grenard.
3. "Sylvester's theorem and an application to the Tschirnhaus transformations" by Violet Wochner.
4. "Demonstration of the Mader harmonic analyser" by A. P. Cowgill.

1. Following a definition given by H. T. Davis in the American Journal, 1924, and with the restriction that $\phi(c)=0$, Mr. Ely studied the operation ${}_cD_x^{v+m}(x)$, $0<v<1$, $m=1, 2, 3, \cdots$; extended the "rationalizing" process of Davis to differential equations of fractional order higher than one, and worked out some examples.

2. Mrs. Grenard discussed polynomials orthogonal to x^{2r}, in the interval $(-1, +1)$ and also when this interval is replaced by a simple closed curve in the complex domain. Difference equations similar to that for the nth Legrendre polynomial were obtained. Corresponding differential equations do not exist.

3. Miss Wochner discussed a theorem of Sylvester's, proved a complementary theorem, and showed how they could be used to perform a Tschirnhaus transformation. A method was given for writing the adjoint of a matrix A, which is in standard form, from the characteristic equation of A. A method for computing high powers of a matrix was suggested.

4. Mr. Cowgill explained the theory of the harmonic analyser of Mader, and demonstrated its use. The curve $y=3 \sin x+4 \cos x+4 \sin 2x-3 \cos 2x-2 \sin 3x-\cos 4x$ accurately plotted and then analysed, gave the equation $y=2.92 \sin x+3.97 \cos x+3.99 \sin 2x-2.97 \cos 2x-1.94 \sin 3x-1.03 \cos 4x$.

M. M. Flood, *Chairman*

HUNTINGTON'S THEOREM ON MOMENTS

By WILLIAM F. OSGOOD, Harvard University

Professor Huntington[1] appears to have been the first to obtain a necessary and sufficient condition for taking moments about a point, in the dynamics of a rigid body in two dimensions. By "taking moments about a point" is meant, writing the equation of moments in the form:

$$I\frac{d^2\theta}{dt^2} = \sum \text{moments}, \tag{1}$$

where I denotes the moment of inertia about the point, and the right-hand side of the equation is the sum of the moments of the forces about the point. His theorem is as follows.

[1] *The theorem of rotation in elementary mechanics*, this Monthly, vol. 21 (1914), pp. 315–320.

Theorem. *It is permissible to take moments about a point Q distinct from the centre of gravity when and only when the vector acceleration of Q, thought of as a point fixed in the body, is collinear with the line joining Q with the centre of gravity, or vanishes.*

Huntington's proof is elegant and succinct, but it makes high demands on the maturity of the student. The following proof, though requiring more computation, is easier for the beginner to understand.

The equation of moments can be written in the form:

$$\frac{d}{dt}\sum m_i \left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right) = \sum \text{moments about the origin}. \tag{2}$$

It is the left-hand side only of this equation, with which we are concerned. Let the body be referred to moving axes, the point $Q:(x_0, y_0)$ being taken as the origin in that system. Then

$$\begin{cases} x = x_0 + \xi\cos\theta - \eta\sin\theta, \\ y = y_0 + \xi\sin\theta + \eta\cos\theta, \end{cases} \tag{3}$$

where x_0, y_0, θ are functions of t which determine the motion. Moreover,

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{dx_0}{dt} - (y - y_0)\dfrac{d\theta}{dt}, \\[2ex] \dfrac{dy}{dt} = \dfrac{dy_0}{dt} + (x - x_0)\dfrac{d\theta}{dt}. \end{cases} \tag{4}$$

Computing dx_i/dt and dy_i/dt from equations (4), we find:

$$x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt} = x_i\frac{dy_0}{dt} - y_i\frac{dx_0}{dt} + \left\{x_i^2 + y_i^2 - x_0x_i - y_0y_i\right\}\frac{d\theta}{dt}.$$

Hence

$$\sum m_i\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right) = M\left(\bar{x}\frac{dy_0}{dt} - \bar{y}\frac{dx_0}{dt}\right) + \left\{I - M\bar{x}x_0 - M\bar{y}y_0\right\}\frac{d\theta}{dt}, \tag{5}$$

where

$$I = M(k^2 + \bar{r}^2),$$

and k denotes the radius of gyration about the centre of gravity.

The most interesting case is that in which the point Q is taken as the instantaneous centre, (x_1, y_1), for the text-book literature is full of mistakes here, as Professor Huntington has pointed out. In this case, if we let Q be at the origin, O, of the (x, y)-system of coordinates at the arbitrary instant of time $t=t$, we shall have, at that instant,

$$x_1 = 0, \quad y_1 = 0, \quad \frac{dx_1}{dt} = 0, \quad \frac{dy_1}{dt} = 0, \tag{6}$$

but the second derivatives, d^2x_1/dt^2 and d^2y_1/dt^2 will not in general vanish.

On differentiating equation (5) with respect to the time and taking account of (6), we find that the left-hand side of (2) reduces to

$$I\frac{d^2\theta}{dt^2}+M\left(\bar{x}\frac{d^2y_1}{dt^2}-\bar{y}\frac{d^2x_1}{dt^2}\right).$$

It is this last term that must vanish, if we are to be allowed to "take moments about the instantaneous centre." It will, of course, vanish whenever the vector acceleration of Q vanishes; but otherwise only when

$$\bar{x}:\frac{d^2x_1}{dt^2}=\bar{y}:\frac{d^2y_1}{dt^2},$$

and this is precisely Professor Huntington's result.

The general case of an arbitrary point $Q:(x_0, y_0)$ can be treated in the same manner. It will be convenient to let Q flash through the origin, O, at an arbitrary instant, $t=t$. Here, $x_0=0$, $y_0=0$, but even the derivatives of the first order will not in general vanish. Still, the left-hand side of (2) reduces to

$$I\frac{d^2\theta}{dt^2}+M\left(\bar{x}\frac{d^2y_0}{dt^2}-\bar{y}\frac{d^2x_0}{dt^2}\right),$$

and Huntington's theorem is proved.

NEW MATHEMATICAL PERIODICALS

By R. C. ARCHIBALD, Brown University

There are probably few mathematicians who are aware that during the past five years at least 32 new mathematical periodicals have been started; they are listed below. Rumania heads the list with seven, followed by Italy with five and Poland with four. Argentina and the United States, each with three, come next; then Germany with two. Czechoslovakia, England, Holland, Hungary, India, Japan, Russia, Switzerland, each have one. Only four of these, the ones whose numbers are marked with a star, have not been inspected and are not in the library here. At least half of those listed may be regarded as research publications, namely, numbers 2, 3, 4, 10, 11, 13, 16, 20, 22, 23, 24, 26, 28, 30, 31, 32.

Rumania

*1. *Buletinul "Asociaţei matematice" a Liceului "C. Alimănestianu" din Olteniţa*, Olteniţa, v. 1, 1928–29.

2. *Bulletin de Mathêmatiques et de Physiques, Pures et Appliquêes de l'Ecole Polytechnique de Bucarest*, Bucharest, v. 1, 1929–30, 238 p.
In French. Mainly articles in pure and applied mathematics.

3. *Bulletin Scientifique de l'École Polytechnique de Timişoara.* Comptes Rendus des Séances de la "Société Scientifique de Timişoara," Timişoara, v. 1, Dec. 1925–1928, 359 p.; v. 2, 1929, 255 p.; v. 3, 1930, 361 p.

   There are many papers, practically all of them being in French.

4. *Mathematica*, Kolozsvár [Cluj], v. 1, 1930, 165 p.; v. 2, 1929, 183 p.; v. 4, 1930, 210 p.

   Sixty-one articles, almost all of them in French, edited by a national committee.

*5. *Revista "Matematică Bârlădeană," Revistă lunară de Matematici Elementare*, Bârlad, v. 1, nos. 1–4, Jan.–Apr., 1926.

6. *Revista Universitară Matematică*, Bucharest, v. 1, 1929, 256 p.

   Taking the place in Rumania that *Nouvelles Annales de Mathématiques* so long filled in France.

*7. *Ziarul Matematic. Revistă lunară pentru uzul Şcoalelar Secondare*, Chişinău, v. 1, nos. 1–5, Jan.–June, 1926.

Italy

*8. *Giornale di Matematica e Fisica della Scuola Media*, Reggio Calabria; there were, apparently, 2 nos. in 1927, 2 nos. in 1928, and 6 nos. in 1929.

9. *Rassegna di Matematica, Fisica e Scienze Naturali, Periodico mensile ad uso degli Studenti delle Scuole Medie*, Gioia dal Colle, v. 1, nos. 1–8, Nov. 1928–June 1929, 144 p.

   No more published.

10. *Rendiconti del Seminario matematico della R. Università di Padova*, Padua, v. 1, 1930, 216 p.

    Contains 10 articles by 8 authors.

11. *Rendiconti del Seminario Matematico e Fisico di Milano*, Milan, v. 1, 1927, 8+129 p.+2 plates; v. 2, 1928, 12+200 p.; v. 3, 1929, 16+267 p.

    The contents include 39 articles.

12. *Rivista di Matematica Pura ed Applicata per gli Studenti delle Scuole Medie*, Palermo, v. 1, Dec. 1925–June 1926, 136 p.; v. 5 was completed in June, 1930. Paged consecutively 1–646, from v. 1, no. 1—v. 5, no. 8.

Poland

13. *Académie Polonaise des Lettres, Comptes Rendus Mensuels des Séances de la Classe des Sciences Mathématiques et Naturelles*, Cracow, v. 1, 1929, 78 p.; v. 2, 1930, nos. 1–7, 55 p.

14. *Mathesis Polska. Miesiecznik poświęcony Naukon Ścislym i ich Metodologji* [Polish Mathesis. Monthly devoted to the science of numbers and methodology], Warsaw, v. 1, 1926, 8+156 p.; v. 2, 1927, 8+136 p.; v. 3, 1928, 8+188 p.; v. 4, 1929, 8+186 p.; v. 5, 1930, nos. 1–8, 170 p.

15. *Parametr. Czasopismo poświecone Nauczaniu Matematyki* [Parameter. Polish periodical of mathematical science], Warsaw, v. 1, 1930, 400 p.
Devoted to the teaching of mathematics in elementary and secondary schools, 10 numbers a year.

16. *Studia Mathematica*, Lemberg, v. 1, 1929, 4+255 p.
Fourteen articles in French, German, and Italian, dealing with topics of functional equations.

Argentina

17. *Boletín del Seminario Matemático Argentino*, Buenos Aires, Sept. 1928–Sept. 1929, nos. 1–5; *Boletín de Seminario Matemático*, no. 6, Feb., 1930; *Universidad de Buenos Aires, Facultad de Ciencias Exactas Físicas y Naturales, Seminario Matemático*. nos. 7–9, March, 1930; 222 p.

18. *Boletin Matemático*, Buenos Aires, v. 1, 1928, 124 p.; v. 2, 1929, 192 p.; v. 3, 1930, 168 p.
Short articles, notes, and solutions of problems.

19. *Boletin Matemático Elemental*, Buenos Aires, v. 1, 1930, 36 p.

United States

20. *Annals of Mathematical Statistics*, American Statistical Association, Ann Arbor, Mich. v. 1, Feb.–Nov., 1930, 363+187 p.
Twenty articles, one with extensive tables.

21. *Mathematics News Letter*, Baton Rouge, La., v. 1, 62 p.; v. 2, 98 p.; v. 3, 184 p.; v. 4, 192 p.; v. 5, nos. 1–4, 96 p., Oct. 1926–Dec. 1930.

22. *University of Oregon Publication, Mathematics Series*, v. 1, nos. 1–2, 1929–30, 40+16 p.
One number is a set of tables and the other a monograph.

Germany

23. *Quellen und Studien zur Geschichte der Mathematik, Abteilung A, Quellen* Berlin, v. 1, 1930: *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau herasugegeben und kommentier von W. W. Struve* 12+197 p.+10 folding plates.

24. *Quellen und Studien zur Geschichte der Mathematik, Abteilung B, Studien* v. 1, parts 1–3, Berlin, Mar. 1929–Sept. 1930, p. 1–412.
Sixteen articles in German and two in English.

Czeckoslovakia, England, Holland, Hungary, India, Japan, Russia, Switzerland.

25. *Aktuárské Vědy. Pojistná Matematika Matematická Statistika.* [Actuarial News. Periodical for Mathematics and Mathematical Statistics], Bohemian Mathematics and Physics Society, Prague, v. 1, 1929–30, 188 p.
Articles in English, French, German, and Bohemian.

26. *Quarterly Journal of Mathematics, Oxford Series*, v. 1, 1930, 4+318 p.
This *Journal* is the successor to *The Quarterly Journal of Mathematics* and *Messenger of Mathematics.* The above mentioned volume contains twenty-five articles.

27. *Mathesis. Tijdschrift voor Wiskunde* [Mathesis. Journal of Mathematics], The Hague, v. 1, 1928, 288 p.; v. 2, 1929, 288 p.; v. 3, 1930, 304 p.

28. *Közslemênyek a Debreceni Tud. Egyetem Matematikai Szeminâriumâbol.* [Communications of the mathematical seminary of the University of Debreczin], Budapest, Pécsett, Sarospatak, Debreczin, 1927–9, nos. 1–4, 106+47+37+1 pl.+80 p.

Four memoirs, by different authors, in Hungarian.

29. *Bulletin of the Mathematical Association, University of Allahabad*, v. 1, 1928, 74 p.

Four articles in English.

30. *Memoirs of the Faculty of Science and Agriculture, Taihoku Imperial University*, Formosa, Japan, v. 2, no. 1 [first number published], Sept. 1929, *Mathematics*, no. 1, 37 p.

Three articles in German.

31. *Journal de la Société Physico-Mathêmatique de Lêningrade*, Leningrad, v. 1, 1927, 342+15 folding plates; v. 2, 1928–29, 382 p.+4 folding plates.

In Russian, French and German.

32. *Commentarii Mathematici Helvetici Editi Societate Mathematica Helvetica*, Zürich, v. 1, 1928–29, 323 p.; v. 2, 1930, 306 p.

Articles in French and German.

Dec., 1930.

SOME DETERMINANTS IN THE THEORY OF DEVELOPABLES

By R. P. BAKER, University of Iowa

If x, y, z, w are functions of t the theory of developables demands the consideration of the matrix

$$\left|\begin{matrix} x, & y, & z, & w \\ x', & y', & z' & w' \\ x'', & y'', & z'', & w'' \\ x''', & y''', & z''', & w''' \end{matrix}\right| .$$

The ratios of the elements of the first row give the points of the system; the ratios of the determinants formed from the first and second row (subject to the Plücker identity), the lines of the system; and the ratios of the determinants of the first three rows, the planes of the system. The determinant is the wronskian which vanishes only for a plane curve.

Now if X, Y, Z, W are the cofactors of x, y, z, w in the wronskian and we form the matrix

$$\begin{vmatrix} X, & Y, & Z, & W \\ X', & Y', & Z', & W' \\ X'', & Y'', & Z'', & W'' \end{vmatrix}$$

we expect and actually obtain the dual representation, namely: X, Y, Z, W give the planes; the first two rows, the lines in planar coordinates; and the third order determinants, the points of the system.

A repetition of the process should lead to a matrix whose first line ξ, η, ζ, ω is proportional to x, y, z, w but apparently ξ contains fourth derivatives, and the wronskian which we expect to occur as the proportionality factor contains only third derivatives. This seems to require explanation. We shall show that these extra derivatives are only apparent and that the theory can be extended to n functions.

1. As to the proportionality of ξ, η, ζ, ω to x, y, z, w we have, by the theory of determinants,

$$\sum xX = 0, \quad \sum x'X = 0, \quad \sum x''X = 0.$$

Differentiating, $\Sigma x'X + \Sigma xX' = 0$, $\Sigma x'X' + \Sigma x''X = 0$; hence $\Sigma xX' = 0$ and $\Sigma x'X' = 0$. Differentiating again, $\Sigma x'X' + \Sigma xX'' = 0$; hence $\Sigma xX'' = 0$. We have also $\Sigma \xi X = 0$, $\Sigma \xi X' = 0$, $\Sigma \xi X'' = 0$; and so

$$\xi : \eta : \zeta : \omega = x : y : z : w.$$

2. As to the proportionality factor, Y which appears in the first line of ξ is

$$\begin{vmatrix} x, & z, & w \\ x', & z', & w' \\ x'', & z'', & w'' \end{vmatrix}.$$

We may denote this by (012) from the fact that it contains derivatives of these orders.

Differentiating, $Y' = (013)$ and $Y'' = (014) + (023)$. The determinant for ξ can be represented as the sum of two determinants whose first columns are

$$\begin{matrix} (012) & & (012) & \\ (013) & \text{and} & (013) & \text{respectively.} \\ (023) & & (014) & \end{matrix}$$

The first determinant is $-xV^2$ for the elements are minors of the wronskian V in reverse order of rows. We now have to prove that the second vanishes identically. Consider the determinant

$$\begin{vmatrix} x, & y, & z, & w, & 0 \\ x', & y', & z', & w', & 0 \\ x'', & y'', & z'', & w'', & 0 \\ 0, & y, & z, & w, & x \\ 0, & y', & z', & w', & x' \end{vmatrix} \equiv \begin{vmatrix} x, & 0, & 0, & 0, & -x \\ x', & 0, & 0, & 0, & -x' \\ x'', & y'', & z'', & w'', & 0 \\ 0, & y, & z, & w, & y \\ 0, & y', & z', & w', & x' \end{vmatrix} = 0.$$

Expanding by the first three rows,

$$\begin{vmatrix} x, & z, & w \\ x', & z', & w' \\ x'', & z'', & w'' \end{vmatrix} \begin{vmatrix} x, & y \\ x', & y' \end{vmatrix} + \begin{vmatrix} x, & y, & w \\ x', & y', & w' \\ x'', & y'', & w'' \end{vmatrix} \begin{vmatrix} z, & x \\ z', & x' \end{vmatrix}$$
$$+ \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} \begin{vmatrix} x, & w \\ x', & w' \end{vmatrix} = 0;$$

and we have a set of multipliers for the elements of the first row of ξ which gives a zero. Now the row of second derivatives may be replaced by any arbitrary quantities, for instance by the corresponding third or fourth derivatives, without affecting the result. Hence the determinant vanishes.

3. In the case of the determinants which should represent the lines in planar Plücker coordinates we have

$$\begin{vmatrix} Z, & W \\ Z', & W' \end{vmatrix} = \begin{vmatrix} [z'''], & [w'''] \\ [z''], & [w''] \end{vmatrix} = \begin{vmatrix} x, & y \\ x', & y' \end{vmatrix} V,$$

where $[z'']$ denotes the cofactor of z'' in the wronskian.

4. To generalize to n functions we study the first column of the determinant for ξ.

In short notation it is

$(0, 1, 2, \cdots k),$

$(0, 1, 2, \cdots k-1, k+1),$

$(0, 1, 2, \cdots k-2, k, k+1) + (0, 1, 2 \cdots k-1, k+2),$

$(0, 1, 2, \cdots k-3, k-1, k, k+1) + 2(0, 1, \cdots k-2, k, k+2) + (0, 1, \cdots k-1, k+3),$

. . . .

$$(0, 2, 3, \cdots, k+1) + \binom{k-1}{1}(0, 1, 3, \cdots, k+1) + \binom{k-1}{2}(0, 1, 2, 4, \cdots).$$

The binomial coefficients are immaterial for our purpose. We notice that the first column contains minors of the wronskian (in reverse order). They lack in turn the symbol $k+1, k \cdots 3, 2, 1$. The last one is the only one without a

1, the penultimate the only one without a 2, and so on. An aggregate selected one from each row will have a pair of common symbols. If not (01), we must use the wronskian minor in the last row. If we select this and if (02) is not a common pair we must take the preceding wronskian minor. Ultimately we have all the wronskian minors or a common pair of symbols in the aggregate.

The property used in the special case $k=2$ was that an aggregate with a pair of common rows formed with the other columns similarly represented formed a vanishing determinant. This obviously generalizes.

The final result is: $\xi_1 = (-1)^{n-1} V^{n-2}$, where n is the number of functions.

For the minors of the third and higher orders the multiplier method must be used to rid the aggregate of determinants containing the derivatives of order n and higher.

5. In these proceedings differentiation has apparently been replaced by a formal symbolic process. The properties actually used are $d(u+v)=du+dv$ and $d(uv)=udv+vdu$; the fact that $dt/dt=1$ was not appealed to. This allows us apparently to generalize the operator to either $\lambda \cdot dx/dt$ or $d(\lambda x)/dt$ where λ is an arbitrary function of t. However in case we do the geometric interpretation depending only on ratios of determinants is unchanged.

A NOTE ON INSTRUCTION IN MECHANICS

By ALEXANDER WUNDHEILER, University of Warsaw

Certain common methods of solution of problems in mechanics contain a source of error. We shall show that if we make use of the energy integral, impossible motions are generally obtained.

The common method of treatment of holonomic mechanical systems, the constraints and the potential energy not involving the time, consists in first writing down the energy integral. This is sufficient, if the system has only one degree of freedom. If not, we then seek other equations, e.g. the Lagrangian ones. It will be shown by some examples that the system containing the energy integral and other equations admits, generally, solutions alien to the given, not integrated, system.

Consider the simple harmonic motion of a particle attracted toward a center proportionally to the distance. The potential function is $\frac{1}{2}mk^2x^2$ and the energy integral is

$$\dot{x}^2 = k(a^2 - x^2), \tag{1}$$

where a is a constant. If we seek the motion corresponding to the initial conditions $t=t_0$, $x=a\neq 0$ we find *two* solutions. The one is $x=\text{const.}=a$, and signifies equilibrium at the point a. The second one is given by the quadrature:

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{k^{1/2}(a^2 - x^2)^{1/2}} = k^{-1/2} \arcsin \frac{x - x_0}{a} \cdot$$

It is here obvious that the first one is wrong, and we prove it by verifying that the differential equation of second order of the motion,

$$\ddot{x} = - kx, \tag{2}$$

does not admit the solution $x=a$. In the present case, the falseness of the first solution is evident (for $a \neq 0$), and there is no difficulty in finding out where the alien solution comes in. In fact, in deducing the energy integral, we multiply (2) by $\dot{x}$, and introduce, therefore, the solution of the equation $\dot{x}=0$, viz. $x=a$.

We shall now consider cases where the falseness of the alien solutions is not so evident, and where their explanation is not so simple. Consider Kepler motion, that is, the central motion of a particle attracted toward a center proportionally to the inverse square of the distance. The energy integral in polar coordinates is

$$\dot{r}^2 + r^2\dot{\phi}^2 = 2\mu r^{-1} + h, \tag{3}$$

and the integral of areas is

$$r^2\dot{\phi} = C. \tag{4}$$

It is well known that all circumferences, the centers of which are the center of attraction, are possible trajectories. We ask if the manner in which the particle describes them is uniquely determined. Putting in the equations (3) and (4) $r=r_0$, we obtain

$$r_0{}^2\dot{\phi}^2 = 2\mu r_0{}^{-1} + h, \quad \dot{\phi} = C r_0{}^{-2}.$$

As h and C are arbitrary constants we infer that $\dot{\phi}$ may have, along such a circumference, an arbitrary but constant value, i.e. that such a circumference may be described by a particle with an arbitrary velocity. Now, this is false. In fact, the intrinsic equation for the normal acceleration, which in the present case takes the form

$$mv^2 r_0{}^{-1} = m\mu r_0{}^{-2}$$

shows that the velocity is completely determined by the normal component of the force and the radius of curvature of the trajectory. We obtain

$$V = (\mu r_0{}^{-1})^{1/2}$$

and the other values of v must be considered as alien solutions, introduced when effecting the computation leading to the energy integral. But the indication of the precise moment where this introduction took place is not quite so simple as in the former example.

We find another quite analogous case in the theory of the spherical pendulum. The reader will verify immediately, after having written down the energy

integral and the integral of areas, that they give for all parallels of the sphere an arbitrary velocity, whereas the intrinsic equations of the motion give a completely determined velocity.

The generalization of these examples does not present any difficulty. But it is easily seen that in the more complicated problems concerning systems of material particles and rigid bodies, our intuitive knowledge of the motion will not suffice to guess which are the alien motions. Consider, for instance, a homogeneous, rigid, heavy rod, one extremity of which slides without friction along a vertical line, and the other one on a horizontal plane. Let the angle of the rod with the vertical be θ. If we examine the motion by means of the integrals of energy and areas, we obtain the result that the stationary motion, $\theta=$ const., is possible in connection with an arbitrary value of $\dot{\phi}$ (ϕ designating the angle of the horizontal projection of the rod with a fixed horizontal direction). It would seem that in the present case it is not quite easy to guess that corresponding to a given value of θ, only one determined value of $\dot{\phi}$ is possible, as is seen when we make use of the Lagrangian equations of the motion.

THE SOLUTIONS OF $x^y=y^x$, $x>0$, $y>0$, $x\neq y$, AND THEIR GRAPHICAL REPRESENTATION

By H. L. SLOBIN, University of New Hampshire

The equation

$$x^y = y^x, \tag{1}$$

is equivalent to

$$y \log x = x \log y \quad \text{or} \quad \frac{\log x}{x} = \frac{\log y}{y}\cdot$$

For convenience we shall write this equation in the form

$$\frac{\log x_1}{x_1} = \frac{\log x_2}{x_2}\cdot \tag{2}$$

If now we consider the graphs of $y=\log x$ and $y=mx$ we note that $y=0$, the x-axis, is parallel to the curve, $y=\log x$, as x becomes infinite, since $dy/dx=1/x$ when $y=\log x$. We may now determine the value of m for which $y=\log x$ and $y=mx$ meet and have the same slope. Since $\log x=mx$ and $m=1/x$ we have $\log x=1$ or $x=e$ (the base of the system of natural logarithms).

It is thus evident that the lines $y=mx$ for $0<m<1/e$ will each have two intersections with $y=\log x$. For each such line we thus determine a pair of numbers x_1 and x_2, the abscissas of the points of intersection of $y=mx$ and $y=\log x$, so that

$$\frac{\log x_1}{x_1} = \frac{\log x_2}{x_2}\,.$$

To determine the solutions of $x^y=y^x$, where both x and y are rational numbers, is merely to determine the pairs of rational numbers x_1 and x_2 satisfying equation (2).

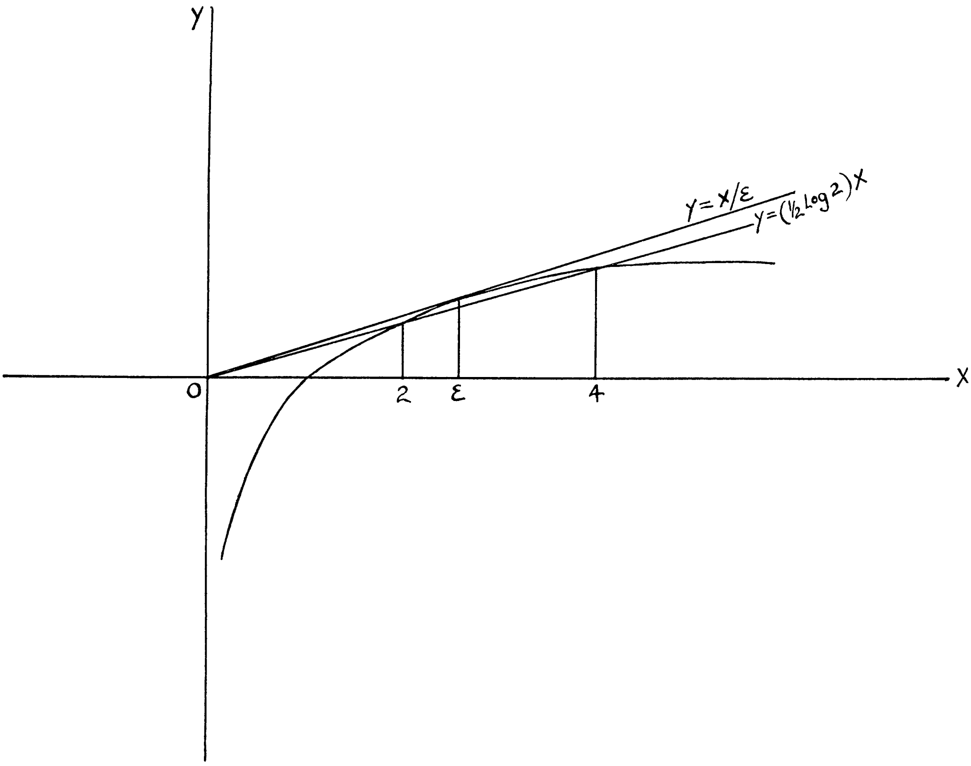


FIG. 1.

If we assume x_1 to be a rational number, then $x_2=x_1r$, where r is undetermined but must be rational if x_2 is to be rational. Hence

$$\frac{\log x_1}{x_1} = \frac{\log (x_1r)}{x_1r} \quad \text{or} \quad \left(1-\frac{1}{r}\right)\log x_1 = \frac{\log r}{r},$$

or

$$x_1 = r^{1/(r-1)}, \qquad x_2 = r^{r/(r-1)}.$$

If we assume $x_2>x_1$, then $r>1$. Therefore letting $r=1+s$, we have

$$(3) \qquad x_1 = (1+s)^{1/s}, \text{ and } x_2 = (1+s)^{(1+s)/s}.$$

If we let

$$s = 1,\ \frac{1}{2},\ \frac{1}{3},\ \cdots,\ \frac{1}{n},$$

we have

$$x_1 = 2,\ \frac{9}{4},\ \frac{64}{27},\ \cdots,\ \left(1+\frac{1}{n}\right)^n \cdots,$$

and

$$x_2 = 4,\ \frac{27}{8},\ \frac{256}{81},\ \cdots,\ \left(1+\frac{1}{n}\right)^{n+1} \cdots.$$

It will be noted that as s was chosen to traverse a set of rational numbers from 1 to 0, x_1 traversed a set from 2 to e, and x_2 traversed a set from 4 to e.

In seeking *all* the pairs of rational numbers x_1 and x_2 that satisfy equations (3), we must confine ourselves to s rational. Hence letting $s=p/q$, where p and q are prime to each other, we have

$$x_1 = \left(\frac{p+q}{q}\right)^{q/p}, \quad x_2 = \left(\frac{p+q}{q}\right)^{(q+p)/p}. \tag{4}$$

Since x_1 and x_2 are to be rational numbers, $(p+q)^{1/p}$ and $q^{1/p}$ must be integers, say n_1 and n_2, respectively. Hence

$$x_1 = \left(\frac{n_1}{n_2}\right)^q \text{ and } x_2 = \left(\frac{n_1}{n_2}\right)^{p+q}. \tag{5}$$

Letting $n_1-n_2=t$, we have

$$(n_2+t)^p=n_1^p=p+q, \qquad n_2^p=q.$$

Hence

$$(n_2+t)^p - n_2^p = p. \tag{6}$$

Expanding (6) we have

$$pn_2^{p-1}t + \frac{p(p-1)}{2!}n_2^{p-2}t^2 + \cdots + t^p = p. \tag{7}$$

Obviously equation (7) cannot be satisfied if n_2 and t are integers. Hence the assumption $s=p/q$ does not give rational values for x_1 and x_2, unless $p=1$ or $q=1$.

If $p=1$, $s=1/q$ and

$$x_1 = \left(\frac{1+q}{q}\right)^q, \text{ and } x_2 = \left(\frac{1+q}{q}\right)^{q+1}. \tag{8}$$

We can now let $q=1, 2, 3$, and then have from (8):

$$x_1 = 2, \frac{9}{4}, \frac{64}{27}, \cdots, \quad x_2 = 4, \frac{27}{8}, \frac{256}{81}, \cdots,$$

which is the aggregate already determined.

If $q=1$, $s=p$ and hence

$$x_1 = (1 + p)^{1/p}, \text{ and } x_2 = (1 + p)^{(1+p)/p}. \tag{9}$$

In this case x_1 and x_2 cannot be rational unless $(1+p)^{1/p}$ is an integer. But it is evident from the graph $y = \log x$ that the value of x_1 must lie between 1 and e, and the *only* possible integer is 2. This value for x_1 corresponds to $p = 1$ and gives $x_1 = 2$ and $x_2 = 4$, which has already been obtained.

The significance of our conclusions is interesting if interpreted graphically. Each of the lines $y = mx$, where $\frac{1}{2}\log 2 \leqq m \leqq e^{-1}$, cuts the curve in two points, and the points whose abscissas are both rational numbers have been determined; but if $0 < m < \frac{1}{2}\log 2$, the abscissas of the points of intersection cannot *both* be rational.

It is also evident that we cannot have two intersections for each value of m save for $0 < m < e^{-1}$.

The problem is capable of expansion to similar problems. Thus the equation

$$(e^x)^{(e^y)} = (e^y)^{(e^x)}$$

cannot have a single pair of values x and y both algebraic numbers (that is rational numbers and algebraic irrational numbers). This is evident since we have $xe^y = ye^x$ or $x/y = e^{x-y}$. If x and y were both algebraic numbers x/y and $x - y$ would be algebraic numbers while e^{x-y} would be a transcendental number. (See the author's paper in the Rendiconti di Circolo Mathematico di Palermo, vol. 38 (1914), p. 353.)

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A THEOREM ON FOCI

By EDWIN J. PURCELL, University of Colorado

THEOREM: *If a real plane algebraic curve of class m has one or more real axes of symmetry, then in general exactly m foci lie on each axis of symmetry; any such set of m collinear foci completely determines the remaining foci.*

Proof: Consider any real line of symmetry of the curve. Let this line be the x-axis in rectangular coördinates. The tangential equation of the curve will have real coefficients and be of the form

$$f(u, v^2) = 0. \tag{1}$$

Letting $v = \pm i/c$ in (1), there will be in general exactly m values of u satisfying (1). Interpreted geometrically, this means that there will be exactly m intersections of isotropic tangents lying on the axis of symmetry. These m in-

tersections are m foci (real or imaginary) of the curve. Since the axis of symmetry is a real line, not the line at infinity, it does not pass through I or J, the circular points at infinity. The m foci on it completely determine the m tangents to the curve from I and the m tangents to the curve from J. The other intersections of these 2 m tangents are the remaining (m^2-m) foci. The theorem follows.

If k foci on an axis of symmetry coincide, then the number of foci on that axis is increased by (k^2-k), since such a focus counts k^2 times. If the curve is tangent to the line at infinity so that the number of real (finite) foci is reduced from m to p, say, then exactly p foci will lie on each axis of symmetry.

The Characteristic Equations of the Adjoint and the Inverse of a Matrix

By H. S. Thurston, University of Alabama

The following theorem may possibly be found somewhere in current mathematical literature, but the writer is unaware of the previous publication of a proof.[1]

Theorem. If a is a non-singular matrix of order n and determinant D, a^{-1} its inverse and A its adjoint, then the roots of the characteristic equation of A are respectively D times the roots of the characteristic equation of a^{-1}, and the latter are the reciprocals of the roots of the characteristic equation of a.

The characteristic functions of a, A, and a^{-1} are respectively

$$\phi(\lambda) \equiv s_n - s_{n-1}\lambda + s_{n-2}\lambda^2 - \cdots + (-1)^n\lambda^n$$

$$\Phi(\lambda) \equiv S_n - S_{n-1}\lambda + S_{n-2}\lambda^2 - \cdots + (-1)^n\lambda^n$$

$$\psi(\lambda) \equiv p_n - p_{n-1}\lambda + p_{n-2}\lambda^2 - \cdots + (-1)^n\lambda^n$$

where s_k, S_k, and p_k are respectively the sums of all k-rowed principal minors of a, A, and a^{-1}.

On page 31 of Bôcher's *Introduction to Higher Algebra*, we find the following theorem: If D' is the adjoint of any determinant D, and M and M' are corresponding m-rowed minors of D and D' respectively, then M' is equal to the product of D^{m-1} by the algebraic complement of M. Although the adjoint of a matrix and the adjoint of a determinant are in form conjugate to each other, the k-rowed principal minors of both have the same value, and from the above theorem we see that

$$S_{n-k} = D^{n-k-1}s_k.$$

---

[1] For the theorem on the reciprocal see Pascal's *Repertorium*, Vol. I Part I, page 117 *Editor*.

Since each element of A is D times the corresponding element of a^{-1}, it follows at once that

$$p_{n-k} = \frac{S_{n-k}}{D^{n-k}} = \frac{s_k}{D}\,.$$

From the relations $S_{n-k}=D^{n-k}p_{n-k}$ and $s_k=Dp_{n-k}$, it is clear by elementary theory of equations that the roots of $\Phi(\lambda)=0$ are D times the roots of $\psi(\lambda)=0$, while those of $\psi(\lambda)=0$ are the reciprocals of those of $\phi(\lambda)=0$.

If a is singular, of rank r, the theorem becomes trivial even in a modified form. The rank of A is 1 or 0 according as $r=n-1$ or $r\leqq n-2$. If $r=n-1$, $\Phi(\lambda)=0$ has one root different from 0; if $r\leqq n-2$, all roots of $\Phi(\lambda)=0$ are 0.

A New Method for Solving the Equation $x^x=c$

By E. C. Kennedy, University of Texas College of Mines

Equations of the type $x^x=c$ may be solved by finding two approximate values of x and getting a better value by straight line interpolation. If the process is repeated several times an accurate result may be obtained.

A far quicker, easier, more accurate method involving less chance of error is described below.

Let $x^x=(P_0+h)^{(P_0+h)}=c$, where P_0 is a rough approximation to the value of x. Then $P_0^{(P_0+h)}(1+h)=c$, approximately. Taking natural logs of both sides and assuming $\log(1+h)=h$ we obtain for our first approximation, P_1,

$$P_1 = (P_0 + h) = (P_0 + \log c)/(1 + \log P_0).$$

To get a second approximation, P_2, we write

$$P_2 = (P_1 + h') = (P_1 + \log c)/(1 + \log P_1).$$

The nth approximation is readily seen to be

$$P_n = (P_{n-1} + \log c)/(1 + \log P_{n-1}).$$

To illustrate the brevity of this method we shall solve the equation $x^x=32.46$. Taking $P_0=3$ (any value in this neighborhood will do) we have

$$P_1 = (3 + \log 32.46)/(1 + \log 3) = 6.480/2.099 = 3.087.$$

Thus $x=3.087$ is the first approximation. (Newton's method gives 3.096). In order to get the maximum degree of accuracy obtainable by a second approximation it is generally necessary to use a 7-place table. Thus

$$P_2 = \frac{3.087 + \log 32.46}{1 + \log 3.087} = \frac{6.5670085}{2.1271997} = 3.0871613,$$

a result correct to at least 7 places. A third trial takes our answer to about 12

places. After the first approximation the accuracy of the result increases amazingly.

By this method it is almost impossible to go wrong for even if an inexcusably poor value of P_0 is taken the error will automatically eliminate itself with one or two more approximations. For example, if $c=100$ we should choose $P_0=$about 3.4 or 3.5 or 3.6 instead of 3, although the latter will serve of course. For two approximations (and seldom more than two are needed) no multiplications and only two divisions are required.

If $c<.692 \cdots$ there is no positive solution in real numbers. If $.692 \cdots <c <1$, then there are two solutions—at least two greater than zero. To get one of these choose P_0 slightly greater than $1/\epsilon$, to get the second choose P_0 slightly less than $1/\epsilon$ and the rest is purely mechanical.

This method is applicable to equations of the form $u^u=c$, where $u=u(x)$. For example,

$$\log (x - 5)^{\log(x-5)} = 7.2, \quad \tan (x - .4)^{\tan(x-.4)} = 9.68, \text{ etc.}$$

may be solved very accurately in four or five minutes. If one were to solve ten or twelve equations of this type it would be best to plot $y=x^x$ carefully, then by making use of the graph and one approximation the root could be found to 5 or 6 places in two minutes.

Newton's method could be used on such equations, but the method described above is much superior. In fact, it does not seem possible that there could be an easier way of solving such equations than that given.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Intergralgleichungen. By G. Kowalewski. Berlin, Walter de Gruyter & Co., 1930. 302 pages.

This book is one of "Göschen's Lehrbücherei," and is not to be regarded as a complete treatise. Rather, it seems designed to occupy the position in German that the several times reprinted "Tract" of Bôcher still fills so well in English. The exposition of Kowalewski is exceedingly detailed, perhaps unnecessarily so, but also, on that account, very easy reading; it is largely algebraic, and the subject matter is governed by the traditional choice. The chapter of applications at the end, however, is hardly sufficient to give the reader any hold, and the "Literaturhinweise" is scanty. A good many special integral equations are

worked through in complete detail, rather as applications or illustrations of the general theory than as indications of possibilities that are not covered by it.

Hence although there are some indications of a group theory and of a theory of functional transformations, the reader is not led into new problems and unsolved questions. He is likely to regard the subject as a circle in which all the roads run parallel to the circumference. The book is an able exposition, and will serve to introduce the student to the subject. But the reader, having finished it, should turn elsewhere, say to Carlemann, in order to be shocked into renewed vitality.

The introduction deals with Abel's equation and other special inversions of integral formulae, like the Fourier integrals, considered as functional transformations; Chapter I is devoted to Volterra integral equations, with a brief analysis of the algebraic or symbolic properties of the integral operator; Chapter II treats similarly the Fredholm type, with incidental reference to elementary divisors; Chapter III gives the theory of symmetric kernels; and Chapter IV discusses briefly two applications—the vibration problem for a string, and, without much proof, the Dirichlet problem for a region interior to a closed curve.

GRIFFITH C. EVANS

Tutorial Exercises in Trigonometry. By Raymond W. Brink and Ella Thorp. Century Company, New York, 1931. 102 pages, 9×11. Paper bound. $1.25.

The need for carefully selected and well-arranged exercises in trigonometry has been realized in the publication of this book. In each set of exercises not only is the student led by definite, graded steps to the more difficult problems, but enough exercises are given to effect a drill on each principle. Then too ten sets of review exercises, besides a final, all-embracing test, serve to organize the work of the preceding lessons.

The form of the book suits its purpose. Each sheet, printed on only one side, presents a neat appearance. Exercises, diagrams, and room for working have been carefully spaced. Furthermore the heading of each page by a concise statement of the topic covered is an aid to both teacher and student. The book should prove interesting and helpful as a supplement to any text in trigonometry.

HARRIET GRIFFIN

Il Passato e il Presente delle Principali Teorie Geometriche. Storia e bibliografia. Fourth edition. By Gino Loria. Padua, Cedam, 1931. xxiii+467 pp. Lire 60.

In this interesting volume, the author presents, in cursive style, an outline of the principle theories of geometry, with emphasis on their historical development. He states briefly the chief results that have been arrived at in each of the various fields of geometry and gives numerous references for the benefit of readers who may wish to pursue any topic further. The work is one for which the author is conspicuously well fitted and to which he brings a wealth of informa-

tion and critical judgment. In spite of unfortunately numerous typographical errors, the book is decidedly well written. For anyone who wishes to take a hasty but interesting trip through the domain of geometry, to have pointed out to him the main topics that have been developed, with their modes of treatment and the outstanding results, this volume will serve as an excellent guidebook.

The present fourth edition is based on the second, which was published in 1896. The first 291 pages constitute a thorough revision of that earlier edition. The remainder of the text traces, in similar fashion, the development of geometry since 1896.

C. H. SISAM

Plane and Spherical Trigonometry. By R. D. Carmichael and E. R. Smith. Boston, Ginn & Company, 1930. xii+198 pages. $1.60.

"There is very little room for novelty in preparing a textbook for an introductory course in trigonometry. The material and methods are fairly well standardized."

It is scarcely to be expected that a trigonometry which carries the two sentences quoted above as the opening statements of its preface will incorporate any theoretical innovations of development or choice of material. The reader may rest assured at once that this textbook contains nothing which has not been fully tested in the class room. The authors have had long and successful experience in both writing and teaching and their book is conservative, teachable, well-written and mathematically sound.

The selection and arrangement of subject-matter is such as would probably be approved as first choice by a majority of those teaching the subject. Measurement of angles, definitions of the trigonometric functions for positive acute angles, elementary identities, general definitions of the functions, solutions of right triangles by means of natural functions, solutions of oblique triangles by means of right triangles, reduction formulas and graphs, addition formulas, solutions of triangles in general, DeMoivre's theorem and related identities, spherical triangles and their applications—this organization and development of material would generally be conceded to be both logical and psychological.

Moreover, the general treatment is such as to permit of considerable flexibility and adaptation to different types of classes and to teachers' preferences. If a very short course is desired, the first three chapters (pp. 1–66) form a complete unit covering the most essential ideas of the subject. If an instructor would prefer to omit the definitions of the functions for acute angles and begin at once with the definitions for general angles, that can easily be done. If he prefers to omit or abbreviate the solutions of right triangles by means of natural functions or the solutions of oblique triangles by means of right triangles, trusting to the later more complete treatment of triangles in Chapter VII, that can be done; if time presses, a number of starred supplementary sections can also be omitted without destroying the continuity of the course

Occasionally statements appear which lead to queries but scarcely to definite criticisms. On page 3 we read "A *straight line* . . . extends indefinitely in two directions. A *straight line segment* is that part of a straight line which lies between any two points on the line. A *ray*, or *half-line*, is that part of a straight line which lies on either side of a point on the line," and on page 4, ". . . a ray has direction, one end-point, but not length. A line segment is generally referred to as a line when the context admits of no doubt as to the meaning." Is it objectionable to say that a ray is longer than any segment of it or that the length of a ray is greater than that of any segment of it, even though the context might admit of no doubt as to the meaning? Would such usage of terms be any more objectionable than to speak of a line segment, which certainly has length, as a line, which certainly does not have length (in the same sense that a ray does not have length) under like conditions?

On page 15, line 1, we find "If two variable quantities, x and y, are so related . . . ." Why not simply "If two variables, x and y, are so related . . . ," since, immediately, in the first line of the next paragraph, the authors say" The variable x is called the independent variable and y the dependent variable or function of x?" Also, on page 43 we find "Such numbers as $\sqrt{2}$, $\frac{2}{3}$, $\cdots$, constants like π . . . and e . . . and many other quantities . . . ." Are $\sqrt{2}$, $\frac{2}{3}$, π and e quantities or definitely and precisely numbers which are sometimes used to state a measure of quantity? Possibly both, but the writer believes that there is a growing tendency not to use the indefinite (and, as sometimes used, obscure) term "quantity" indifferently for all sorts of concepts (number, variable, function, etc.) but to use the definite and precise term most appropriate whenever possible.

Again, on page 22, the student is asked to "check" certain identities by substituting 30° for the angle and finding the numerical values of the members. While "check" is often used in this limited sense, should not such use be discouraged, because of the danger that students may come to believe that substitution of a single numerical value forms a complete check? The exercise can be stated readily in terms which carry no such implication.

No material errors, however, were noticed, exercises are adequate and well graded and special attention seems to have been given to the arrangement of numerical work throughout the book.

It is hardly necessary to add, except by way of compliment, that the printer's part has been very well done. The imprint of the Athenaeum Press on a mathematical book is assurance of quality.

U. G. MITCHELL

Advanced Mathematics for Students of Engineering and Physics, I and II. By D. Humphrey. Oxford University Press, London, 1929. Part I, iv+120 pages, and Part II, iv+175 pages, bound in a single volume.

In this book the author aims to give those students of engineering and physics who have completed elementary calculus a knowledge of and training

for manipulative skill in as many branches of mathematics required in their other courses as he can present in a book of moderate size. He considers that most books on practical mathematics obscure and limit the mathematics involved by devoting too much space to the working out of practical problems and that manipulative skills in pure mathematics should be acquired before applications are attempted. The book, based on the work done at the Polytechnic, London, has been designed in accordance with these views.

The text is divided into seventeen chapters. Two chapters, containing forty-four pages, are on integration; five, containing eighty-eight pages, on differential equations; the remaining ten, containing one hundred forty-one pages, on series, complex quantities, applications to geometry, solid geometry, calculus of finite differences, determination of laws, numerical solution of equations, determinants, Fourier series and harmonic analysis, and vectors. The four hundred or so exercises, which are accompanied by answers, are almost entirely of an abstract nature; the few which are not deal only with very simple applications.

In the selection and arrangement of his material the author has not attempted to please everyone nor has he gone out of his way to make it particularly attractive to those students who require constant motivation in their courses. Probably everyone will agree that the emphasis on differential equations is rightly placed. Since each reader has his own tastes and feels his own special needs most, the amount of space allotted to the other topics is open to discussion of such length that it will not be considered here. On the whole, the reviewer finds the shorter topics, as well as the longer ones, clearly, directly, and accurately presented and, he believes, with a degree of rigor and completeness corresponding to the academic status of the student for whom they are intended. He feels, however, that the insertion of some discussion of the validity of infinite processes would make them more clear to the student and that alterations in two or three objectionable statements, of which the most important is the statement that the differential of $f(x)$ is the *actual* change when x is increased by dx, would contribute accuracy well within the student's powers of appreciation. The reviewer now wishes to prove his worth as such by adding to the above minor criticisms his support of the author in the matter of motivating material. There is a strong movement afoot to increase the amount of secondary material included in a course for the purpose of linking it up with other subjects to which it may apply. So strong has this tendency become that there is some danger of its being overdone. Indeed, accusations have been made that some text-books and some courses consist entirely of so-called motivating material to the exclusion of tangible principles on which to base the work. Such material is undoubtedly of great value in numerous connections, but the wisdom of using it beyond the first or second college year in well organized professional courses, such as engineering, is questionable. The engineer who after his first two years of training has to be further wheedled into preparing for the profession ought to be encouraged to try something else.

The reviewer recommends the book for closer examination by instructors who offer courses including the topics mentioned and who desire little motivating material in their texts and by independent students who desire a knowledge of and manipulative skill in these topics. It is well worthy of consideration as a mathematical contribution to the literature of engineering.

EARL L. MICKELSON

The Volterra Integral Equation of Second Kind. By Harold Thayer Davis. Indiana University Studies Nos. 88, 89, 90. 1930. 71 pages. Price $1.00 in paper cover, $1.25 in cloth.

This is the third of a series of monographs by the author on the general field of integral equations. The subject of discussion is the equation

$$u(x) = f(x) + \lambda \int_a^x K(x, t)u(t)dt$$

in the unknown function $u(x)$.

In Part I the classical method of successive substitutions is shown to lead to Volterra's resolvent kernel, and the case where $K(x, t)$ is a function of $(x-t)$ is examined by means of functions permutable with unity. Differential and integral operators of infinite order are introduced and are used to solve the equation when $K(x, t)$ is expandible as

$$\sum_{n=0}^{\infty} a_n(x)(x - t)^{n-1}/(n - 1)!.$$

Part I closes with a discussion of the case $K(x, t) = g(x, t)/(x-t)^\alpha$, $0<\alpha<1$, using fractional differentiation and integration.

Part II considers the singular equation, with either the constant limit infinite, or the kernel having an infinite discontinuity of type other than considered at the close of Part I. Results of Evans and Love are set forth and extended, the theorems covering several cases where there exist families of solutions.

The author has inserted several illustrative examples and problems, the latter containing just sufficient difficulty to whet the reader's appetite, without destroying it. The list of references is numbered to correspond with the complete bibliography in the first of this series,[1] and includes six entries added since that was published. A portrait of Vito Volterra makes an appropriate frontispiece, and adds to the appearance of the book.

The work is hardly to be recommended to the beginner, for, although no previous knowledge of integral equations is presumed, the large number of ideas introduced would tend to overwhelm him. However, it should carry a person with some familiarity with the subject and a good grounding in analysis, up to the frontier where research is being carried on.

L. S. KENNISON

[1] *The Present Status of Integral Equations*, Indiana University Studies, No. 70 (1926).

Types of Mathematical Thinking. Von Zahlen und Figuren. Proben mathematischen Denkens, für Liebhaber der Mathematik. Ausgewählt und dargestellt von Hans Rademacher und Otto Toeplitz. Julius Springer, Berlin, 1930; v+164 pages, 129 figures.

The purpose of this book is, as indicated in the sub-title, to give to the layman, through a number of examples of typical mathematical reasoning, a feeling for the real character of mathematical science. A brief table of contents will give some idea of the range of problems which the authors have been able to bring within the range of readers possessing no technical knowledge whatever beyond a few of the most important of the theorems of plane geometry and the rudiments of algebra. Here it is:

On the series of prime numbers. The continuous tracing of a net of curves (each branch once and only once). Some minimal properties (rectangles with given areas, inscribed triangles, inscribed polygons). Incommensurable, segments and irrational numbers. A minimal property of the triangle joining the feet of the altitudes of a given triangle, after H. A. Schwarz. The same property, after L. Fejér. Some points in the theory of aggregates. Plane sections of the right circular cone. The Waring problem. On closed knotted curves. Is the decomposition of a number into its prime factors unique? The four color problem. The regular polyhedra. Pythagorean numbers, and a glimpse of the Fermat problem. The Pferch circle of a given (finite) set of points—i.e. the circle with least radius enclosing the set. Approximation of irrational numbers by rationals. The drawing of straight lines by linkages. Perfect numbers. Euler's proof that the sequence of primes is infinite. Fundamental remark on maximum problems (warning against assumption of existence of the maximum, with illustration). Figure of greatest area with given perimeter (Steiner's linkage method). Periodic decimals. A characteristic property of the circle. Curves of constant breadth. Indispensibleness of the compass in the constructions of elementary geometry. A property of the number 30 (namely, that it is the greatest number such that all numbers below it with which it has no common factors are prime numbers).

This indication of contents brings to light a most varied group of topics, many of them classic, some involving still unanswered questions, some quite modern. It cannot indicate the grace with which they are treated, the simplicity, the naturalness, the interesting historical introductions, or the ever alert playing up of viewpoints or methods which, arising naturally in the problems, are characteristic of mathematical thinking. Difficulties of proofs are reduced to a minimum by concrete introductions and by judiciously selected preliminary lemmas. But rigorous methods are used in all cases where proofs are given, for otherwise the purpose of the undertaking would be vitiated.

The book should find a wide distribution among laymen, for the number of persons who are curious as to the nature of mathematical thought, and who are capable of understanding it, is constantly increasing. Probably there are many readers of the MONTHLY to whom questions have been addressed which this

book is well suited to answer. But it will also prove of great interest to the mathematician himself, because of the insight it gives into a variety of problems and points of view, at least some of which are likely to have novelty for any given person, because its style is a model of clarity and simplicity for those engaged in instruction, and because it is a mine of material for anyone called upon for talks, either to laymen or to students' mathematical clubs.

The authors, eminent as productive scientists, have here engaged in a work of popularization on a most dignified plane. They have not been concerned with puzzles or recreations, nor with persuading a reluctant public of the value of mathematics. They have given a genuine glimpse of the science itself, in a number of its phases, free from technical intricacies and symbolism. Without doubt the book deserves a place in every mathematical library. A translation into English, making the book available to still wider circles of laymen in this country, would be of great value.

O. D. KELLOGG

Éléments de Mathématiques Financières. By R. Thiry. Paris, Librairie Vuibert 1930. viii+87 pages.

The content of the book consists of material which the author has presented frequently as a course at l'Institut d'Enseignement Commercial Supérieur de Strasbourg. This course is one for the *general* students of commerce, and not primarily for those who will specialize in the mathematics of investment. The author's object was to present in this book the minimum essentials of the theory of the subject, together with a few of its most important applications. It is interesting to note that the author, in common with many persons in other parts of the world, particularly in the United States, considers these minimum essentials of the mathematics of investment an indispensable part of the equipment of any person who wishes to deal intelligently with the modern business world.

There is little of novelty in the 30 pages of theory with which the author starts the book. He treats simple interest, simple discount, compound interest, and annuities certain in orthodox fashion. The most general annuity problem which he considers is one where the interest period involved is an integral multiple of the payment interval of the annuity. The author does not introduce any of the customary actuarial abbreviations $a_{\overline{n}|}$, $s_{\overline{n}|}$, etc., but leaves his formulas in their inconvenient explicit algebraic forms. Apparently, he does not adopt the desirable logical attitude of considering simple discount, or simple interest payable in advance, as an entity by itself. He considers it as an *approximation to simple interest* which is convenient in certain applications but which, nevertheless, is slightly unjust to the borrower.

After the brief consideration of theory, the remainder of the text (about 30 pages) is devoted to applications. There is one chapter dealing with the amortization of debts, with remarks included concerning the sinking fund method—which, it is interesting to note, is referred to as the "système américain." A second chapter is devoted to a discussion of loans (including bonds as a

special case) considered from the standpoint of an investor who buys the title to some of the debtor's payments. The main object of the chapter is to introduce an interpolation method for determining the yield which such an investor obtains from his purchase. In the case of a bond which is a part of an issue which is redeemable in installments or by drawings, the author makes extensive use of a notion which he refers to as the *mean* price of a bond of the issue, at a given yield. Also, he introduces the companion notion of the *mean* yield, in case a bond of such an issue is bought at a specified price.

In an appendix there are twelve pages of brief extracts of the usual tables employed in the mathematics of investment. In this appendix there is also given a brief discussion of the theory of exponents and of geometric progressions. Logarithms are not explicitly used in the book. Throughout the book the author gives a sufficient number of illustrative examples, solved in the text, to clarify the general processes introduced. However, there are no exercises included for the reader to solve. The mathematical manipulations in the text are performed accurately, but the methods employed present no novelty.

The reviewer believes that the book as a whole would be interesting to an American reader not because of any particular novelty in its content, but because it furnishes an illustration of the courses in the mathematics of investment which are presented to the general student in French schools of business administration.

WILLIAM L. HART

The Logic of Discovery. By R. D. Carmichael. The Open Court Publishing Company, Chicago, 1930. ix+280 pages. Price $2.00.

The scope of this book is somewhat broader than might be inferred from the title. The volume, parts of which have been previously published in various periodicals, gives a careful analysis of some phases of the philosophy of mathematics, with especial reference to their possible influence on the philosophy of other sciences. As stated in the preface, no technical knowledge of mathematics on the part of the reader is presupposed.

The first chapter, "The Logic of Discovery," discusses the possibility of developing a systematic analysis of those mental processes by which an investigator is led to infer the probable truth of a proposition before he is able to give a demonstrative proof of its validity.

In this chapter the reader's attention is directed to a theory which is certainly very interesting, and which, if firmly established, must have profound implications in the philosophy of science. In discussing "the locus of the essential step in discovery," the author says that "It is in the formation of the conjecture itself or goes back even farther to the formation of the hypothesis out of which comes the proposition to be tested, whether by experiment or by reasoning. It may even be found in a more remote place in the process of discovery than this, its chief element resting in a principle partaking somewhat of a metaphysical nature (as in the general principle of relativity) or in an ideal of a purely abstract character (as in Descartes' doctrine of clarity). It is as if the

mind were seeking to impose itself upon nature, insisting that whatever explanations we may finally adopt, they shall be such as satisfy the requirements of a norm set up by the mind itself. Certain of these demands may be impossible of realization. One then constructs a norm of a modified sort. The essential step in discovery is in the construction of definite hypothesis in the form of a particular or a general law or proposition and in the formulation of a principle or norm lying back of the hypothesis and contributing effectively to giving it existence." (Pages 13, 14.) This conception of scientific truth as depending upon the mental processes of the observer as well as upon the objective facts of nature is one which is repeatedly emphasized throughout the book.

In Chapter II, "What is the Place of Postulate Systems in the Further Progress of Thought?," the author explains briefly the use of postulate systems, and shows that although this name is most commonly used in connection with purely mathematical theories, essentially the same principle is involved in the development of certain branches of other sciences, notably in rational mechanics.

Chapter III, "On the Nature of Systems of Postulates," probably requires of the non-mathematical reader a greater degree of intellectual effort than any other in the book. The author has, however, succeeded admirably in minimizing the difficulties of the subject. A simple postulate system is studied in detail. The exposition is carefully planned to yield a large amount of information with a minimum of technique.

Undoubtedly some authorities will consider that Professor Carmichael's explanation of what is meant by "rigorous thinking" (on pages 76 and 77) places too much emphasis on the postulational formulation of the foundations and not enough on the necessity for careful discrimination in the choice of methods by which one builds on this foundation. Just what mental processes imply logical necessity is a question on which there is by no means unanimous agreement among scholars. The author's discussion does, however, serve to show the necessity for explicit statement of all hypotheses in a logical exposition of any scientific theory.

In Chapter IV, "Concerning the Postulational Treatment of Empirical Truth," the author discusses in greater detail than previously the possibility of a more explicit use of the postulational method in the development of scientific theories. Economics is given particular attention, and an ingenious plan (due to Rueff) for the reduction of this science to a purely deductive form is described. The possibility of such a treatment of ethics is also considered.

In discussing the procedure to be followed if it is found that a particular system of postulates leads to results which are invalid for a set of objects under investigation, the author says: "We shall need to replace our given system by one which is not equivalent to it; moreover, we shall have to replace it by one having no concrete interpretation (of any sort) in common with the one which we seek to modify so as to avoid contradiction—a fact which will be apparent on a little reflection." (Pages 89, 90.) It may serve to make the situation somewhat clearer if it is noted that in the part of the sentence following the word "moreover" the author evidently has in mind the case in which the new system

(but not necessarily the original one) is categorical. Without this qualification, the statement is incorrect, as may be shown by a simple example.[1] On the whole, however, the exposition is logical and clear.

In Chapter V, "The Structure of Exact Thought," and in Chapter VI, "The Notion of Doctrinal Function," the author continues the discussion of postulate theory. Here special emphasis is given to the fact that the development of a mathematical science associated with a system of postulates is entirely independent of any particular concrete interpretation of the system. The theory is illustrated by application to two specific postulate systems, one of which states the fundamental properties of finite groups.

Chapter VII, "Hypothesis Growing into Veritable Principle," contains an account of the evolution of scientific ideas from vague speculative conjectures, based on little or no experimental evidence, to veritable principles of science. An excellent example of this process is furnished by the atomic theory, the history of which the author traces from the philosophical speculations of the Greeks to the electronic theory of matter.

The chapter also contains an able criticism of Delacre's view of value of abstract theory in scientific research, and a discussion of Vaihinger's philosophy of the "As If."

Chapter VIII, "What is Reasoning?," will perhaps be the most interesting in the book to a reader with an interest in psychology. This chapter is closely connected with the first. It gives an interesting discussion of the rôle of the subconscious mind in mathematical discovery.

The author shows in a convincing manner that the theory advanced in Rignano's *Psychology of Reasoning* is entirely inadequate to account for such logical processes as mathematical induction, or the type of thought involved in the construction of a doctrinal function.

The closing chapter, "The Larger Human Worth of Mathematics," is devoted to a discussion of some aspects of the cultural value of mathematics.

The book is written in a very readable style and is heartily recommended to anyone who is anxious to become familiar with the interrelations of mathematics, the physical and social sciences, and philosophy.

FRED W. PERKINS

---

[1] A system S comprising the postulates numbered I–VI, inclusive, (on pages 62 and 63) and VII_2 (page 72) with $n=2$, is consistent. A second system S', obtained from S by replacing VII_2 by VII_1 (page 72) with $n=2$, is consistent and non-categorical. If the set of objects to be studied are the seven letters $A, B, C \cdots G$, arranged in m-classes as indicated on page 64, then the system S is inapplicable, but the system S' is applicable. Nevertheless S and S' have a concrete interpretation in common, namely the thirteen letters $A, B, C, \cdots, M$ arranged in m-classes as indicated on page 71.

That the statement given in the text is correct when qualified as noted in this review, is readily established by noting that if the original system and a new categorical system have one concrete interpretation in common, then (since all interpretations of the new system are isomorphic) the original system must be applicable to all concrete interpretations of the new system; in other words, the new system cannot be valid for any set of objects for which the original system is invalid.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3509. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Through the vertices A, B, C of a triangle parallel lines are drawn cutting a given straight line Δ at α, β, γ; the parallels to BC, CA, AB drawn through α, β, γ form a triangle $A_1B_1C_1$ equal to ABC, (Monthly, 1929, p. 424). Prove that the isogonal conjugates to B_1C_1, C_1A_1, A_1B_1, in the angles $(\Delta, A\alpha)$, $(\Delta, B\beta)$, $(\Delta, C\gamma)$ respectively are concurrent.

3510. *Proposed by William Sell, University of Alabama.*

Ellipses, as many as possible, are drawn wholly within a circular plane region R. Each ellipse has semi-axes a and b. No two ellipses overlap, and each is tangent to at least two others, but otherwise they are placed at random. The probability of the inclination (of any major axis to a fixed line) having a particular value is the same for all values from zero to 2π. Find the limit of the quotient of the area of R divided by the sum of the areas of the ellipses, as R increases without limit.

3511. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given an irregular tetrahedron and a triangle; show how to pass a plane which shall cut from the tetrahedron a triangle similar to the given triangle.

3512. *Proposed by J. Rosenbaum, Milford, Conn.*

Prove that in the tetrahedron of Problem 3482 (March, 1931), the common center of the two spheres is also the centroid of the tetrahedron.

3513. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$f(s, p) = \sum_{k=1}^{p}(-1)^{p+k}(2k-1)_sH_{p-k}$$

where

$${}_sH_r \equiv \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \quad {}_sH_0 \equiv 1.$$

Prove: (a) If p is odd, then $f(s, p)$ is positive for all values of s;

(b) If p is even, then $f(s, p)$ is positive, equal to zero, or negative according as s is less than, equal to, or greater than 3.

3514. *Proposed by J. P. Ballantine, University of Washington.*

Let D_n denote a determinant of order n whose elements are all zeros and ones, and which has 2 ones and $n-2$ zeros in every row and column. Show that:

(a) For every n, $D_n = \pm 2^m$, where m and n are both even or both odd, or $D_n = 0$.

(b) If $D_5 = 0$, then two rows are identical, and conversely.

(c) If two rows of D_n are alike, then two columns are alike, and conversely.

(d) $3m \leqq n$,

(e) Show that property (*b*) does not hold except for D_2, D_3, and D_5.

3515. *Proposed by E. P. Bogdanoff, Harbin City, China.*

Give an elementary proof, without the use of the calculus, that the equation

$$2^x = 4 \cdot x$$

has two real roots only, and calculate each of them.

Determine the location of the imaginary roots, and compute the pair having the least absolute value.

SOLUTIONS

299 [1914, 267; 1931, 171]. *Proposed by B. F. Finkel, Drury College.*

A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased, that it may rest with its base in the common surface of the fluids.

From Walton's *Hydrostatical Problems.*

Solution by Harry D. Ruderman, Brooklyn, N. Y., and the Proposer

Let σ_1 be the density of the upper fluid; σ_2, the density of the lower fluid; ρ_1, the density of the cone in the first position; and ρ_2, the density of the cone in the second position.

Since the cone in the second position is to have the surface of its base coincident with surface of the lower fluid, the density of the cone in the second position must be the same as the density of the lower fluid, that is, $\rho_2 = \sigma_2$, and hence, if the altitude of the cone does not exceed the depth of the upper fluid, $\rho_2 - \rho_1 = \sigma_2 - \sigma_1$ is the amount the density of the cone must be increased to satisfy the conditions of the problem.

In the contrary case let h_1 be the length of the whole altitude of the cone and h_2 the length of that portion of the altitude immersed in the second fluid when the cone is in its first position. Let r_1 be the radius of the base of the cone and r_2, the radius of the section of the cone formed by the common surface of the fluids.

Now the mass of the cone must be equal to the masses of the two fluids displaced by the cone.

The mass of the cone satisfying the first condition of the problem is $\frac{1}{3}\pi r_1^2 h_1 \rho_1$. The mass of the first liquid displaced by the cone is

$$\tfrac{1}{3}\pi(r_1^2 + r_1r_2 + r_2^2)(h_1 - h_2)\sigma_1$$

and the mass of the second liquid displaced is $\frac{1}{3}\pi r_2^2 h_2\sigma_2$.

Hence

$$\tfrac{1}{3}\pi r_1^2 h_1\rho_1 = \tfrac{1}{3}\pi(r_1^2 + r_1r_2 + r_2^2)(h_1 - h_2)\sigma_1 + \tfrac{1}{3}\pi r_2^2 h_2\sigma_2;$$

whence $r_1^2 h_1\rho_1 = (r_1^2 + r_1r_2 + r_2^2)(h_1 - h_2)\sigma_1 + r_2^2 h_2\sigma_2$. From similar triangles, $r_2/r_1 = h_2/h_1$ or $r_2 = r_1h_2/h_1$ substituting r_1h_2/h_1 for r_2 in the above equation and solving for ρ_1 we have

$$\rho_1 = \sigma_1\frac{[h_1^2 + h_1h_2 + h_2^2]}{h_1^3}(h_1 - h_2) + \frac{h_2^3}{h_1^3}\sigma_2$$

or

$$\rho_1 = \sigma_1(h_1^3 - h_2^3)/h_1^3 + \frac{h_2^3}{h_1^3}\sigma_2$$

From the equation $\rho_2 = \sigma_2$ above, we subtract the equation for ρ_1 and we have

$$\rho_2 - \rho_1 = \sigma_2 - \frac{h_2^3}{h_1^3}\sigma_2 - \frac{h_1^3 - h_2^3}{h_1^3}\sigma_1$$

or

$$\rho_2 - \rho_1 = (\sigma_2 - \sigma_1)(h_1^3 - h_2^3)/h_1^3 .$$

Also solved by B. D. Roberts.

3239 [1927, 98). *Proposed by Alex S. Wiener, Cornell University.*

Solve the following simultaneous equations for u, v, w, x, y, and z:

$$\begin{aligned}(u - a_1)(x - b_1) - (v - b_1)(w - a_1) &= 0,\\ (u - a_2)(z - b_2) - (v - b_2)(y - a_2) &= 0,\\ (w - a_3)(z - b_3) - (x - b_3)(y - a_3) &= 0,\\ u^2 + v^2 = w^2 + x^2 = y^2 + z^2 &= r^2.\end{aligned}$$

II. Solution by Otto Dunkel, Washington University.

This second solution is given since the first one [1929, 288] is deficient in some respects and since it involves unnecessarily tedious algebraic computations. This solution applies to the more general problem in which the circle is replaced by any conic, and it does not require a change of coordinates. It may be of interest since it proves incidentally some known geometric theorems. We first restate the generalized problem in geometric form. Given any conic S and three points in its plane $A_1(a_1, b_1)$; $A_2(a_2, b_2)$; $A_3(a_3, b_3)$; to find the coordinates (y, z); (w, x); (u, v); of the vertices C_1, C_2, C_3 of a triangle inscribed in S such that C_1C_2 passes through A_3; C_2C_3, through A_1; C_3C_1, through A_2. We shall discard the trivial case of a vertex of $A_1A_2A_3$ lying upon S.

An exceptional case which was not mentioned in the first solution will be considered first. If $A_1A_2A_3$ is a self-polar triangle with respect to S there is an infinite number of solutions obtained by taking any point of S for one vertex of the required triangle. For, if A_3 is the vertex inside the conic, let C_1 be a point on S, and draw C_1A_3 cutting S again in C_2. Draw A_1C_2 cutting S again in C_3. Since A_1A_2 is the polar of A_3, the pencil $A_2(C_2\,A_3\,C_1\,A_1)$ is harmonic; since A_2A_3 is the polar of A_1, the pencil $A_2(C_2\,A_3\,C_3\,A_1)$ is harmonic. Hence A_2C_1 and A_2C_3 lie on the same straight line, or A_2C_1 and A_1C_2 meet in C_3 on S. Similarly A_2C_2 and A_1C_1 meet in C_3' on S, and C_3C_3' passes through A_3. Hence any one of the four triangles with vertices at C_1, C_2, C_3, C_3' is a required triangle.

We take up now the case where A_1, A_2, A_3 do not lie in a straight line and do not form a self-polar triangle. Let $A_1'A_2'A_3'$ be the triangle such that $A_2'A_3'$ is the polar of A_1; $A_3'A_1'$, of A_2; $A_1'\,A_2'$, of A_3. Denote by $P(xx)=0$ the equation of the conic S, and by $P(ix)=0$ the equation of the polar of $A_i(a_i,\,b_i)$. These two equations contain both x and y but for brevity of writing we drop the y in the notation. We have $P(ij)=P(ji)$ and $P(ii)\neq 0$, $i=1, 2, 3$. Obviously the equations of the lines joining corresponding vertices are

$$
\begin{aligned}
A_1A_1' &: P(31)P(2x) - P(12)P(3x) = 0,\\
A_2A_2' &: P(12)P(3x) - P(23)P(1x) = 0, \qquad (1)\\
A_3A_3' &: P(23)P(1x) - P(31)P(2x) = 0.
\end{aligned}
$$

These equations give immediately the theorem that the triangles $A_1A_2A_3$ and $A_1'\,A_2'\,A_3'$ are perspective.

Let A_1A_1' cut $A_2'\,A_3'$ in A_1'', and let A_2'', A_3'' be defined similarly. The equation of $A_2''\,A_3''$ will now be derived. This equation may be written in two ways

$$
\begin{aligned}
-\,P(12)P(3x) + P(23)P(1x) - kP(2x) &= 0,\\
P(23)P(1x) - P(31)P(2x) - lP(3x) &= 0.
\end{aligned}
$$

It is obvious that each of these equations is the equation of $A_2''\,A_3''$ if $k=P(31)$ and $l=P(12)$. Hence we have

$$
\begin{aligned}
A_2''A_3'' &: P(12)P(3x) - P(23)P(1x) + P(31)P(2x) = 0,\\
A_3''A_1'' &: P(23)P(1x) - P(31)P(2x) + P(12)P(3x) = 0, \qquad (2)\\
A_1''A_2'' &: P(31)P(2x) - P(12)P(3x) + P(23)P(1x) = 0.
\end{aligned}
$$

These equations tell us that the pencils of lines such as $A_2''\,A_3''$, $A_3''\,A_1''$, $A_1'\,A_2'$, A_3A_3' are harmonic. The two lines $A_2''\,A_3''$, $A_3''\,A_1''$ and the conic determine a set of conics having four points in common. The equations of such conics may be written

$$P^2(12)P^2(3x) - [P(23)P(1x) - P(31)P(2x)]^2 - kP(xx) = 0.$$

We find the conic of the set which passes through A_3 by setting $x=a_3$, $y=b_3$,

and we thus find $k=P^2(12)P(33)$, since $P(33)\neq 0$. Hence the equation of this conic is

$$(3)\qquad P^2(12)[P^2(3x)-P(33)P(xx)]-[P(23)P(1x)-P(31)P(2x)]^2=0.$$

Since $P(3x)=0$ is the polar of A_3 with respect to $P(xx)=0$, the first bracket is the product of two linear factors α and β. Hence (3) has the form $P^2(12)\alpha\beta-\gamma^2=0$, where α, β, γ each vanish for a_3, b_3. Therefore (3) is the equation of two straight lines intersecting in A_3 and passing through pairs of points in which $A_2''A_3''$, $A_3''A_1''$ cut S. Denote these two lines by $C_1A_3C_2$ and $C_1'A_3C_2'$, where C_1, C_1' are the intersections of $A_2''A_3''$ with S, and C_2, C_2', those of $A_3''A_1''$ with S. In a similar manner we define C_3 and C_3' on $A_1''A_2''$, and we then have the following sets of three points on a straight line

$$(4)\qquad \begin{array}{c} C_2A_1C_3,\ C_3A_2C_1,\ C_1A_3C_2,\\ C_2'A_1C_3',\ C_3'A_2C_1',\ C_1'A_3C_2', \end{array}$$

where all the C's lie on S.

The equations (2) taken in turn with $P(xx)=0$ determine the coordinates of the points C_1, C_2, C_3 independently of the manner in which they were obtained. If $A_1A_2A_3$ is self-polar $P(12)=P(23)=P(31)=0$, and we have no equations. But this is the case of an infinite number of solutions which we have already considered and which is easily handled. If the three points A_1, A_2, A_3 lie in a straight line, $P(1x)$, $P(2x)$ and $P(3x)$ are linearly dependent, and we must examine the equations (2) in this regard. Consider the first one and suppose that it is identically zero; then by setting the x and y in it equal to the coordinates of the vertices of the triangle $A_1A_2A_3$ we find in turn $P(23)P(11)=2P(12)P(31)$, $P(31)P(22)=0$, $P(12)P(33)=0$. Or $P(23)=P(31)=P(12)=0$, and this means that $A_1A_2A_3$ is self polar. Hence no equation of (2) is identically zero, if the triangle is not self-polar; and these equations determine the solution in all cases except the case of an infinite number of solutions. We now see that there are an infinite number of solutions only when the given triangle is self-polar.

The equations (2) show that the straight line

$$P(12)P(3x)+P(23)P(1x)+P(31)P(2x)=0$$

is the axis of perspectivity on which the corresponding sides of $A_1'A_2'A_3'$ and $A_1''A_2''A_3''$ meet. All of the above results may be obtained by synthetic projective geometry methods.

3391 [1929, 448]. *Proposed by Otto Dunkel, Washington University.*

The ratio of the shortest diagonal to a side of an ordinary regular polygon of 19 sides satisfies an equation of the 9th degree with rational coefficients. The remaining roots are the corresponding ratios with alternately plus and minus signs for the remaining regular polygons of star form. By a known theorem the roots of this equation may be obtained by solving first a cubic with rational

coefficients and then solving three other cubics whose coefficients are rational functions of the roots of the first cubic. Derive with as little computation as possible a set of such equations.

Solution by the Proposer.

In the case of the convex regular polygon of 19 sides the equation for the ratio of the shortest diagonal to a side may be written down from the solution of 3322 [1929, 291–293] by equating the lengths of the diagonals $A_1A_n = y_n$ for $n=10$ and $n=11$. Using the formula in the note on that solution for y_n, we obtain the equation

$$(1)\qquad y^9 - y^8 - 8y^7 + 7y^6 + 21y^5 - 15y^4 - 20y^3 + 10y^2 + 5y - 1 = 0,$$

with the root $y_3/y_2 = y$. This equation results from the use of the formula in the note

$$(2)\qquad y_n' = yy_{n-1}' - y_{n-2}',$$

where $y_n' = y_n/y_2$. The star formed polygon with the side y_j, $j=3, 4, \cdots, 10$, has y_{2j-1} for its corresponding diagonal, and it is to be shown that $(-1)^j y_{2j-1}/y_j$ is also a root of (1). Let $A_1', A_2', \cdots, A_{19}'$ denote the successive vertices of the polygon, where $A_1'A_2' = A_1A_j$, $A_1'A_3' = A_1A_{2j-1}$, $A_1'A_n' = A_1A_{(n-1)j-(n-2)}$. If r is the radius of the circle circumscribing the polygon and $\theta = \pi/19$, $A_1'A_n' = 2r \sin (n-1)(j-1)\theta$. We have

$$\sin (n-1)(j-1)\theta + \sin (n-3)(j-1)\theta = 2 \sin (n-2)(j-1)\theta \cos (j-1)\theta,$$
$$= \frac{\sin (n-2)(j-1)\theta \sin 2(j-1)\theta}{\sin (j-1)\theta}$$

or

$$\frac{A_1'A_n'}{A_1'A_2'} + \frac{A_1'A_{n-2}'}{A_1'A_2'} = \frac{A_1'A_{n-1}'}{A_1'A_2'}\,\frac{A_1'A_3'}{A_1'A_2'},$$

$A_1'A_2' = 2r \sin (j-1)\theta$, $A_1'A_3' = 2r \sin 2(j-1)\theta$, $0 < 2(j-1)\theta \leqq 18\pi/19$. $A_1'A_2'$ and $A_1'A_3'$ are both positive, but $A_1'A_n'$ may be negative for certain values of n. Hence the equation (2) is satisfied when $A_1'A_n'/A_1'A_2'$ is substituted for y_n' and $A_1'A_3'/A_1'A_2'$ for y. But since $10(j-1)\theta + 9(j-1)\theta = (j-1)\pi$, we have $A_1'A_{11}' = (-1)^j A_1'A_{10}'$. Since y_{11}' contains only odd powers of y and y_{10}' only even powers, a solution of (1) is $(-1)^j A_1'A_3'/A_1'A_2' = (-1)^j A_1A_{2j-1}/A_1A_j = (-1)^j 2 \cos (j-1)\theta$.

It will be more convenient to consider the equation

$$(3)\qquad y^9 + y^8 - 8y^7 - 7y^6 + 21y^5 + 15y^4 - 20y^3 - 10y^2 + 5y + 1 = 0,$$

whose roots are the negatives of the roots of equation (1). Hence the roots of (3) are

$$
(4) \qquad \begin{aligned}
&r_1 = 2\cos 2\theta, && r_8 = -2\cos 3\theta, && r_7 = -2\cos 5\theta,\\
&r_2 = 2\cos 4\theta, && r_3 = 2\cos 6\theta, && r_5 = -2\cos 9\theta,\\
&r_4 = 2\cos 8\theta, && r_6 = -2\cos 7\theta, && r_9 = -2\cos \theta,\\
&r_k = \epsilon^k + \epsilon^{-k}, && \epsilon = e^{2\theta i}, && \theta = \pi/19.
\end{aligned}
$$

The order of writing these roots is determined as follows: the numbers in the first line below are congruent, modulus 19, to the corresponding numbers in the second line

$$
(5) \qquad \begin{array}{ccccccccc}
1, & 2, & 4, & 8, & -3, & -6, & 7, & -5, & 9,\\
2^0, & 2^1, & 2^2, & 2^3, & 2^4, & 2^5, & 2^6, & 2^7, & 2^8.
\end{array}
$$

Now set

$$(6) \qquad z_1 = r_1 + r_8 + r_7, \quad z_2 = r_2 + r_3 + r_5, \quad z_3 = r_4 + r_6 + r_9;$$

and let us find the equation whose roots are the z's. We have at once

$$(7) \qquad z_1 + z_2 + z_3 = -1,$$

since the left side is the sum of the roots of (3). We next calculate the products z_1z_2, z_2z_3, z_1z_3. This may be conveniently done as follows for z_1z_2. From (4) it will be seen that the subscripts of the r's are the exponents of ϵ, and we have merely to write the sum and the difference of every pair of subscripts taking one from z_1 and its mate from z_2. Thus we get

$$3, 10, 9, 4, 11, 10, 6, 13, 12, 1, 6, 5, 2, 5, 4, 4, 3, 2.$$

The numbers 10, 11, 12, 13 are then replaced by the absolute values of their residues with respect to 19, i.e., 9, 8, 7. 6. We then count the number of times the sets of subscripts, 1, 8, 7; 2, 3, 5; 4, 6, 9 occur. Hence

$$(8) \qquad z_1z_2 = z_1 + 2z_2 + 3z_3; \quad z_2z_3 = z_2 + 2z_3 + 3z_1; \quad z_3z_1 = z_3 + 2z_1 + 3z_2.$$

Then from (7) we obtain $\Sigma z_iz_j = -6$. In order to evaluate $z_1z_2z_3$ we eliminate z_3 from the expression for z_1z_2 by means of (7), and thus find $z_1z_2 = -z_2 - 2z_1 - 3$. Multiplying by z_3 and then eliminating the two products by means of (8) we obtain

$$(9) \qquad z_1z_2z_3 = -7(z_1 + z_2 + z_3) = 7.$$

Hence the z's satisfy the equation

$$(10) \qquad z^3 + z^2 - 6z - 7 = 0.$$

All the roots of this equation are real, and it is obvious that it has one and only one positive root. In order to identify the roots we observe that $\frac{1}{2}z_2$ contains two positive terms $\cos 4\theta$, $\cos 6\theta$, and the negative term $-\cos 9\theta$, and that $\cos 4\theta > \cos 9\theta$. Hence the positive root is z_2, and z_1 and z_3 are each negative. From $\frac{1}{2}(z_1 - z_3) = (\cos\theta - \cos 3\theta) + (\cos 2\theta - \cos 5\theta) + (\cos 7\theta - \cos 8\theta)$, we see that each parenthesis on the right is positive. Hence $z_1 > z_3$.

We shall now obtain the three equations for the three sets of r's. We have $r_1+r_7+r_8=z_1$. Also $r_1r_7=r_8+r_6$; $r_1r_8=r_9+r_7$; $r_7r_8=r_{15}+r_1=r_4+r_1$. We add and subtract the two subscripts in the product to get the two terms on the right. Hence $\Sigma r_ir_j=r_1+r_8+r_7+r_4+r_6+r_9=z_1+z_3$. Then $r_1r_7r_8=r_8^2+r_6r_8=r_{16}+r_{14}+r_0+r_2=2+r_2+r_3+r_5=2+z_2$. Hence the equation for r_1, r_7, r_8 is

$$y^3 - z_1y^2 - (1 + z_2)y - (2 + z_2) = 0, \quad r_1, r_7, r_8. \tag{11}$$

The other two equations are obtained in the same manner

$$\begin{aligned} y^3 - z_2y^2 - (1 + z_3)y - (2 + z_3) &= 0, \quad r_2, r_3, r_5, \\ y^3 - z_3y^2 - (1 + z_1)y - (2 + z_1) &= 0, \quad r_4, r_6, r_9. \end{aligned} \tag{11'}$$

There are other simple ways of deriving equation (10). From the equation used to establish (9) and from two similar equations we have

$$z_2 = -2 - \frac{1}{z_1 + 1}; \quad z_3 = -2 - \frac{1}{z_2 + 1}; \quad z_1 = -2 - \frac{1}{z_3 + 1}. \tag{12}$$

It may also be shown, without the use of (10), that

$$z_2 = 4 - z_1^2; \quad z_3 = 4 - z_2^2; \quad z_1 = 4 - z_3^2. \tag{13}$$

It suffices then to find one root of (10), say z_1, and then z_2 may be calculated from the first equation of either (12) or (13).

It will be easily seen that the equation

$$y_n' = yy_{n-1}' - y_{n-2}'$$

gives a geometrical interpretation of the usual process for reducing a reciprocal equation of even degree to an equation of half its degree.

3431 [1930, 261]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters a and b designate respectively two fixed, non-complanar lines, the angle between which is α, and the perpendicular distance between which is l; the letters P and Q designate two points on a third line c such that the distance PQ is constant; moreover, points P and Q of line c move on lines a and b, respectively. Show: (I) that the locus (relative to the frame formed by lines a and b) of any third point S of line c is an ellipse whose plane is parallel to lines a and b and whose semi-major and semi-minor axes are

$$\left\{\frac{(n-m)^2 - l^2}{4\sin^2\alpha} + nm\frac{(n-m)^2 - l^2}{(n-m)^2}\right\}^{1/2} + \frac{\{(n-m)^2 - l^2\}^{1/2}}{2\sin\alpha} \tag{1}$$

and

$$\left\{\frac{(n-m)^2 - l^2}{4\sin^2\alpha} + nm\frac{(n-m)^2 - l^2}{(n-m)^2}\right\}^{1/2} - \frac{\{(n-m)^2 - l^2\}^{1/2}}{2\sin\alpha}, \tag{2}$$

wherein n and m represent the linear segments QS and PS, respectively; (II) that the angles which lines a and b make with the semi-major and the semi-minor axes, respectively, are

$$\gamma = 45° - \tfrac{1}{2}(\alpha + \beta), \tag{3}$$

and

$$\delta = 45° - \tfrac{1}{2}(\alpha - \beta), \tag{4}$$

wherein β is given by the relation

$$\tan \beta = (n - m) \cot \alpha/(n + m). \tag{5}$$

Note: The linear segments n and l of expressions (1) and (2) are regarded as positive for all configurations; m in these expressions is positive or negative according as point S divides segment PQ externally or internally; angles α and β, in equations (3), (4), and (5), are regarded as positive for all configurations; γ and δ may be either positive or negative, depending upon the position of point S on line c.

Solution by Otto Dunkel, Washington University.

Let a and b cut their common perpendicular in A and B, where $AB = l$; and let the plane through S perpendicular to AB cut the latter in O. Then $OA/OB = m/n$, and S moves in this fixed plane. Let the projections on this plane of P, Q, a, b, c be P', Q', a', b', c', and set $P'S = p$, $Q'S = q$. Then P' and Q' lie on the fixed lines a' and b', respectively, and we have

$$\begin{aligned} &p/q = m/n, \qquad q - p = \left[(n - m)^2 - l^2\right]^{1/2} \\ &p = \frac{m}{n - m}(q - p), \quad q = \frac{n}{n - m}(q - p). \end{aligned} \tag{1}$$

Thus the problem reduces to finding the locus of S, where S, P', Q' are fixed points on the moving line c' such that P' and Q' move on the fixed straight lines a' and b', respectively. It will now be shown how to reduce this construction for S to the ordinary trammel construction for an ellipse. Let a circle (C) be passed through P', Q', O for a given position of c', and suppose that this circle is now rigidly attached to c' at P' and Q' and moves with it. Since the fixed chord $P'Q'$ of (C) subtends the angle α, which is the angle between a' and b', at every point of a certain arc of (C) and since P' and Q' lie on a' and b', the point O must lie on (C) for every position of c'. Through S and C, the center of (C), draw a straight line cutting (C) in X and Y. Draw also the lines $x = OX$, $y = OY$ on the fixed plane, and suppose that XYS is also rigidly attached to the moving system of c' and (C). The arc XP' subtends a constant angle at points on part of the circumference of (C) and this angle is the same in the initial position as the properly chosen angle between a' and x, intersecting in the point O on (C). Since P' lies always on a', X must lie always on x, since it does so for the initial posi-

tion. Similarly Y moves on y. We now have a straight line carrying fixed points on it X, Y, S so that X moves on x, Y on y, and x and y meet in a right angle at O. This is the trammel construction for the ellipse. Consider now the simple case where n and m are both positive and $n>m$; then p and q are positive and $q>p$. Suppose also that X and Y have been so chosen that YS has a greater length than XS. Then the semi-major axis has the length of YS on the x axis, and the semi-minor axis has the length of XS on the y axis. Let r be the radius of (C), M the mid-point of $P'Q'$, β the acute angle CSM, γ the angle between x and a', and δ the angle between y and b'. Then $\angle MCP'=\alpha$, $\angle XCP'=2\gamma$ and $\angle YCQ'=2\delta$, and we easily find

$$(2)\qquad \begin{gathered} 2r\sin\alpha = q - p, \qquad \tan\beta = \frac{q-p}{q+p}\cot\alpha, \\ \gamma = 45^\circ - \tfrac{1}{2}(\alpha+\beta), \quad \delta = 45^\circ - \tfrac{1}{2}(\alpha-\beta). \end{gathered}$$

In this case, if α is less than $90°$, β may be considered as an acute positive angle, and then δ and γ are positive and acute. If, however, $\alpha>90°$, we would naturally take β acute and negative and then γ and δ would be both negative. The lengths of the semi-axes are easily found from the equations

$$(3)\qquad \begin{gathered} YS - XS = 2r = (q-p)/\sin\alpha, \quad YS\cdot XS = qp \\ YS + XS = \left[\left(\frac{q-p}{\sin\alpha}\right)^2 + 4pq\right]^{1/2}. \end{gathered}$$

These results (1), (2), (3) give YS, XS, and $\tan\beta$ in terms of m and n which agree with the values in the problem. There is no difficulty in tracing the changes as S is taken at other points on c' relative to the segment $P'Q'$. When S lies within the segment $Q'P'$, q and p have opposite signs, as also m and n. Thus in this case the expression (2) of the problem becomes negative. If S lies between Y and C, or outside the segment on the Y side, the major axis lies upon y and not upon x as in the other case, so that γ now gives the angle between the minor axis and b.

3465 [1930, 551]. *Proposed by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.*

Prove that for all positive integral values of n

$$\sum_{i=n}^{2n}(-1)^{2n-i}\frac{(2i+2)!}{i!(2n-i)!(2i-2n)!} = 2^{n-1}(n+1)(2n+1)(2n^2+7n+4).$$

I. *Solution by W. Randolph Church, Hightstown, New Jersey.*

We shall first prove that

$$(1)\qquad \sum_{j=0}^{n}(-1)^j\binom{4n-2j+2}{2n+2}\binom{2n}{j} = 2^{2n-2}(2n^2+7n+4).$$

It is known (Netto, *Lehrbuch der Combinatorik*, §158) that

$$\sum_{s=0}^{q}\binom{p}{r+s}\binom{q}{s}=\binom{p+q}{r+q}.$$

Setting $p=4n-2j$, $r=2n$, $q=2$ in this formula, we have

$$\sum_{s=0}^{2}\binom{4n-2j}{2n+s}\binom{2}{s}=\binom{4n-2j+2}{2n+2}.$$

Substituting this in the left side of (1) we have

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\binom{4n-2j+2}{2n+2}\binom{2n}{j}&=\sum_{j=0}^{n}(-1)^j\sum_{s=0}^{2}\binom{2}{s}\binom{4n-2j}{2n+s}\binom{2n}{j}\\&=\sum_{s=0}^{2}\binom{2}{s}\sum_{j=0}^{n}(-1)^j\binom{4n-2j}{2n+s}\binom{2n}{j}.\end{aligned}$$

But it is also known (ibid, §160) that

$$\sum_{k=0}^{n}(-1)^k\binom{p}{k}\binom{2p-2k}{2p-m}=2^m\binom{p}{m}.$$

Setting $p=2n$, $m=2n-s$ in this formula and substituting we have

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\binom{4n-2j+2}{2n+2}\binom{2n}{j}&=\sum_{s=0}^{2}\binom{2}{s}2^{2n-s}\binom{2n}{2n-s}\\&=2^{2n}+2\cdot2^{2n-1}\cdot2n+2^{2n-2}\cdot n(2n-1)\\&=2^{2n-2}(2n^2+7n+4).\end{aligned}$$

which completes the proof of (1).

Substituting $j=2n-i$ in (1), and changing the limits of summation accordingly, we obtain the formula

$$(2)\qquad \sum_{i=n}^{2n}(-1)^{2n-i}\binom{2i+2}{2n+2}\binom{2n}{i}=2^{2n-2}(2n^2+7n+4).$$

Writing (2) in terms of factorials, and introducing the factors $(2n+1)(2n+2)$ on each side, we have

$$\sum_{i=n}^{2n}(-1)^{2n-i}\frac{(2i+2)!(2n)!(2n+1)(2n+2)}{(2n+2)!(2i-2n)!i!(2n-i)!}=2^{2n-1}(n+1)(2n+1)(2n^2+7n+4),$$

which reduces immediately to

$$\sum_{i=n}^{2n}(-1)^{2n-i}\frac{(2i+2)!}{i!(2n-i)!(2i-2n)!}=2^{2n-1}(n+1)(2n+1)(2n^2+7n+4),$$

which is the given relation.

II. *Solution by Otto Dunkel, Washington University.*

This solution will employ the methods of the calculus of finite differences, and we first present some facts which will be used. The difference operator Δ may be written $\Delta = U-1$, $U^k f(x) = f(x+k)$. If we set $x^{(n)} = x(x-1) \cdots (x-n+1)$, then it is easily shown that

$$\Delta^p x^{(n)} = n(n-1) \cdots (n-p+1) x^{(n-p)}. \tag{1}$$

We now prove that

$$\Delta_x^n f(2x) = \left[\sum_{j=0}^{n} 2^{n-j} \binom{n}{j} \Delta_y^{n+j} \right] f(y), \quad y = 2x, \tag{2}$$

where the subscripts x and y indicate that the Δ applies only to the variable indicated by the subscript. Thus

$$\begin{aligned} \Delta_x f(2x) &= f(2x+2) - f(2x) = f(y+2) - f(y) \\ &= (U^2 - 1) f(y) = (U+1)(U-1) f(y) = 2\Delta(1 + 2^{-1}\Delta) f(y). \end{aligned}$$

Hence

$$\begin{aligned} \Delta_x^n f(2x) &= 2^n \Delta^n (1 + 2^{-1}\Delta)^n f(y), \\ &= \left[\sum_{j=0}^{n} 2^{n-j} \binom{n}{j} \Delta^{n+j} \right] f(y). \end{aligned}$$

We may write the sum on the left in the problem

$$\sum_{i=0}^{2n} (-1)^{2n-i} \binom{2n}{i} \phi(i), \quad (2n)!\phi(i) = (2i+2)^{(2n+2)}; \tag{3}$$

since $\phi(i) = 0$ if $i = 0, 1, \cdots n-1$. Now write (3) in the form

$$\left[\sum_{i=0}^{2n} (-1)^{2n-i} \binom{2n}{i} U^i \right] \phi(0) = (U-1)^{2n} \phi(0) = \Delta^{2n} \phi(0). \tag{4}$$

Since $\phi(i)$ is of degree $2n+2$ in i, $\Delta^{2n+j}\phi(i) = 0$ if $j > 2$. Hence by (2) we have

$$\Delta^{2n}\phi(0) = 2^{2n} \left[\Delta^{2n} + n\Delta^{2n+1} + \frac{n(2n-1)}{4} \Delta^{2n+2} \right] \phi(0),$$

where on the left the argument 0 is increased by two units in each operation of Δ, while on the right the increase is only one unit. Applying (1) to each of the three Δ operations on the right, we get after slight reductions

$$2^{2n} \frac{(2n+2)!}{(2n)!} \left[1 + 2n + \frac{n(2n-1)}{4} \right] = 2^{2n-1}(n+1)(2n+1)(2n^2 + 7n + 4).$$

III. *Solution by the Proposer.*

The member on the left may be written

$$2(n+1)(2n+1)\sum_{j=0}^{n}(-1)^{n+j}\binom{2n+2j+2}{2j}\binom{2n}{n-j}.$$

In this the summed terms are the coefficient of x^{2n} in the expansion of $(1-x^2)^{2n}(1+x)^{-(2n+3)}$. This coefficient is equal to that of x^{2n} in the expansion of $(1-x)^{2n}(1+x)^{-3}$, that is to say,

$$\sum_{i=0}^{2n}\binom{i+2}{2}\binom{2n}{i}.$$

This readily reduces to $2^{2n-2}(2n^2+7n+4)$. Hence, the result given in the problem.

3470 [1931, 50]. *Proposed by F. L. Wren, George Peabody College for Teachers.*

If the hypotenuse of a right triangle be divided into n equal parts and the vertex of the right angle be joined to these points of equal division, then, if d_i be the length of the lines so drawn, we have

$$\sum_{i=1}^{n-1}d_i^2=\frac{(n-1)(2n-1)}{6n}h^2,$$

where h is the length of the hypotenuse.

I. *Solution by Beatrice Aitchison, Johns Hopkins University.*

Take the x-axis along the side of the right triangle of length a and the y-axis along the side of length b. Then the coordinates of the ith point of division of the hypotenuse are $[(n-i)a/n, ib/n]$. Hence

$$\sum_{i=1}^{n-1}d_i^2=\sum_{i=1}^{n-1}(n-i)^2a^2/n^2+\sum_{i=1}^{n-1}i^2b^2/n^2,$$

$$=\frac{a^2+b^2}{n^2}\sum_{i=1}^{n-1}i^2=\frac{(n-1)(2n-1)}{6n}h^2.$$

II. *Generalization by R. Goormaghtigh, Bruges, Belgium.*

If the side BC of a triangle ABC, whose sides are a, b, c be divided into n equal parts and the vertex A be joined to these points, then if d_i be the length of the lines so drawn,

$$\sum_{i=1}^{n-1}d_i^2=(n-1)bc\cos A+\tfrac{1}{6}n^{-1}(n-1)(2n-1)a^2.$$

For

$$d_i^2=c^2(n-i)n^{-1}+b^2in^{-1}-a^2i(n-i)n^{-2}$$

and

$$\sum_{i=1}^{n-1} d_i^2 = c^2n^{-1}\sum_{i=1}^{n-1}(n-i) + b^2n^{-1}\sum_{i=1}^{n-1} i - a^2n^{-1}\sum_{i=1}^{n-1} i + a^2n^{-2}\sum_{i=1}^{n-1} i^2.$$

Hence

$$\sum_{i=1}^{n-1} d_i^2 = \tfrac{1}{2}(n-1)(b^2+c^2-a^2) + \tfrac{1}{6}n^{-1}(n-1)(2n-1)a^2$$
$$= (n-1)bc\cos A + \tfrac{1}{6}n^{-1}(n-1)(2n-1)a^2.$$

Also solved by Frank Ayres, R. P. Agnew, Jane Bent, E. M. Berry, Alice Bromwell, A. G. Clark, Mannis Charosh, Ralph Deutsch, Edward Fleischer, J. D. Hill, James Hamilton, L. S. Johnston, J. F. Locke, Clara Mize, G. T. Miller, J. H. Neelley, A. Pelletier, P. L. Rea, C. A. Rupp, O. J. Ramler, A. W. Randall, F. Underwood, F. B. Wiley, Paul Wernicke, Kamcheung Woo, G. A. Yanosik, B. F. Yanney and the Proposer.

3471 [1931, 50]. *Proposed by W. R. Ransom, Tufts College.*

In assigning dormitory rooms, a college gives preference to pairs of students in this order:

$$AA,\ AB,\ AC,\ BB,\ BC,\ AD,\ CC,\ BD,\ CD,\ DD,$$

in which AA means two seniors, AB means senior and junior, etc. Determine numerical values to assign to A, B, C, D so that the set of numbers $A+A$, $A+B$, $A+C$, $B+B$, etc., corresponding to the order indicated above, will be in descending magnitude. Find the general solution, and also the solution in least integers.

Solution by B. F. Yanney, College of Wooster.

The essential inequalities given, are as follows, the others being redundant:

$$(1) \qquad 2A > A+B > A+C > 2B,\ \ B+C > A+D > 2C > B+D.$$

Obviously, inequalities 1, 2, and 4 of (1) imply that

$$A > B > C > D,$$

whence we may write:

$$(2) \qquad C = D + k_1,\ \ B = D + k_1 + k_2,\ \ A = D + k_1 + k_2 + k_3,$$

where the k's are all positive. These expressions substituted in inequalities 3, 4, 5, 6 of (1) give, respectively,

$$(3) \qquad k_3 > k_2,\ \ k_1 > k_3,\ \ k_2 + k_3 > k_1,\ \ k_1 > k_2.$$

For the last inequality in (3), we may write $k_1 = k_2 + e$, which substituted in inequality 3 gives $k_3 > e$. For this last write $k_3 = e + e_1$. Then from inequality 2 we get $k_1 > e + e_1$, for which write $k_1 = e + e_1 + e_2 = k_2 + e$. Therefore $k_2 = e_1 + e_2$. This

in inequality 1 gives $k_3 > e_1+e_2$, for which write $k_3 = e_1+e_2+e_3 = e+e_1$. Therefore $e = e_2+e_3$. We have now secured the following values for the k's:

$$k_1 = e_1 + 2e_2 + e_3; \quad k_2 = e_1 + e_2; \quad k_3 = e_1 + e_2 + e_3. \tag{4}$$

Now, letting D = an arbitrary number n, and substituting (4) in (2), we have the following general solution:

$$\begin{aligned} D &= n, \\ C &= n + e_1 + 2e_2 + e_3, \\ B &= n + 2e_1 + 3e_2 + e_3, \\ A &= n + 3e_1 + 4e_2 + 2e_3. \end{aligned} \tag{5}$$

We obtain the smallest integral values, if positive, by letting $n = e_1 = e_2 = e_3 = 1$; we have $D=1$, $C=5$, $B=7$, $A=10$.

Also solved by H. T. R. Aude, W. F. Cheney, A. G. Clark, S. A. Corey, R. F. Clash, Ralph Deutsch, Laurence Hampton, Theodore Lindqist, G. T. Miller, F. Underwood, G. A. Yanosik, F. L. Wilmer, and the Proposer.

3472 [1931, 51]. *Proposed by Morgan Ward, California Institute.*

Let S_n denote the sum of the nth powers of the roots of $F(x) = x^3 - Px^2 + Qx - 1$, where P, Q are integers; p, a prime of the form $3n+2$ chosen so that $F(x)$ is irreducible modulo p, and μ, the least value of n such that $S_n \equiv S_0$, $S_{n+1} \equiv S_1$, $S_{n+2} \equiv S_2$, mod p. Prove that if n is not divisible by μ, $S_n \equiv 0$ mod p, when and only when $S_{3n} \equiv 3$ mod p.

Solution by the Proposer.

In view of the identity $S_{3n} - 3 \equiv (S_n^2 - 3S_{-n})S_n$, it is sufficient to show that

$$S_n^2 - 3S_{-n} \equiv 0 \bmod p \tag{1}$$

only when μ divides n. If (1) is true, the polynomial $x^3 - S_n x^2 + S_{-n}x - 1$ whose roots are the nth powers of the roots of $F(x) = 0$ is congruent modulo p to

$$(x - \tfrac{1}{3}S_n)^3 + \frac{S_n{}^3}{27} - 1,$$

and hence is reducible modulo $p = 3n+2$, since, with respect to this modulus, every integer is a cubic residue.

The congruence $x^3 - S_n x^2 + S_{-n}x - 1 \equiv 0 \bmod p$ thus has a solution of the form

$$x = \alpha^n \equiv k \bmod p \tag{2}$$

where α is some root of $F(x) = 0$, and k is an integer.

Owing to the irreducibility of $F(x)$ modulo p, (2) implies that $\beta^n \equiv k$, $\gamma^n \equiv k \bmod p$, where β and γ are the remaining roots of $F(x) = 0$. Hence

$$\alpha^n \beta^n \gamma^n \equiv k^3 \bmod p, \quad \text{or} \quad k^3 \equiv 1,\ k \equiv 1 \bmod p.$$

But if $\alpha^n \equiv 1 \bmod p$, then $\alpha^{n+1} \equiv \alpha$, $\alpha^{n+2} \equiv \alpha^2$, $\alpha^{n+3} \equiv \alpha^3 \bmod p$ and $S_n \equiv S_0$, $S_{n+1} \equiv S_1$, $S_{n+2} \equiv S_2 \bmod p$. Accordingly, from the minimal character of μ, μ divides n.

3473 [1931, 51]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the incircle is tangent to CA at D and to CB at E. Through a variable point P on DE, AP and BP are drawn meeting CB at X and CA at Y.

Find the envelope of the line XY.

Solution by E. M. Berry, Lynchburg College.

The following is a solution of the generalized problem: A triangle ABC has its sides tangent to a conic with D and E points of tangency on CA and CB respectively. Through a variable point P on DE, AP and BP are drawn meeting CB at X and CA at Y, find the envelope of the line XY.

We shall use homogeneous point and line coordinates. Take CDE as the triangle of reference. Let $x_1=0$, $x_2=0$, $x_3=0$ be the equations of the sides EC, ED, and DC respectively then the vertices of the triangle CDE are $C(0:1:0)$, $D(1:0:0)$ and $E(0:0:1)$.

The general equation of the conic touching DC at D and EC at C is

$$(1) \qquad 2c_1x_1x_3 + c_2x_2^2 = 0.$$

The tangent to (1) at the point $(y_1:y_2:y_3)$ is given by

$$(2) \qquad c_1x_1y_3 + c_2x_2y_2 + c_1x_3y_1 = 0.$$

Let $[u_1:u_2:u_3]$ be the line coordinates representing the line

$$u_1x_1 + u_2x_2 + u_3x_3 = 0,$$

then $[c_1y_3:c_2y_2:c_1y_1]$ are coordinates of the line (2).

It is easy to show that the equation of the conic (1) in line coordinates is

$$(3) \qquad 2c_2u_1u_3 + c_1u_2^2 = 0.$$

Let $[a_1:a_2:a_3]$ be the line AB, where we have

$$(4) \qquad 2c_2a_1a_3 + c_1a_2^2 = 0$$

since AB is also tangent to the conic, and let P be the point $(z_1:0:z_3)$. Then A and B are the points $(a_2:-a_1:0)$ and $(0:a_3:-a_2)$. The coordinates of AP and BP are $[-a_1z_3:-a_2z_3:a_1z_1]$ and $[a_3z_3:-a_2z_1:-a_3z_1]$ respectively. Then the points X and Y are $(0:a_1z_1:a_2z_3)$ and $(a_2z_1:a_3z_3:0)$ whence the coordinates of XY are $[a_3z_3^2 : -a_2z_1z_3:a_1z_1^2]$. Using equation (4) we see that the coordinates of XY satisfy equation (3), from which we see that the envelope of XY is the original conic.

A Note by the Editors. The theorem of this problem is the converse of that of 3458 [1930, 508], a proof of which is given [1931, 350] by analytical methods and

also by the use of Brianchon's theorem. The truth of one theorem follows easily from that of the other.

Another synthetic proof is as follows. As P moves along DE, the range of points X generated on BC is projective with the range Y on AC. Hence XY envelopes a conic. When P is at D, we have AC tangent to the envelope at D, and similarly BC is tangent to it at E. When P is at G, the intersection of DE with AB, we have AB tangent to the envelope. This suffices to show that the envelope is the original conic. The point of tangency F of AB with the envelope is easily seen to be the harmonic conjugate of G with respect to A, B.

Also solved by Rufus Crane, R. Goormaghtigh, J. D. Hill, A. Pelletier, A. W. Randall, Wallace Smith, H. E. Stelson, G. A. Yanosik, F. L. Wren, Kam-cheung Woo, and the Proposer.

3474 [1931, 51]. *Proposed by R. E. Gaines, University of Richmond, Va.*

From a point P on the ellipse $b^2x^2+a^2y^2=a^2b^2$ two chords are drawn, one through $(ae, 0)$ and the other through $(-a, 0)$; for what value of e and what position of P will these two chords trisect the area of the ellipse?

Solution by A. Pelletier, Montreal, Canada

Let O be the center of the ellipse; $A'A$, its major axis; PA' and PC, the required chords, where PC passes through the focus F on OA. Consider the circle on $A'A$ as a diameter which projects orthogonally into the ellipse. Then the points P' and C' on the circle which project into P and C must be such that $P'A'$ and $P'C'$ trisect the area of the circle. The angle $P'OA=\alpha$ is the eccentric angle of the required point P. The area enclosed by the arc $P'A$ of the circle and its two chords AA', $A'P'$ is equal to $\frac{1}{2}\pi a^2-\frac{1}{3}\pi a^2=\frac{1}{6}\pi a^2$. It is also equal to $\frac{1}{2}a^2\alpha+\frac{1}{2}a^2\sin\alpha$, and hence

$$(1)\qquad \alpha+\sin\alpha=\frac{\pi}{3}\cdot$$

The angle α is approximately .53627 or $30°43'33''$. The coordinates of P are $(a\cos\alpha, b\sin\alpha)$.

Obviously $A'P'=C'P'$, and hence $P'O$ bisects the angle $A'P'C'$. Then $\angle OA'P'=\angle OP'A'=\angle OP'F=\frac{1}{2}\alpha$; also $\angle AFP'=3\alpha/2$. Hence

$$(2)\qquad e=\frac{OF}{OA'}=\frac{P'F}{P'A'}=\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)}=\frac{1}{1+2\cos\alpha}=.36775 \text{ approximately.}$$

Also solved by R. Goormaghtigh, O. J. Ramler, F. L. Wilmer, and the Proposer.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

Send all reports of club activities to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

The Pi Mu Epsilon Mathematical Fraternity

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1930–1931

Alpha of Iowa, Iowa State College

The officers for 1930–1931 were: Frances L. Fish, Director; Richard Apple, Vice Director; Donald Paul Needham, Treasurer; Pauline Evarts, Secretary; E. C. McCracken, Instructor in Physics, Librarian; Professor D. L. Holl, Faculty Advisor; Dr. E. R. Smith, Head of the Department of Mathematics, Permanent Secretary.

We now have sixty five active members. We had two initiations the past year, one on November 25th, at which time we took in ten new members, and the other on May 12th, when we initiated ten.

At the annual college Honor's Day in the fall, 1930, we awarded a check of $25 and a certificate to Henry Dale Bossert, a junior architectural engineering student. We give this prize each year to the student having the highest scholastic average who has completed the course in calculus. He is also taken into Pi Mu Epsilon.

The meetings and programs were as follows:

October 28, 1930: "Analytic solutions of differential equations" by Dr. Holl.

November 25, 1930: Initiation and banquet.

December 10, 1930: "Discontinuous functions" by Burnette Backhaus; "An existence theorem" by Robert Cochran.

January 28, 1931: "The use of mathematics in other sciences" by Miss Colpitts; "Review of the life and works of Steinmetz" by George Gross.

February 19, 1931: "Projective Geometry" by Mr. Jean Hempstead.

March 11, 1931: "Expressions of an integral in terms of some differential operators" by Mr. L. F. Robertson.

April 22, 1931: "A study of the value of i^i and its historical significance" by Bernice Brown.

May 12, 1931: Initiation and banquet.

June 4, 1931: "Mathematical derivation of the formula describing the motion of Foucault's pendulum" by Charles Wells.

FRANCES L. FISH, *Director*

Pi Mu Epsilon of the University of Montana

The Montana chapter of Pi Mu Epsilon has been enjoying a successful year. Meetings have been held regularly once a month.

The following officers for the year 1930–1931 were elected on June 4, 1930: Paul Treichler,

Director; Albert Besancon, Vice Director; Elsie Magnuson, Secretary; Professor G. D. Shallenberger, Treasurer.

Initiation for the new members was held at the annual banquet on March 17, 1931. The following took the pledge of membership: Kathryn Coe, '33; Emma Bravo, '33; Horace Worden, '33; Joe Lasby, '33; Franklin Long, '32; Gene Sunderlin, '32; William White, '30; C. Smith, '28.

There are now thirty active members. As a partial requirement for membership, certain papers have been read before Pi Mu Epsilon.

The meetings and programs were as follows:

November 12, 1930: "De Moivre's theorem" by Franklin Long.
November 19, 1930: "Table of integrals" by William White.
December 3, 1930: "Higher plane curves" by Kathryn Coe.
December 10, 1930: "History of calculus" by Professor N. J. Lennes.
January 14, 1931: "Rotation of axes" by Horace Worden.
January 28, 1931: "Historical development of plane analytic geometry" by Emma Bravo.
February 4, 1931: "Sets of axioms" by William White.
February 11, 1931: "Cubic and biquadratic" by Dex Chevalier.
February 18, 1931: "Fictions in thought" by Burr Lennes (a report of *Philosophy as If* by Vathinger).
February 25, 1931: "Quadratic functions in two variables," "Discriminant," by M. Monaco.
March 11, 1931: "Review of *Pastures of Wonder* by Keyser" by Gene Sunderlin.
April 8, 1931: "Involutes and evolutes" by Dick Thomas.
May 20, 1931: "Probability" by C. Smith.

The annual banquet was held on March 17, the birthday of the Montana chapter. A picnic was held on April 26.

ELSIE MAGNUSON, *Secretary*

Pi Mu Epsilon of the University of Kentucky

The officers of the Kentucky Alpha chapter of Pi Mu Epsilon for the year 1930–1931 were as follows: Professor H. H. Downing, Director; Assistant Professor M. C. Brown, Vice Director; Assistant Professor D. E. South, Treasurer; Associate Professor F. E. LeStourgeon, Secretary; Instructor Sallie Pence, Librarian.

The chapter has twenty-seven active members. Four new members were taken in during the year. Initiation of new members was held on December 18, and again on May 7. During the year there were, besides the eight regular meetings at which papers were read, three business meetings, and a banquet on the evening of May 7 in honor of the initiates.

The meetings and programs were as follows:

October 23, 1930: "A formula in analytic geometry" by Professor C. G. Latimer.
November 20, 1930: "Curvature of Einstein space" by Professor E. L. Rees.
February 19, 1931: "Discontinuous functions" by Professor F. E. LeStourgeon.
February 26, 1931: "Cosines of multiple angles in terms of cosines of the simple angles" by Professor H. H. Downing.
March 26, 1931: "Contingent functions" by Professor D. E. South.
April 15, 1931. "The course of deep bore holes" by Professor O. T. Koppius of the department of physics.
April 30, 1931: "Newton's quadratures of curves" by Mr. R. C. Bullock.
May 7, 1931: "Algebraic geometry (Something of its methods and results)" by Professor Mayme I. Logsdon of the University of Chicago.

ELIZABETH LESTOURGEON, *Secretary*

Alpha of Oklahoma, University of Oklahoma

This Alpha Chapter of Pi Mu Epsilon reports the following officers for the year 1930–1931 who have held office since May, 1930: Mr. Earl LaFon, Director; Mr. John C. Brixey, Vice Di-

rector; Miss Dora McFarland, Secretary; Mr. Charles C. Edmondson, Treasurer; Professor J. O. Hassler, Librarian.

Of the thirty five members on the roll of this chapter, fourteen are faculty members, ten are graduate students in mathematics or natural science, nine are seniors, and two are juniors. Two elections were held during the year, at which sixteen new members were admitted; six in November and ten in April. Each time an initiation dinner was held at the Faculty Club. These dinners took the place of the regular meetings and were the only social meetings of the year.

Regular meetings were held on the second and fourth Thursdays of each month. During the year the following papers and discussions were given:

October 23, 1930: "The three-cusped hypocycloid" by Mr. Springer.

November 13, 1930: "Topics in the theory of numbers" by Mr. Ransbarger, Mr. Edmondson, Mr. LaFon.

December 1, 1930: "A historical discussion of the quadratic, cubic, and quartic equations" by Mr. Dorsett.

January 8, 1931: "Certain canonical forms and their properties for quadratic, cubic, and quartic equations" by Dr. Elsie McFarland.

February 26, 1931: "Topics from continued fractions" by Mr. John Brixey.

March 1, 1931: "Tangential equations" by Mr. Springer.

March 26, 1931; "Fallacies in recent attempts to trisect an angle" by Mr. Guthrie.

April 9, 1931: "Some differential geometry by vector analysis" by Mr. LaFon.

April 23, 1931: "On the nature of projective differential geometry" by Dr. Hassler.

DORA MCFARLAND, *Secretary*

Beta of Missouri, Washington University

The Missouri Beta chapter reports a very successful year under the following officers elected at the meeting held on May 17, 1930:

Professor Edmond Siroky, Director; Elizabeth Harris, Southwestern Bell Telephone Co., Vice Director; Professor Jessica M. Young, Secretary; Harold K. Crowder, Assistant Secretary; Instructor Bayard R. Brick, Treasurer; Ford Pennell, Librarian.

We also have an executive committee. The student members of this committee were: Philip Mills Arnold, Lillian Hoagland, Roland E. Miller, Helen Stammer.

There were fifty three active members during the year 1930–1931. Thirty new members were initiated on April 25, 1931, distributed as follows: from the College of Liberal Arts—eight; Schools of Engineering and Architecture—twenty; School of Graduate Studies—two.

The ten meetings and programs held during the year were as follows:

October 9, 1930: "Old text books in our library" by Philip M. Arnold; "Mathematical puzzles" by John C. Lebens.

November 4, 1930: "The algebra of logic" by Charles O. Quade; "Vector algebra" by Richard Torrence.

December 8, 1930: "Diffraction patterns" by Charlotte Wieghard; "The fourth dimension" by Lillian Hoagland.

January 8, 1931: "The Arabians and mathematics" by Rosella Dodt; "Mechanical integration" by William E. Stephens. (This paper was illustrated with models.) Mr. Stephens's paper was the prize paper for the year.

February 10, 1931: "Generating functions" by Professor Otto Dunkel; "The empty column" by Walter W. Schmidt.

March 9, 1931: "Some mathematical recreations" by F. Richard Singer; "Balmer series" by Jessie Best.

March 18, 1931: Passing of amendment to by-laws raising the initiation fee to $5.00. Election of new members.

April 13, 1931: (Special meeting.) An open meeting held in conjunction with the Washington University Chapter of the Society of the Sigma Xi. Professor J. A. Schouten, of the Delft

Technical School in Holland, gave a popular lecture on "Generalized idea of parallelism and applications to modern geometry and physics." About 200 persons attended the lecture and the reception following. Four guests, including Director General Ingold, honored us by coming down from the Missouri Alpha Chapter.

April 25, 1931: Initiation and annual banquet; "Some theorems related to the incenters and excenters of triangles" by Instructor Pearl Colby Miller.

May 16, 1931: Business meeting and social gathering. Award of a prize of $10.00 to William E. Stephens for the paper, presented before the chapter during the year, that was of most general interest to student members. Treasurer's report. Election of officers and student members of executive committee for the year 1931–1932. Refreshments and a social hour were enjoyed following each meeting. The students showed a marked interest in the meetings throughout the year. The program committee (Professor Eugene Stephens, Chairman) should be congratulated on having so many papers presented by students.

J. M. YOUNG, *Secretary*

Alpha of Oregon, The University of Oregon

The officers for 1930–1931 were: Mildred M. Wharton, Director; Arthur Johnson, Vice Director; Helen Elliott, Secretary; Robert Holmquist, Treasurer; Kenneth Kienzle, Executive Chairman.

The officers were elected May 28, 1930 by secret ballot. The organization had thirty seven active members in 1930–1931. Three members of the mathematics faculty and nine graduate assistants were included in that number.

The primary aim of Pi Mu Epsilon has become the aim of our organization, namely: "scholarship for the individuals in the organization in all subjects and particularly in mathematics."

A student to be elected to membership must be in the upper half of his class in general scholarship, and in the upper third in mathematical scholarship. He must have completed at least two terms of differential and integral calculus and be taking his final term of the subject. Members of the mathematical faculty, major and minor graduate students in mathematics and major and minor undergraduate students in mathematics subject to the above requirements are eligible.

The meetings and programs were as follows.

November 24, 1930 "Mathematics and music" by Kenneth Kienzle.

January 12, 1931: "Discussion and presentation of the mathematical principles involved in the use and working of a calculating machine" by Edna Keepers.

February 9, 1931: "Presentation and discussion of the basis, history and properties of non-Euclidean Geometry" by Dr. R. R. Davis.

April 6, 1931: "Mathematics and its relation to biological growth" by George Schlesser.

On October 30, 1930 and on May 14, 1931, the club held business meetings. On May 19, 1931, the Oregon Alpha chapter of Pi Mu Epsilon was installed.

Throughout the year our attention has been centered about our petition to the national honorary fraternity. In April we received word from Professor Louis Ingold of the University of Missouri of the granting of our charter; and on May 19, twenty-nine charter members were installed in the Oregon Alpha chapter. At that time, twenty-one new members, graduate assistants and faculty members were initiated also. The President of our University, Dr. Arnold Bennett Hall, addressed us at a banquet following our ceremony.

The year will be closed with a picnic for all new and old members.

MILDRED M. WHARTON, *Director*

B.

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Boston University

The mathematics club of Boston University aims to promote good fellowship among the students interested in mathematics, to encourage the study of mathematics in the University, and to

discuss the more interesting aspects of mathematics. Any person interested in higher mathematics is eligible for membership. The club has about thirty active members.

The officers are students in the College of Liberal Arts who are majoring or minoring in mathematics. Elections are held at the April meeting by ballot and by majority vote. The officers for the year 1930–1931 were: Alden P. Cleaves, President; Muriel M. Sutherland, Vice President; Marion L. Goodwin, Secretary; Samuel Feldman, Assistant Secretary; Warren C. Dean, Treasurer; Ruth H. Schelin, Assistant Treasurer; Carleton H. Foss, Gladys E. Knowlton, Lucien B. Taylor, Executive Committee.

Regular meetings are held the first and third Thursday noons of each month. The speakers at these meetings are usually students.

The meetings and programs were as follows:
October 2, 1930: "Mathematics in pure economics" by Professor Robert E. Bruce.
October 16, 1930: "Mathematics in nature" by Ruth Schelin.
November 6, 1930: "Properties of parabolas geometrically" by Gladys Knowlton.
November 20, 1930: "Mathematics of Thales" by Warren Dean.
December 4, 1930: "Life of Euler" by Doris Atkinson.
January 8, 1931: "Projective ornaments" by Ora Park.
February 19, 1931: "Geometrical fallacies" by Thomas Homkowycz.
March 5, 1931: "Life of Descartes" by Francis Blackwell.
March 19, 1931: "Paper folding" by Carleton Foss.
April 16, 1931: "Living mathematics" by Eleanor Johnson.
April 30, 1931: "The mathematics of field artillery" by Professor Warren O. Ault.

The annual five dollar prize for the best paper was awarded to Thomas Homkowycz. Honorable mention was given to Carleton Foss and Gladys Knowlton.

Each semester the club holds a supper, social, and theatre party. The first was held on Wednesday, December 10. The entertainment, following the supper, consisted of music, card tricks, and a talk by Professor Bruce about his trip to Angcor, the ancient capital of Cambodia (illustrated with slides). The play, "Street Scene" concluded the evenings program. At the second party, held on Tuesday, April 7, Professor Bruce spoke on "Early numerals in India" and exhibited several snapshots which he had taken. Miss Mildred B. Mitten, '24, one of the charter members of the club, was present and told us a little of the early activities of the club.

The club affords an excellent opportunity for informal acquaintance between students and teachers.

MARION L. GOODWIN, *Secretary*

The Mathematics Club of the New Jersey College for Women

The officers for 1930–1931 were: Beatrice Velten, President; Louise Killheffer, Vice President; Helen Carpenter, Secretary; Marian Bruen, Treasurer. These officers were nominated by a nominating committee consisting of the officers of the preceding year and voted upon by the members of the club at a regular meeting in May 1930.

The aim of the club is to stimulate interest in the field of mathematics and to afford opportunities for investigation. Membership is limited to those who have taken or are taking a course in calculus. Our present membership is thirty-five.

The formal papers for the year were: "Modern discoveries in geometry with respect to the orthopole" by Dr. Richard Morris, March, 1931; "Graphical methods in analysis" by Mary Crandall, May, 1931; "Power series" by Beatrice Velten, May, 1931.

The topics discussed by student members were: "Babylonian mathematics" by Lucy Macaluso, October, 1930; "Egyptian mathematics" by Louise Killheffer, October, 1930; "Greek mathematics" by Marie Kozesnick, October, 1930; "Preservation of mathematics by Hindus and Arabians" by Doris Swain, November, 1930; "Apollonius, Euclid, and Archimedes" by Helen Shack, November, 1930; "Beginnings of analytic geometry" by Verna Wilson, November, 1930; "Leibnitz's and Newton's controversy on calculus, by Caroline Jung, November, 1930; "Modern

geometry" by Lillian Solomon, November, 1930; "Archimedes' proof of the area of a segment of a parabola" by Beatrice Velten, January, 1931; "Determination of an angle of a right triangle without the use of tables" by Marian Bruen, February, 1931; "How to inscribe a rectangle within another rectangle" by Margaret Thompson, February, 1931; "Proof of Morley's theorem relating to a triangle" by Janet Mather and Virginia Stevens, March, 1931; "Proofs and verification of some area formulas of the triangle" by Edna Schnitzler, Mae Blydenburgh, Irene Stevens, Alice Maier, Helen Shack, Margaret McCallum, March and April, 1931; "Groups" by Marian Bruen, April, 1931; "Applications of groups to trigonometry" by Irene Stevens, April, 1931.

The other activities of the club were: A mathematics club tea in October, 1930; a Christmas party in December, 1930; a mathematics club social and dance in May, 1931; and joint dinners of the two mathematics clubs in December, 1930 and April, 1931.

We also had lectures by visiting speakers. The speakers and the topics discussed were: Professor R. C. Archibald of Brown University spoke on "Mathematics prior to the Greeks" in December, 1930; Dr. S. A. Schelkunoff of the Bell Telephone Laboratories spoke on "Telephony and the application of mathematics to some of its problems" in April, 1931.

HELEN H. CARPENTER, *Secretary*

The Mathematics Club of the University of Buffalo

This club was organized in September, 1929 for the purpose of stimulating an interest in the history and problems of mathematics. All students majoring in mathematics and others interested are eligible for membership in the club. We had twenty-five members. Miss Margaret Morgan was our president.

The meetings and programs were as follows:

October 1930: Organization and election of officers; "Review of 'Flatland' by A. Square" by Miss Montague.

November 1930: "Squaring the circle" by Miss Agnes Higgins; "Magic squares" by Dr. Gehman.

February 1931: "Short cuts in computation" by Miss Montague and Professor Harrington.

April 1931: "The fourth dimension" by Mr. John Greenwood of Technical High School.

HARRIET F. MONTAGUE

The Mathematics Club of Wellesley College

The aim of the mathematics club for the year 1930–1931 has been no different from that of other years—that is, it has been to bring Wellesley's students of mathematics into more intimate contact socially pleasurable and instructively valuable. Membership in the club is open to all students of mathematics with the exception of Freshmen. This year the active members number sixty-eight.

The officers for this year, elected by ballot last May 20, 1930 at the last meeting of the club for that year, were:

Melita Holly, President; Katherine Atwood, Vice President; Virginia Francis, Treasurer; Emily Neal, Secretary; Claudia Jessup, Junior Executive; Miss Lennie P. Copeland, Faculty Advisor.

Unfortunately, the president was forced to resign after Christmas because of ill health. Since then, Katherine Atwood has been acting president.

The year's programs have been interesting and varied. In the course of the meetings, a number of topics have been discussed by student members of the club.

The meetings and programs were as follows:

October 17, 1930: "Mathematics and botany" (illustrated with drawings) by Barbara Little; "Probability" by Frances Fletcher; "Numbers, an explanation of a system based on twelve instead of ten" by Barbara Trask.

November 14, 1930: "Number systems" by Esther Van Artsdalen; "Codes and ciphers" by Barbara Bicknell; "Zeno's paradoxes" by Claudia Jessup; "The role of '*e*' in Physics" by Miss Gabriel Asset, Assistant in the Department of Physics.

December 5, 1930: The program was presented by Miss Helen Merrill and Miss Marion Stark, professors of mathematics here, who entertained the club with false mathematical proofs, etc., and ended their program with Miss Stark's reading some mathematical poetry; an article, "Some excitement in mathematics" by Stephen Leacock; and the story of A, B, and C—"The human element in mathematics."

The club ended its year's work on April 17, 1931 with the presentation of a play—"Modern mathematics looks up his ancestors"—written in verse by Miss Marion Stark. To the play were invited not only members of the club but all in the College interested in mathematics.

EMILY NEAL, *Secretary*

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Rhind Mathematical Papyrus is a magnificent work and should be in every college library. The edition is absolutely limited and no more will be available when this edition is exhausted. The price of $15.00 to members of the Association and $20.00 to non-members has been found inadequate to cover the expense of storage, commissions, postage, etc. Therefore, after January 1, 1932, the price to members will be $20.00 and to non-members $25.00, postage prepaid. Orders received before that date will be filled at the old prices. Members, individual and institutional, should order directly through Secretary W. D. Cairns, Oberlin College, Oberlin, Ohio. Non-members should order through the Open Court Publishing Co., 339 East Chicago Avenue, Chicago, Illinois.

Professor W. B. Carver, of Cornell University, will be the editor-in-chief of this Monthly beginning with the January, 1932 issue. From now on all manuscripts submitted for publication should be sent to him at White Hall, Cornell University, Ithaca, N. Y.

Assistant Professor C. T. Bumer, of the Ohio State University, has been appointed assistant professor of mathematics at Kenyon College.

Assistant Professor Evelyn T. Carroll, of Wells College, has been promoted to an associate professorship of mathematics.

Associate Professor H. H. Downing, of the University of Kentucky, has been promoted to a professorship of mathematics.

Dr. W. W. Flexner has been appointed lecturer in mathematics at Bryn Mawr College.

Dr. H. C. Gossard, a charter member of the Mathematical Association, was inaugurated as president of the New Mexico Normal University on October 8, 1931.

Dr. Deborah M. Hickey has been appointed professor of mathematics at the Mississippi Delta State Teachers College.

Mr. E. H. Hildebrandt has been appointed assistant professor of mathematics at De Pauw University.

Professor Temple R. Hollcroft, of Wells College, will spend his sabbatical year, 1931–32, in travel and study abroad.

Assistant Professor W. G. Hubert has been promoted to an associate professorship at the College of the City of New York.

Mr. Lonnie Langston has been promoted to an assistant professorship of mathematics at the Texas Technological College.

Associate Professor H. F. MacNeish, of the College of the City of New York, has been transferred to Brooklyn College.

Dr. Gertrude I. McCain, of East Radford, Virginia, has been appointed professor of mathematics at Marymount College, Salina, Kansas.

Dr. J. E. Merrill has been appointed assistant professor of astronomy at the University of Illinois.

Dr. A. K. Mitchell, of Yale University, has been appointed to an assistant professorship at Trinity College, Hartford.

Professor U. G. Mitchell, of the University of Kansas, has been appointed head of the department of mathematics, succeeding Professor C. H. Ashton who asked to be relieved of administrative duties because of ill health.

Assistant Professor L. T. Moore, of Yale University, has been appointed assistant professor at Brooklyn College of the City of New York.

E. J. Oglesby, of Washington Square College, New York, has been appointed professor of engineering mathematics in the University of Virginia.

Dr. Anna Pell-Wheeler, will resume her position as head of the department of mathematics at Bryn Mawr College in the fall of 1931.

Assistant Professor G. A. Pfeiffer, of Columbia University, has been promoted to an associate professorship of mathematics.

Z. M. Pirenian, instructor at Alabama Polytechnic Institute, has been appointed assistant professor at the University of Florida.

Professor C. L. Poor, of the department of celestial mechanics of Columbia University, has retired.

Associate Professor J. F. Ritt, of Columbia University, has been promoted to a professorship of mathematics.

Dr. Hazel E. Schoonmaker has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Dr. G. W. Starcher, of Ohio University, has been promoted to an assistant professorship of mathematics.

Assistant Professor J. S. Taylor, of the University of Pittsburgh, has been promoted to a professorship of mathematics.

Assistant Professor V. B. Teach, of the Armour Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. Earl Thompson has been promoted to an associate professorship of the teaching of mathematics at the Texas Technological College.

Associate Professor Bird M. Turner, of West Virginia University, has been promoted to a professorship of mathematics.

Dr. R. S. Underwood has been promoted to a professorship of mathematics at the Texas Technological College.

Dr. A. Marie Whelan has been promoted to an assistant professorship of mathematics at Hunter College.

Dr. S. D. Zeldin has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

The following appointments to instructorships in mathematics are announced:

Brooklyn College, Miss Margaret M. Young.
Brown University, Mr. Max Astrachan.
University of Chicago, Dr. C. W. Mendel, Dr. W. T. Reid.
Columbia University, Dr. A. C. Berry.
Cornell University, Mr. A. H. Black.
Hunter College, Miss Polly P. Nelson.
University of Kansas, Miss Eula Johnson, Miss Winona Venard.
Long Island University, Mr. K. G. Fuller.
Marquette University, Mr. Stephen Lewandowski.
Ohio State University, Dr. Laurens Earle Bush.
Princeton University, Mr. J. L. Vanderslice.
Stanford University, Mr. H. M. Bacon.
Trinity College, Mr. W. H. Mitchell.
Wells College, Miss Nancy Cole.

Professor A. A. Michelson, of the University of Chicago, the distinguished physicist, died May 9, 1931, at the age of seventy-eight.

Dr. Bessie I. Miller, formerly professor of mathematics at Rockford College, and a charter member of the Association, died February 4, 1931.

Professor Claude Irwin Palmer, head of the Department of Mathematics and Dean of the Students at the Armour Institute of Technology, died April 9, 1931, at the age of sixty. Professor Palmer was a charter member of the Mathematical Association of America.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars,

to be known as the .... Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: .... Signature ....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

LIBRARY CATALOG

VOLUME XXXVIII, 1931

NUMBER 8, OCTOBER

PART II

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

PREFATORY NOTE

This Catalog is published for the information of the members of the Association. Many are so situated that they do not have easy access to certain books and periodicals which are available in the Association Library; and they are invited to make free use of the available material, under the liberal rules which are printed below. Small though the Library is as yet, it will be found to contain many items that have significance for persons interested in collegiate mathematics.

The arrangement of this Catalog follows that of the Catalog of the American Mathematical Society, there being two sections, the first containing titles of Periodicals and Publications of Governments, Learned Societies, Educational Institutions, etc.; and the second section containing the titles of other books and pamphlets. The first group is arranged according to the *Union List of Serials*, Gregory, 1927 (sponsored by the American Library Association).

LIBRARY RULES

1. Applications for books may be made in person or by letter to W. D. CAIRNS, Secretary-Treasurer, 33 Peters Hall, Oberlin, Ohio. To these applicants books will be sent by express at the expense of the borrower.

2. A book may be kept four weeks from the date of leaving the library, but the loan may generally be renewed by writing to the Secretary-Treasurer in advance of the due date.

3. Borrowers should return books in person or by prepaid express.

I. PERIODICALS, GOVERNMENT PUBLICATIONS, AND PUBLICATIONS OF SOCIETIES, ACADEMIES, UNIVERSITIES, TECHNICAL SCHOOLS

R. ACCADEMIA NAZIONALE DEI LINCEI, Rome.
Atti, s6, v1, n9+, 1924+ (v7 n1, v9 n12 missing)
ACTA LITTERARUM AC SCIENTIARUM. *See* SZEGED. TUDOMÁNY-EGYETEM, Acta.
ACTA MATHEMATICA, Stockholm. 17+, 1893+
AMERICAN JOURNAL OF MATHEMATICS, Baltimore. 38 n3+, 1916+ (v39 n3 missing)
AMERICAN MATHEMATICAL MONTHLY, Oberlin. 1+, 1894+
AMERICAN MATHEMATICAL SOCIETY, New York.
Bulletin, v6+, 1899+ Indexes; 1891–1904, 1904–1914, 1914–1924.
Catalogue of the Library, 1910, 1926.
Colloquium lectures. See Colloquium publications.
Colloquium publications, 1+, 1903+ (1. Boston (1903) fourth colloquium, H. S. WHITE, Linear systems of curves on algebraic surfaces; F. S. WOODS, Forms of non-euclidean space; E. B. VAN VLECK, Selected topics in the theory of divergent series and continued fractions, 1905. 2. New Haven (1906) fifth colloquium, E. H. MOORE, Introduction to a form of general analysis; E. J. WILCZYNSKI, Projective differential geometry; M. MASON, Selected topics in the theory of boundary value problems of differential equations, 1910. 3. Princeton (1909) sixth colloquium, G. A. BLISS, Fundamental existence theorems; E. KASNER, Differential-geometric aspects of dynamics, 1913. 4. Madison (1913) seventh colloquium, L. E. DICKSON, On invariants and the theory of numbers; W. F. OSGOOD, Topics in the theory of functions of several complex variables, 1914. 5. Cambridge (1916) eighth colloquium, G. C. EVANS, Functionals and their applications, selected topics including integral equations; O. VEBLEN, Analysis situs, 1918 and 1922. 6. G. C. EVANS, The Logarithmic potential. Discontinuous Dirichlet and Neumann problems, 1927. 7. E. T. BELL, Algebraic arithmetic, 1927. 8. L. P. EISENHART, Non-Riemannian geometry, 1927. 9. G. D. BIRKHOFF, Dynamical systems, 1927. 10. A. B. COBLE, Algebraic geometry and theta functions, 1929. 11. D. JACKSON, The theory of approximation, 1930. 12. S. LEFSCHETZ, Topology, 1930.)
F. KLEIN, Lectures on mathematics delivered at Evanston in 1893. Republished in 1911.
Mathematical papers read at the International Mathematical Congress at Chicago in 1893. New York, 1896.
Registers, 1916–1918, 1926, 1928.
Transactions, 1+, 1900+
AMERICAN PHILOSOPHICAL SOCIETY, Philadelphia.
Proceedings, 54+, 1915+ (Various numbers missing)
AMERICAN PHYSICAL SOCIETY. Physical Review. *See* PHYSICAL REVIEW.
Physics, *See* PHYSICS.
ANNALI DELLE UNIVERSITA TOSCANE, Pisa. n s v4–10, 1919–1926.
ANNALI DI MATEMATICA PURA ET APPLICATA, Bologna. s4, v1+, 1923+
General index.
ANNALS OF MATHEMATICS, PRINCETON, N. J. s2 v12+, 1910+
ANNUARIO DI CATANIA, *See* CIRCOLO MATEMATICO DI CATANIA. Annuario.
ARCHIVES DE MATHÉMATIQUES PURES ET APPLIQUÉS, Liège. v1 n1, Jl. 1907||
ASSOCIATION DES PROFESSEURS DE MATHÉMATIQUES, Paris.
Bulletin, 1+, 1910+
Suppléments, 1924+

BENARES MATHEMATICAL SOCIETY, Benares, India.
Proceedings, v1+, 1919+ (v10 missing)
BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATEMATICHE, Turin.
18–19, 1916–1917; s2 v1–2, 1918–1921.
BOLLETTINO DE MATEMATICA (Conti). With Sezione Storico-Bibliografia (continuation of Loria's Bolletíno di Bibliografia e Storia della Scienze Mathematiche) Bologna. n s v1+, 1922+
BROWN UNIVERSITY, Providence, R. I.
Bulletin of the Department of Mathematics, n1. (Suggestions for students of mathematics; mathematics and life activities, 1922.)
BRÜNN. MASARYK UNIVERSITÉ. Faculté des Sciences.
Publications, 1921+ (Various numbers missing)
BUENOS AIRES. UNIVERSIDAD NACIONAL. Seminario Matemático Argentino.
Boletin, 1+, 1928+

CALCUTTA MATHEMATICAL SOCIETY.
Bulletin, 9+, 1917+ (v20 n2–4 missing)
CALIFORNIA. UNIVERSITY.
Publications in Mathematics, v1 n1. (H. W. Stager, On numbers which contain no factors of the form $p(kp+1)$, 1921.)

CARUS MATHEMATICAL MONOGRAPHS, 1+, 1925+ (n1, G. A. BLISS, Calculus of variations, 1925; n2, D. R. CURTISS, Analytic functions of a complex variable, 1926; n3. H. L. RIETZ, Mathematical statistics, 1927; n4. J. W. YOUNG, Projective geometry, 1930.)

CARUS MONOGRAPHS. *See* CARUS MATHEMATICAL MONOGRAPHS.

ČASOPIS PRO PĚSTOVÁNÍ MATHEMATIKY A FYSIKY, Prague. 57+, 1927+

CHRISTIAAN HUYGENS, Amsterdam. 1+, 1921+

CIRCOLO MATEMATICO, Palermo.
- Annuario, 1914.
- Rendiconti, 35+, 1913+
- Supplemento, 11–12, 1919–1921; 15, 1926–1928.

CIRCOLO MATEMATICO DI CATANIA, Catania.
- Annuario, 1921–1923.
- *See also* ESERCITAZIONI MATEMATICHE; NOTE E MEMORIE DI MATEMATICA.

COLORADO COLLEGE.
- Publications, 25, 51–52, 104. (25. F. H. LOUD, Notes on the computation of logarithms, 1907. 51–52. FLORIAN CAJORI, A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity, 1910. 104. C. H. SISAM, On non-ruled octic surfaces whose plane sections are elliptic, 1919.)

DEUTSCHE MATHEMATIKER-VEREINIGUNG, Leipzig.
- Jahresbericht, 30+, 1921+ (v38 n1–4, v39 n1–4 missing)

EDINBURGH UNIVERSITY. Mathematical Department.
- Research Papers, n2–3, 7. (n2. PIERRE HUMBERT, On some results concerning integral equations. n3. E. T. WHITTAKER, On the continued fractions which represent functions of Hermite and other functions defined by differential equations. n7. E. L. INCE. On certain theorems in continued fractions equivalent to Riemann's and other transformations of the *P*-function.)

ENGINEERING EDUCATION. *See* JOURNAL OF ENGINEERING EDUCATION.

ENSEIGNEMENT MATHÉMATIQUE, Paris. 18+, 1916+ (v18 n1, v20 n5–6 missing)

ESERCITAZIONI MATEMATICHE (Circolo Matematico) Catania. 1–3, 1921–1923.

FUNDAMENTA MATEMATICAE, Warsaw. 1+, 1920+

GIESSEN. UNIVERSITÄT. Mathematisches Seminar.
- Mitteilungen, 1–15, 1921–1927.

GIORNALE DI MATEMATICA FINANZIARIA, Turin. 1+, 1919+ (v11 n1–5 missing)

GIORNALE DI MATEMATICHE (Battaglini) Naples. 53–56, 1915–1918.

HAMBURG. UNIVERSITÄT. Mathematisches Seminar.
- Abhandlungen, 1+, 1921+

IBERICA, Barcelona. 3+, 1916+ (Various numbers missing)

ILLINOIS. UNIVERSITY.
- Bulletin, v10 n10, v18 n12. (v10 n10. E. H. WILLIAMS, The electron theory of magnetism, 1912. v18 n12. ARNOLD EMCH. Mathematical models, 1920.)

INDIAN MATHEMATICAL SOCIETY, Poona.
- Journal, 7+, 1915+ (v10 n1, v17 n11 missing)
- Register of members, 1925–1926.

INDIANA. UNIVERSITY.
- Studies, 70. (H. T. DAVIS, The present status of integral equations, 1926)

INTERMÉDIARE DES MATHÉMATICIENS, Paris. 20–27, 1913–1920‖ s2 v1+, 1922+ (Various numbers missing)

ISIS, Bruges. 5+, 1923+ (Various numbers missing)

JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK, Berlin. 44+, 1913+

JAPANESE JOURNAL OF MATHEMATICS (National Research Council of Japan) Tokyo. 1+, 1924+

JOHNS HOPKINS UNIVERSITY. American Journal of Mathematics. *See* AMERICAN JOURNAL OF MATHEMATICS.

JOURNAL DE L'ÉCOLE POLYTECHNIQUE. *See* PARIS. ÉCOLE POLYTECHNIQUE. JOURNAL.

JOURNAL DE MATHÉMATIQUES ELÉMENTAIRES (Vuibert) Paris. 40–43, 1916–1919. (Various numbers missing)

JOURNAL OF ENGINEERING EDUCATION. (Society for the Promotion of Engineering Education) Lancaster, Pa. 3–4, 1912–1914; 8+, 1917+

JOURNAL OF MATHEMATICS AND PHYSICS. (Massachusetts Institute of Technology) Cambridge, Mass. 1+, 1921+ (v3 n1–3, v9 n3 missing)

LA PLATA. UNIVERSIDAD NACIONAL. Facultad de Ciencias Fisico-Matemáticas.
- Contribución al Estudio de las Ciencias Fisicas y Matemáticas.
  - Serie Matemática. Included in Publicaciones listed below.
- Publicaciones, 1921+

LENINGRADSKOE FIZIKO-MATEMATICHESKOE OBSHCHESTVO, Leningrad.
Zhurnal, v1 n1–2, 1926–1927.
LONDON. MATHEMATICAL ASSOCIATION. Mathematical Gazette. *See* MATHEMATICAL GAZETTE, London.
LOUISIANA. UNIVERSITY.
Bulletin. (S. T. SANDERS, The mathematical process, and some of its disciplines, 1920.)

MARTIN'S MATHEMATICAL MAGAZINE. *See* MATHEMATICAL MAGAZINE, Erie, Pa.
MARTIN'S MATHEMATICAL VISITOR. *See* MATHEMATICAL VISITOR, Erie, Pa.
MASARYK UNIVERSITÉ. *See* BRÜNN. MASARYK UNIVERSITÉ.
MASSACHUSETTS INSTITUTE OF TECHNOLOGY. Journal of Mathematics and Physics. *See* JOURNAL OF MATHEMATICS AND PHYSICS.
MATEMATISK TIDSSKRIFT, Copenhagen.
A 1+, 1919+
B 1+, 1919+ (1921 A & B n4 missing)
MATHEMATICAL ASSOCIATION, London. Mathematical Gazette. *See* MATHEMATICAL GAZETTE, London.
MATHEMATICAL ASSOCIATION OF AMERICA, Oberlin, Ohio.
American Mathematical Monthly. *See* AMERICAN MATHEMATICAL MONTHLY.
Carus Mathematical Monographs. *See* CARUS MATHEMATICAL MONOGRAPHS.
List of Officers and Charter Members, 1916.
Register of Officers and Members, 1919, 1922, 1924, 1926, 1927, 1929.
The National Committee on Mathematical Requirements, The reorganization of mathematics in secondary education, 1923. This work contains investigations by V. BLAIR, A. R. CRATHORNE, J. C. BROWN, R. SCHORLING, C. B. UPTON, R. C. ARCHIBALD, D. E. SMITH, and J. A. FOBERG.
R. C. ARCHIBALD, Benjamin Peirce, 1809–80, Biographical sketch and bibliography; Reminiscences by C. W. ELIOT, A. L. LOWELL, W. E. BYERLY, and A. B. CHACE.
Rhind Mathematical Papyrus, 1927–1929. Vol. I. Free translation and commentary, by A. B. CHACE and H. P. MANNING. Bibliography of Egyptian mathematics, by R. C. ARCHIBALD, pp. 121–206, 1927. Vol. II. Photographs, transcription, transliteration, literal translation, by A. B. CHACE, L. S. BULL, and H. P. MANNING. Bibliography of Egyptian and Babylonian mathematics (Supplement), by R. C. ARCHIBALD. 1929.
R. C. ARCHIBALD, Bibliography of Egyptian mathematics, with special reference to the Rhind Mathematical Papyrus. Reprinted from the Rhind Mathematical Papyrus, Vol. I. 1927.
— Bibliography of Egyptian and Babylonian mathematics, supplement to the bibliography in Vol. I. Reprinted with corrections from Rhind Mathematical Papyrus, Vol. II. 1929.
MATHEMATICAL ASSOCIATION OF JAPAN FOR SECONDARY EDUCATION, Tokyo.
Journal. 1+, 1919+
MATHEMATICAL GAZETTE, London. 7+, 1913+ (Various numbers missing)
MATHEMATICAL MAGAZINE (Artemas Martin) Erie, Pa. 1–2 n12, 1882–1910||
MATHEMATICAL MONTHLY (Runkle) Cambridge, Mass. 1–3, 1858–1861.
MATHEMATICAL VISITOR (Artemas Martin) Erie, Pa. 1–2 n4, 1877–1894||
MATHEMATICS TEACHER, New York. 1+, 1908+ (Various numbers missing)
MATHEMATISCHE ZEITSCHRIFT, Berlin. 1 n1, 1918; 31 n5, 1930.
MATHESIS (La Société Mathématique de Belgique) Brussels. 36+, 1922+ (v40 n8, v42 n2 & 10, v 38 n2 missing)
MATHESIS POLSKA, Warsaw. 5+, 1930+
MICHIGAN. UNIVERSITY.
Studies: Humanistic series, v11. (Contributions to the history of science, Part 1, Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi with an introduction, critical notes and English version by L. C. KARPINSKI, New York, 1915.)
Studies: Scientific series, v2. (W. B. FORD, Studies on divergent series and summability, New York, 1916.)
MISSOURI. UNIVERSITY.
University of Missouri studies, v1 n4. (L. H. MCFARLAN, The transformation of the Euler condition in the calculus of variations; F. O. DUNCAN, An extension of the theory of envelopes, 1926.)
MITTEILUNGEN ZUR GESCHICHTE DER MEDIZIN UND DER NATURWISSENSCHAFTEN, Munich. 22+, 1923+ (bd 22 n3–4 missing)
MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, Vienna. 31+, 1921+
MONIST, Chicago. 24, 1914; 26+, 1916+ (v24 n2–4, v26 n1, v39 n3 missing)

NATIONAL RESEARCH COUNCIL. Department of Education. Japanese Journal of Mathematics. *See* JAPANESE JOURNAL OF MATHEMATICS.
NIEUW ARCHIEF VOOR WISKUNDE. (Wiskundig Genootschap te Amsterdam) Amsterdam. s2 v8+, 1907+ (v10 n3–4 missing)
NIEUW TIJDSCHRIFT VOOR WISKUNDE, Groningen. 9+, 1921+
NIPPON SUGAKU-BUTURIGAKKIWAI, Tokyo.
Kizi, s2 v8–9, 1916–1918; s3 v1+, 1919+ (s2 v8 n1–10, s3 v11 n7 missing)
NORSK MATEMATISK FORENING, Oslo.
Skrifter, 1–10, 1921–1922; 15–17, 1924–1927.

NORSK MATEMATISK TIDSSKRIFT (Norsk Matematisk Forening), Oslo. 1+, 1919+ (v2 n4 missing)
NORTH CAROLINA. UNIVERSITY.
Extension Bulletin, v2 n11. (ARCHIBALD HENDERSON, Relativity, A romance of science, 1923.)
NOTE E MEMORIE DI MATEMATICA. (Circolo Matematico, Catania) Catania. 1 n1–3, 1921.
NOUVELLES ANNALES DE MATHÉMATIQUES, Paris. 77–79, 1918–1920.

OPEN COURT, Chicago. 29+, 1915+ (819, 827, 867, 869, 871, 877 missing)

PARIS. ÉCOLE POLYTECHNIQUE.
Journal, s2 v22–27, 1922–1929.
PARIS. ASSOCIATION DES PROFESSEURS DE MATHÉMATIQUES. *See* ASSOCIATION DES PROFESSEURS DE MATHÉMATIQUES.
PENNSYLVANIA. UNIVERSITY.
Publications; Series in Mathematics, n1. (R. J. ALEY, Contributions to the geometry of the triangle; P. R. HEYL, Properties of the locus r = constant, in space of n dimensions, 1897.)
PREIODICO DI MATEMATICA. *See* PERIODICO DI MATEMATICHE.
PERIODICO DI MATEMATICHE, Bologna. s3 v12–14, 1915–1917|| s4+, 1921+ Supplement v19–20, 1916–1917. (Various numbers missing)
Earlier numbers have title: Periodico di Matematica.
PHYSICAL REVIEW (American Physical Society) Ithaca. 7–9, 1916–1917; 11+, 1918+ (Various numbers missing)
PHYSICS (American Physical Society), Minneapolis. 1+, 1931+
PHYSICO-MATHEMATICAL SOCIETY OF JAPAN. Proceedings. *See* NIPPON SUGAKU-BUTURIGAKKIWAI Tokyo. Kizi.
POPULAR ASTRONOMY, Northfield, Minn. 24–26, 1916–1918; 32+, 1924+ (Various numbers missing)
PRACE MATEMATYCZNO-FIZYCZNE, Warsaw. 31–32, 1920, 1922; 34 pt. 2, 1925–1926.

R. ACCADEMIA NAZIONALE DEI LINCEI, Rome. *See* ACCADEMIA NAZIONALE DEI LINCEI, Rome.
REVISTA DE MATEMATICAS, Buenos Aires. 1–5, 1919–1924||
REVISTA DEL CENTRO ESTUDIANTES, Buenos Aires. 9 n31–32, 1922.
REVISTA MATEMATICA DIN TIMISOARA. Timisoara. 1+, 1921+.
REVISTA MATEMATICA HISPANO-AMERICANA, Madrid. 1–7, 1919–1925, s2, 1+, 1926+
Published under the auspices not only of the Spanish Mathematical Society but also of the Laboratorio y Seminario Matematico.
REVUE DE L'ENSEIGNEMENT DES SCIENCES, Paris. 10, 12, 14, 1916–1920|| (Various numbers missing)
REVUE SEMESTRIELLE DES PUBLICATIONS MATHÉMATIQUES. (Societé Mathématique d'Amsterdam) Amsterdam. 22+, 1914+
ROME. UNIVERSITA. Facoltà di Scienze. Seminario Matematico.
Rendiconti, 1–2, 4–6, 1914–1922; s2 v1+, 1924+
ROYAL ASTRONOMICAL SOCIETY OF CANADA, Toronto.
Journal, 10+, 1916+ (v10 n1–2, v14 n2–7, v16 n10 missing)
ROYAL SOCIETY OF CANADA, Ottawa.
Transactions, s3 v18+, 1924+
ROYAL SOCIETY OF EDINBURGH, Edinburgh.
Proceedings, 40+, 1920+
ROZHLEDY MATEMATICKO-PRIRODOVÊDECKÉ, Prague. 7+, 1927/28+
RUNKLE'S MATHEMATICAL MONTHLY. *See* MATHEMATICAL MONTHLY (Runkle) Cambridge, Mass.

SCHOOL AND SOCIETY, New York. 1+, 1915+ (Various numbers missing)
SCHOOL SCIENCE AND MATHEMATICS, Chicago. 1, 1903; 5+, 1915+
SCIENCE, New York. n s 37+, 1913+ (Various numbers missing)
SCIENCE PROGRESS in the twentieth century, London. 45, 1917; 50–51, 1918–1919; 55, 57–58, 1920; 62–64, 1921–1922; 66+, 1923+
SCIENTIA, Bologna. 34+, 1923+ (v34 n1–6, v39 n1, v47 n3 missing)
SEMINARIO MATEMÁTICO ARGENTINO. *See* BUENOS AIRES. UNIVERSIDAD NACIONAL. Seminario Matemático Argentino.
SENDAI. TOHOKU TEIKOKU DAIGAKU.
Science Reports, s1 v9 n1–2, 1920; s10 n1, 3, 1921.
SOCIEDAD CIENTIFICA ARGENTINA, Buenos Aires.
Catalogo de las Publicaciones Periodicas Existentes en la Biblioteca, 1927.
SOCIEDAD MATEMATICA ESPANOLE, Madrid.
Revista, 2–6, 1912–1917||
Continued as Revista Matematica Hispano-Americana.
LA SOCIÉTÉ MATHÉMATIQUE DE BELGIQUE. Mathesis. *See* MATHESIS (La Société Mathématique de Belgique)
SOCIÉTÉ MATHÉMATIQUE DE GRÈCE, Athens.
Bulletin, 1–9, 1919–1928.
SOCIÉTÉ PHYSICO-MATHÉMATIQUE DE LENINGRADE. *See* LENINGRADSKOE FIZIKO-MATEMATICHESKOE OBSHCHESTVO.
SOCIÉTÉ SCIENTIFIQUE DE BRUXELLES.
Annales, 43+, 1923+ (45 n1 & 46 missing)

SOCIETY FOR THE PROMOTION OF ENGINEERING EDUCATION. Journal of Engineering Education. *See* JOURNAL OF ENGINEERING EDUCATION.
SPHINX-OEDIPE, Nancy. 1–21, 1906–1925. (Various numbers missing)
STRASBOURG. UNIVERSITÉ. Institut de Mathématiques.
Publications, 2+, 1921+
SZEGED. R. UNIVERSITAS HUNGARICA FRANCISCO-JOSEPHINA. *See* SZEGED. TUDOMÂNY-EGYETEM.
SZEGED. TUDOMÂNY-EGYETEM. Acta Litterarum ac Scientiarum.
Sectio Scientiarum Mathematicarum, 1+, 1922+

TEXAS. UNIVERSITY.
Bulletin, 222, 243, 1719, 1905, 2063. (222. E. L. DODD, On ordinary plane and skew curves, 1912; 243. H. Y. BENEDICT and J. W. CALHOUN, The teaching of plane geometry, 1912; 1719. J. W. CALHOUN and C. D. RICE, Texas mathematics teachers bulletin, 1917; 1905. E. L. DODD, Some dangers in establishing a pension system and the proper precautions, 1919; 2063. J. W. CALHOUN and H. J. ETTLINGER, Texas mathematics teachers bulletin, 1920.)
TOHOKU IMPERIAL UNIVERSITY. Science Reports. *See* SENDAI. Tohoku Teikoku Daigaku. Science Reports.
TOHOKU MATHEMATICAL JOURNAL, Sendai. 1+, 1911+ (v1 n1–2 missing)
TOKYO. IMPERIAL UNIVERSITY. Faculty of Science.
Journal. Section 1. Mathematics, astronomy, physics, 1+, 1925+
TOULOUSE. UNIVERSITÉ. Faculté des Sciences de Toulouse.
Annales, s3 v1+, 1909+
TSING HUA JOURNAL, Pekin. 1+, 1916+

UNIONE MATEMATICA ITALIANA.
Bollettino, 1+, 1922+ (v1 n4–5 missing)
U. S. BUREAU OF EDUCATION.
Bulletin, 1911 n3, 6–8, 12–13, 16; 1912 n2, 4, 13–14, 29; 1913 n7; 1914 n45; 1915 n31, 35, 39; 1917 n10, 27; 1921 n6, 21. (1911 n6–8, 12–13, 16; 1912 n2, 4, 13–14. International Commission on the Teaching of Mathematics: American report, Edited by D. E. SMITH, W. F. OSGOOD, and J. W. A. YOUNG; 1911 n3. W. C. RUEDIGER, Agencies for the improvement of teachers in service, 1912 n29. D. E. SMITH and C. GOLDZIHER, Bibliography of the teaching of mathematics, 1913 n7. C. D. KINGSLEY, College entrance requirements; 1914 n45. J. C. BROWN, Curricula in mathematics, a comparison of courses in the countries represented in the International Commission; 1915 n31. A comparative study of the salaries of teachers and school officers; 1915 n35. E. H. TAYLOR, Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International Commission; 1915 n39. I. L. KANDEL, The training of elementary school teachers in mathematics in the countries represented in the International Commission; 1917 n10. W. S. MONROE, Development of arithmetic as a school subject; 1917 n27. R. C. ARCHIBALD, The training of teachers of mathematics for the secondary schools of the countries represented in the International Commission; 1921 n6. G. F. ZOOK and S. P. CAPEN, Opportunities for study at American graduate schools; 1921 n21. G. F. ZOOK, Higher education, 1918–20)
U. S. COAST AND GEODETIC SURVEY.
Special Publications, 52, 57. (52. O. S. ADAMS, General theory of the Lambert conformal conic projection, 1918; 57. General theory of polyconic projections, 1919.)
UNIVERSIDAD DE LA PLATA. *See* LA PLATA. UNIVERSIDAD NACIONAL.
UNTERRICHTSBLÄTTER FÜR MATHEMATIK UND NATURWISSENSCHAFTEN (Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts) Berlin. 24 n5–6; 28+, 1922+ (31 n6, 24 n1 missing)

VEREIN ZUR FÖRDERUNG DES MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHTS. *See* UNTERRICHTSBLÄTTER FÜR MATHEMATIK UND NATURWISSENSCHAFTEN.
VIRGINIA POLYTECHNIC INSTITUTE.
Bulletin. (J. E. WILLIAMS and R. B. H. BEGG, A transition curve, 1923)

WASHINGTON. UNIVERSITY.
Publications in Mathematical and Physical Sciences, v1 n1 (E. T. BELL, An arithmetical theory of certain numerical forms, 1915.)
WIADOMOSCI MATEMATYCZNE, Warsaw. 24–26, 1920–1922.
WISKUNDIG GENOOTSCHAP TE AMSTERDAM: Nieuw Archief voor Wiskunde. *See* NIEUW ARCHIEF VOOR WISKUNDE.
WISKUNDIG GENOOTSCHAP TE AMSTERDAM: Wiskundige Opgaven met de Oplossingen. *See* WISKUNDIGE OPGAVEN MET DE OPLOSSINGEN.
WISKUNDIGE OPGAVEN MET DE OPLOSSINGEN. (Genootschap der mathematische Wetenschappen; Wiskundig Genootschap te Amsterdam) Amsterdam. 10+, 1907+

ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND MECHANIK, Berlin. 1+, 1921+
ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, Leipzig. 45+, 1914+ (48 n5, 9; 58 n8–10 missing)

II. OTHER BOOKS AND PAMPHLETS

ADAMS, H. Engineers Handbook. Philadelphia, 1907.
ADAMS, O. S. See Section I of this Catalog, under U. S. COAST AND GEODETIC SURVEY.
ALEY, R. J. See Section I of this Catalog, under PENNSYLVANIA. UNIVERSITY.
ALTSHILLER-COURT, N. College Geometry. Richmond, 1925.
ANDOYER, H. and LAMBERT, J. H. Cours d'astronomie. Paris, 1924.
Cours de méchanique celeste. Paris, 1923.
ARCHIBALD, R. C. Euclid's book on divisions of figures. Cambridge, 1915.
See also Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA, and U. S. BUREAU OF EDUCATION.
AUERBACH, M. and WALSH, C. B. Plane Geometry. Philadelphia, 1927.

BACHMAN, F. P. See FLEXNER, A.
BAGCHI, H. A course of geometrical analysis. Calcutta, 1926.
BAILEY, F. H. See WOODS, F. S.
BAKER, H. F. Principles of geometry. Vol. 3, solid geometry. Cambridge, 1923.
BALL, W. R. Récréations mathématiques et problèmes. Paris, 1927.
BALLY, E. Géométrie synthétique des unicursales. Paris, 1920.
Géométrie générale synthétique. Paris, 1922.
BARTLETT, D. P. Method of least squares. Boston, 1915.
BARTLETT, J. and others. Algebra review exercises. Boston, 1928.
BAUER, E. La théorie de Bohr. Paris, 1922.
BECK, E. G. Real mathematics. London, 1922.
BEGG, R. B. H. See Section I of this Catalog, under VIRGINIA POLYTECHNIC INSTITUTE.
BELL, E. T. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY, and WASHINGTON. UNIVERSITY
BENEDICT, H. Y. See Section I of this Catalog, under TEXAS. UNIVERSITY.
BENNY, L. B. Mathematics for students of technology. Oxford, 1927.
BERKELEY, L. M. Addition-subtraction logarithms. New York, 1930.
BERNARD, D. M. Plane geometry. Richmond, 1927.
BETZ, WILLIAM. Algebra for today. Boston, 1929.
BIRKHOFF, G. D. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
BIRTWISTLE, G. The principles of thermodynamics. Cambridge, 1925.
BLACKBURN, P. P. See WHITE, C. J.
BLAIR, V. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
BLAKE, E. M. The method of indeterminate coefficients and exponents applied to differential equations. Diss., Columbia; New York, 1893.
BLATTACHARYA, D. Vector calculus. Calcutta, 1920.
BLIGH, N. M. The evolution and development of the quantum theory. New York, 1926.
BLISS, G. A. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY, and CARUS MATHEMATICAL MONOGRAPHS.
BOLTON, L. An introduction to the theory of relativity. New York, 1921.
BOLZA, O. Vorlesungen über Variationsrechnung. Vols. 1–2. Leipzig, 1908, 1909.
BOUSSINES, J. Cours de physique mathématique. Paris, 1922.
BOWDEN, J. Theory of integers. New York, 1903.
BRASCH, F. E. Sir Isaac Newton, 1727–1927. Baltimore, 1928.
BRESLICH, E. R. First year mathematics. Chicago, 1915.
BRINK, R. W. Plane trigonometry. New York, 1928.
BROAD, C. D. Sir Isaac Newton, Annual lecture on a master mind. London, 1927.
BROOKMAN, T. A. The family expense account. Chicago, 1914.
BROWN, J. C. See FINNEY, H. A.
See also Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA, and U. S. BUREAU OF EDUCATION.
BROWNE, R. T. The mystery of space. New York, 1919.
BRUNS, H. Wahrscheinlichkeitsrechnung und Kollektivmasslehre. Leipzig, 1906.
BRYAN, W. L., CARMICHAEL, R. D., MACMILLAN, W. D., HUFFORD, M. E., and DAVIS, H. T. A debate on the theory of relativity. Chicago, 1927.
BUCHANAN, H. E. and EMMONS, L. C. Advanced algebra. Cambridge, Mass. 1925.
BUHL, A. Géométrie et analyse des intégrales doubles. Paris, 1920.
BULL, L. S. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
BURGESS, R. W. The mathematics of statistics. Cambridge, 1927.
BYERLY, W. E. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.

CAHEN, E. Théorie des nombres. Paris, 1914.
CAPEN, S. P. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.

Cairns, W. D. Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen. Diss., Göttingen, 1907.
Cajori, F. The conceptions of limits and fluxions from Newton to Woodhouse. Chicago, 1919.
    History of mathematical notations, Vol. I. Notations in elementary mathematics, Chicago, 1928.
    Are the heavens full . . . Reprint fr. Scientific Mo., Oct. 1926, Vol. 23, pp. 346–355.
    Four old astr. observ. . . . Reprint fr. Scientific Mo., Apr. 1928, Vol. 26, pp. 372–376.
    See also Section I of this Catalog, under Colorado College.
Calhoun, J. W. See Section I of this Catalog, under Texas. University.
Calhoun, J. W., White, E. V., and Simpson, T. McN., Jr. Algebra for junior and senior high schools. Richmond, Va., 1930. Same. Second course. Richmond, Va., 1930.
Cambridge Tracts in Mathematics and Mathematical Physics. See Steward, G. C.
Carey, F. S. and Grace, S. F. Four place tables. London, 1927.
Carmichael, R. D. See Bryan, W. L.
Carmichael, R. D. and Weaver, J. H. The Calculus. Boston, 1927.
Carslaw, H. S. Non-Euclidean geometry and trigonometry. London, 1916.
    Plane trigonometry. London, 1930.
    Fourier's series and integrals. 3rd ed. London, 1930.
Cassirer, E., trans. by Swabey, W. C. and M. C. Substance and function and Einstein's theory of relativity. Chicago, 1923.
Castelnuovo, G. See Enriques, F.
Chace, A. B. See Section I of this Catalog, under Mathematical Association of America.
Chappell, E. Five figure mathematical tables. New York, 1915.
Clark, J. R. See Otis, A. S., Rugg, H. O. and Schorling, R.
    See Schorling, R.
Clark, J. R., Otis, A. S. and Hatton, C. Modern-school arithmetic, Books I and II. Yonkers, 1929.
    First steps in teaching number. Yonkers, 1929.
    Modern-school arithmetic. Book III. Yonkers, 1930.
Cline, I. M. Tropical cyclones. New York, 1926.
Coble, A. B. See Section I of this Catalog, under American Mathematical Society.
Collins, J. V. Practical algebra, second course. New York, 1911.
Colpitts, J. T. Entire functions defined by certain power series. Diss., Cornell; Princeton, 1926.
Colpitts, J. T. and Roberts, M. M. Analytic geometry. New York, 1918; rev. ed., 1926.
Comrie, L. J. See Milne-Thomson, L. M.
Corral, J. I. Nuevas formulas de transformacion de coordenadas. Havana, 1928.
    Cantidades complejas, Tomo I; Relatividad eliptica, Tomo II. Havana, 1929.
    Mecanica No-Newtoniana de tipo eliptico, (2 copies) Madrid, 1929.
Crathorne, A. R. See Section I of this Catalog, under Mathematical Association of America.
Crawley, E. S. and Evans, H. B. Analytic geometry. New York, 1928.
Crenshaw, B. H. and Harkin, D. C. College algebra. (2 copies.) Philadelphia, 1929.
Curtiss, D. R. See Section I of this Catalog, under Carus Mathematical Monographs.

Dalaker, H. H. and Hartig, H. E. The calculus. New York, 1930.
Danmar, W. World cognition, New York, 1923.
Darnell, L. See Lyman, E. A.
Davidson, J. and Pressland, A. J. A second geometry, Oxford, 1926.
Davis, H. T. See Bryan, W. L.
    See also Section I of this Catalog, under Indiana. University.
Dence, C. J. See Rushman, C. E.
Dickson, L. E. Algebras and their arithmetics. Chicago, 1923. (2 copies.)
    See also Section I of this Catalog, under American Mathematical Society.
Dodd, E. L. See Section I of this Catalog, under Texas. University.
Downey, W. F. See Milne, W. J.
Dowsett, J. F. Advanced constructive geometry, Oxford, 1927.
Drumaux, P. L'évidence de la théorie d'Einstein. Paris, 1923.
Duncan, F. O. See Section I of this Catalog, under Missouri. University.
Durell, F. Fundamental sources of efficiency. Philadelphia, 1914.

Eisenhart, L. P. See Section I of this Catalog, under American Mathematical Society.
Eliot, C. W. See Section I of this Catalog, under Mathematical Association of America.
Elson, W. H. See Short, R. L.
Elwell, F. H. See Miner, G. W.
Emch, A. See Section I of this Catalog, under Illinois. University.
Emmons, L. C. See Buchanan, H. E.
Encyklopädie der mathematischen Wissenschaften. Band 2, Teil 3, Heft 4; Band 3, Teil 1, Heft 6, 7; Band 3, Teil 2, Heft 7; Band 3, Teil 3, Heft 5; Band 5, Teil 2, Heft 5. Leipzig, 1920–22.
Engel, F. See Lie, S.
Enriques, F. La storia della logica.
    (ed.) La storia e la filosofia delle matematiche: (1) Gli elementi d'Euclide, Libri I–IV, Rome, 1925; (2) Matematiche, scienze naturali e medicina nell'antichità classica, by L. Heiberg, trans. by

G. Castelnuovo, Rome, 1924; (3) Principii di filosofia naturale, teoria della gravitazione, by I. Newton, trans. by F. Enriques, and U. Forti, Rome, 1925; (4) Il "Metodo" di Archimede, by E. Rufini, Rome, 1926.
ENRIQUES, F.—CHISINI, O. Teoria geometrica delle Equazioni. Vols. 1 & 2. Bologna, 1915, 1918.
EPSTEEN, S. Untersuchungen über lineare Differentialgleichungen 4. Ordnung und die zugehörigen Gruppen. Diss., Zurich, 1901.
ESPERTI, V. Derivazione Grafica. Bari, 1929.
ETTLINGER, H. J. See Section I of this Catalog, under TEXAS. UNIVERSITY.
EVANS, G. C. Mathematical introduction to economics. New York, 1930.
See also Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
EVANS, G. W. and MARSH, J. A. First year mathematics. New York, 1916.
EVANS, H. B. See CRAWLEY, E. S.

FANSHAWE, J. E. J. Education for tolerance. New York, 1928.
FERGUSON, D. F. and PIGOTT, H. E. Areas and volumes. New York.
FINE, H. B. Calculus. New York, 1927.
FINNEY, H. A. and BROWN, J. C. Modern business arithmetic. New York, 1916.
Modern business arithmetic, complete. New York, 1916.
FITZPATRICK, J. Exercises d'arithmétique. Vols. 1–2. Paris, 1914.
FLAMANT, P. Sur une équation différentielle fonctionnelle linéaire. Diss. Strasbourg; Palermo, 1924.
FLEXNER, A. and BACHMAN, F. P. Public education in Maryland. New York, 1916.
FOBERG, J. A. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
FORD, W B. See Section I of this Catalog, under MICHIGAN. UNIVERSITY.
FORSYTH, C. H. Logarithms and anti-logarithms. Ann Arbor, 1915.
Mathematical analysis of statistics. New York, 1924.
FORTI, U. See ENRIQUES, F.
FOWLER, R. H. Statistical mechanics. Cambridge, 1929.
VON FRANTZIUS, F. Book of truth and facts. Chicago, 1916.
FRICKE, R. Differential- und Integralrechnung. 2 vols. Braunschweig. 1905.
FUETER, R. Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen. Berlin, 1927.

GALBRUN, H. Théorie de la relativité. Paris, 1923.
GALE, A. S. and WATKEYS, C. W. Elementary functions. New York, 1920.
GHYKA, M. C. Esthétique des proportions dans la nature et dans les arts. Paris, 1927.
GIBBS, R. W. M. Algebra to the quadratic. Oxford, 1927.
GIFFORD, E. Natural sines. Manchester, 1914.
GODFREY, C. and SIDDONS, A. W. Four figure tables. Cambridge, 1927.
GOGGIN, W. J. and TONER, J. V. Accounting, principles and procedure, with laboratory manual to Part I. Cambridge, Mass., 1930.
GOLDZIHER, C. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
GOURSAT, E. Leçons sur l'intégration des équations. Paris, 1921.
Leçons sur le problème de Pfaff. Paris, 1922.
GRACE, S. F. See CAREY, F. S.
GRAUSTEIN, W. C. Eine reelle Abbildung analytischer komplexer Raumkurven. Diss. Bonn; Leipzig, 1913.
See also OSGOOD, W. F.
GÜELL, VICOMTE, L'espace: la relation et la position. Paris, 1924.

HAERTTER, L. D. Plane geometry. New York, 1929. (2 copies.)
Solid geometry. New York, 1930. (2 copies.)
HALLETT, G. H., Jr. Linear order in three dimensional Euclidean and double elliptic spaces. Diss., Pennsylvania; Lancaster, 1920.
HALSTED, G. B. Euclides vindicatus. Chicago, 1920.
HAMMER, E. Lehr- und Handbuch der ebenen und sphaerischen Trigonometrie. Stuttgart, 1923.
HANCOCK, H. Calculus of variations. Cincinnati, 1904.
HARKIN, D. C. See CRENSHAW, B. H.
HART, W. L. The mathematics of investment. Chicago, 1924.
College algebra, Chicago, 1926.
HART, W. W. Geometry Tests. New York.
See also WELLS, W.
HARTIG, H. E. See DALAKER, H. H.
HASTINGS, C. S. New methods in geometrical optics. New York, 1927.
HATTON, C. See CLARK, J. R.
HAWKES, H. E., LUBY, W. A., and TOUTON, F. C. Second course in algebra. Boston, 1918.
Complete school algebra. Boston, 1919.
Solid geometry. Boston, 1922.
New plane geometry. Boston, 1929.

HAZLETT, O. C. On the classification and invariantive characterization of nilpotent algebras. Diss., Chicago; Baltimore, 1916.
HEATH, T. L. The Copernicus of antiquity. London, 1920. (2 copies.)
HEDRICK, E. R. Constructive geometry. New York, 1916.
HEIBERG, J. L. See ENRIQUES, F.
HENDERSON, A. See Section I of this Catalog, under NORTH CAROLINA. UNIVERSITY.
HEYL, P. R. See Section I of this Catalog, under PENNSYLVANIA. UNIVERSITY.
HOFFMAN, J. V. Grundzüge der Körperlehre. Dresden, 1926.
HORNBROOK, A. R. Laboratory methods of teaching mathematics. New York, 1895.
HOVEY, A. S. The pyramid and solids. Helena, Mont., 1925.
HOWARTH, H. E. See SHAPLEY, H.
HUFFORD, M. E. See BRYAN, W. L.
HUMBERT, P. See Section I of this Catalog, under EDINBURGH UNIVERSITY.
HUNT, B. Community arithmetic. New York, 1916.
HURWITZ, W. A. Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung. Diss., Göttingen, 1910.

INDEX DE RÉPERTOIRE BIBLIOGRAPHIQUE DES SCIENCES MATHÉMATIQUES. 3rd ed. Amsterdam, 1916.
INCE, E. L. See Section I of this Catalog, under EDINBURGH UNIVERSITY.
INSOLERA, F. Corso di matimatica finanziaria. Torino-Genova, 1923.

JACKSON, C. S., WHIPPLE, F. J. W., and ROBERTS, L. A twentieth century arithmetic. London, 1915.
JACKSON, D. Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung. Diss., Göttingen, 1911.
See also Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
JACKSON, L. L. See YOUNG, J. W. A.
JOHNSON, G. The arithmetical philosophy of Nicomachus of Gerasa. Diss., Pennsylvania; Lancaster, 1918.
JOHNSON, R. A. and YOUNG, J. W. Modern Geometry. Boston, 1929.
JONES, S. I. Mathematical wrinkles. Gunter, Texas, 1912.

KANDEL, I. L. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
KARAPETOFF, V. Engineering mathematics, Vols. 2, 3, 4, 5. New York, 1916.
KARPINSKI, L. C. See Section I of this Catalog, under MICHIGAN. UNIVERSITY.
KASNER, E. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
KEAL, H. M., PHELPS, N. S., and LEONARD, C. J. Technical mathematics with tables. Vols. 1, 2 & 3. New York, 1923.
KEMPE, A. Der grosse Fermatsche Satz. Amsterdam, 1913.
KENNELLY, A. E. Fonctions hyperboliques. Paris, 1922.
KEYSER, C. J. The human worth of rigorous thinking. New York, 1916.
Mathematical philosophy. New York, 1922.
Mole philosophy. 1927.
KILLAM, S. D. Über graphische Integration von Funktionen einer komplexen Variabeln mit speziellen Anwendungen. Diss., Göttingen, 1912.
KINGSBURY, H. B. and WALLACE, R. R. Geometry workbook. Milwaukee, 1930.
KINGSLEY, C. D. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
KINGSTON, H. R. Metric properties of nets of plane curves. Diss., Chicago; Baltimore, 1916.
KLEIN, F. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
KLEIN, F. and WOLFF, G. Archimedes. Berlin, 1927.
KNOPP, K. Theory and application of infinite series. London, 1928.
KNOTT, C. G. Napier memorial volume. London, 1915.
KRATHWOHL, W. C. See PALMER, C. I.
KUHN, H. W. and MORRIS, C. C. The mathematics of finance. Cambridge, 1926.

LACAZE, H. Cours de cinématique théorique. Paris, 1920.
LAMBERT, J. H. See ANDOYER, H.
LANDAU, E. Grundlagen der analysis. Leipzig, 1930.
LANG, E. See SPILLMAN, W. J.
LEATHEM, J. G. The mathematical theory of limits. London, 1925.
LEBON, E. Table de caractéristiques de base 30030, Tome I, Fasc. I. Paris, 1920.
LECAT, M. Bibliographie de la Relativité. Bruxelles, 1924.
Théorie des déterminants supérieurs. Louvain, 1926.
LECORNU, L. Cours de méchanique. Vol. 2. Paris, 1915.
LEFSCHETZ, S. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
LEHMER, D. N. Synthetic projective geometry. Boston, 1917. (2 copies.)
LENNES, N. J. A survey course in mathematics. New York, 1926.
See also SUTTON, C. W.
LEONARD, C. J. See KEAL, H. M.
LEVEUGLE, R. Précis de calcul. Paris, 1920.

LIE, S. Gesammelte Abhandlungen. Herausgegeben von dem Norwegeschen Mathematischen Verein durch F. Engel und P. Heegaard. 2 vols. Leipzig, 1922–24.
Gesammelte Abhandlungen. Herausgegeben von F. Engel. Vierter Band, und Anmerkungen. Leipzig, 1929.
LIETZMANN, W. Methodik des mathematischen Unterrichts. III Teil. Leipzig, 1924.
LINDELL, S. A. See SCHORLING, R.
LINEBARGER, C. E. See YOUNG, J. W. A.
LINFIELD, M. B. Espace discret paramëtrique et non paramëtrique. Diss., Strasbourg; Paris, 1925.
LING, G. H., WENTWORTH, G., and SMITH, D. E. Elements of projective geometry. Boston, 1922.
LIPKA, J. Natural families of curves in a general curved space of n-dimensions. Diss., Columbia; Lancaster, 1912.
LOUD, F. H. See Section I of this Catalog, under COLORADO COLLEGE.
LOWELL, A. L. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
LUBY, W. A. See HAWKES, H. E.
LYCHE, M. R. T. Etudes sur l'équation fonctionnelle d'Abel dans le cas des fonctions réelles. Diss., Strasbourg; Palerme, 1927.
LYMAN, E. A. and DARNELL, L. Elementary algebra, first course. Chicago, 1917.
Elementary algebra. Chicago, 1917. (2 copies.)

MACMILLAN, W. D. See BRYAN, W. L.
MALO, E. Sur la décomposition des nombres en leurs facteurs. Nancy, 1920.
MANNING, H. P. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
MARSH, J. A. See EVANS, G. W.
MARTIN, L. A. Textbook of mechanics. Vol. VI, thermodynamics. New York, 1916.
MATHÉMATIQUES. Rapports sur l'enseignment. Bruxelles, 1911.
MASON, M. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
MATHEWSON, L. C. Elementary theory of finite groups. Boston, 1930. (2 copies.)
MCCLENON, R. B. and RUSK, W. J. Introduction to the elementary functions. Boston, 1918.
MCFARLAN, L. H. See Section I of this Catalog, under MISSOURI. UNIVERSITY.
MCLENNON, E. Cosmical evolution. Corvallis, 1916.
MICHEL, F. and POTRON, M. Composition des mathématiques. Paris, 1922.
MIE, G. La théorie Einsteinienne de la gravitation. Paris, 1922.
MILNE, E. A. The aims of mathematical physics. Oxford, Eng., 1929.
MILNE, W. J. and DOWNEY, W. F. First year algebra. New York, 1924.
MILNE, W. J. and WESCOTT, G. J. B. First course in calculus. London, 1918.
MILNE-THOMSON, L. M. and COMRIE, L. J. Standard four-figure mathematical tables, Editions A and B. London, 1931.
MINER, G. W., ELWELL, F. H., and TOUTON, F. C. Business arithmetic. Boston, 1923.
MOLENBROEK, P. Leerboek der stereometrie. Groningen, 1923.
MOLENBROEK, P. and WIJDENES, P. Planimétrie, 1, 2. Groningen, 1924.
MONGE, G. Géométrie descriptive, Vols. 1–2. Paris, 1922.
MONROE, W. S. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
MONTESSUS, R. Exercises et leçons de méchanique. Paris, 1915.
MOORE, F. H. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
MORGAN, F. M. See YOUNG, J. W.
MORRIS, C. C. See KUHN, H. W.
MOULTON, F. R. New methods in exterior ballistics. Chicago, 1926.
MUDGETT, B. D. Statistical tables and graphs. Cambridge, Mass., 1930.
MULLINS, G. W. and SMITH, D. E. Freshman mathematics. Boston, 1927.
MUSCHELISVILI, N. Applications des intégrales. Tiflis, 1922.

NEWTON, I. See ENRIQUES, F.
NATIONAL COMMITTEE OF FIFTEEN, Provisional report on geometry syllabus. Chicago, 1911.
Final report on geometry syllabus. Chicago, 1912.
NYBERG, J. A. First course in algebra. New York, 1924.
Second course in algebra. New York, 1926.

OPPERMAN, A. Prémiers éléments d'une théorie du quadrilatère complet. Paris, 1919.
OSGOOD, W. F. Elementary calculus. New York, 1921.
See also Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY, and U. S. BUREAU OF EDUCATION.
OSGOOD, W. F. and GRAUSTEIN, W. C. Plane and solid analytic geometry. New York, 1921.
OTIS, A. S. Statistical method in educational measurement. New York, 1925.
See also CLARK, J. R.
OTIS, A. S. and CLARK, J. R. Modern plane geometry. New York, 1927.
Modern solid geometry. New York. 1928.

PALMER, C. I. and KRATHWOHL, W. C. Analytic geometry. New York, 1921.
PARSONS, G. L. Elementary integral calculus. Cambridge, 1926.
PASSANO, L. M. Calculus and graphs. New York, 1921.
PETROVITCH, M. Les spectres numeriques. Paris, 1919.
PHELPS, N. S. See KEAL, H. M.
PIERCE, B. Curves, functions, and forces. Boston, 1846.
Algebra. Boston, 1850.
Plane and spherical trigonometry. Boston, 1852.
PIERPONT, J. Functions of a complex variable. Boston, 1914
PIGGOTT, H. E. See FERGUSON, D. F.
PINCHERLE, S. Lezioni di calcolo. Bologna, 1920.
PLASSMANN, J. Fixstern Beobachtungen. Berlin, 1927.
PLESSNER, A. See SCHLESINGER, L.
DE POMPIGNON, A. Note sur le calcul tensoriel. Paris, 1923.
POTRON, M. Exercises de calcul différentiel et intégral. Paris, 1927.
See also MICHEL, F.

RABINOVITCH, I. E. The foundations of the Euclidean geometry as viewed from the standpoint of kinematics. Diss., Johns Hopkins; New York, 1903.
REICHE, F. The quantum theory, trans. by Hatfield, H. S. and Brose, H. L. Rev. ed. Nottingham, Eng. 1930.
REYMOND, A. History of the sciences in Graeco-Roman antiquity. New York, 1927.
RICE, C. D. See Section I of this Catalog, under TEXAS. UNIVERSITY.
RIDER, P. R. The space problem of the calculus of variations in terms of angle. Diss., Yale; Baltimore, 1917.
RIETZ, H. L. (ed.) Handbook of mathematical statistics. Boston, 1924. (2 copies.)
See also Section I of this Catalog, under CARUS MATHEMATICAL MONOGRAPHS.
ROBB, A. A. Time and space. Cambridge, 1921.
ROBBINS, E. R. New plane and solid geometry. New York, 1916.
ROBERTS, L. See JACKSON, C. S.
ROBERTS, M. M. See COLPITTS, J. T.
ROMAN, I. The transmission of waves through a symmetric optical system. Diss., Chicago, 1921.
RORAY, N. L. Industrial arithmetic. Philadelphia, 1916. (2 copies.)
Industrial arithmetic for girls. Philadelphia, 1917.
ROUGIER, L. La matière et l'énergie. Paris, 1921.
ROUTH, E. J. Die Dynamik der Systeme Starrer Körper. Leipzig, 1898.
ROY, M. M. Recherches sur les surfaces portantes en aerodynamique. Diss., Strasbourg; Paris, 1923.
RUEDIGER, W. C. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
RUFINI, E. See ENRIQUES, F.
RUGG, H. O. and CLARK, J. R. Fundamentals of high school mathematics. New York, 1920.
RUSHMAN, C. E. and DENCE, C. J. High school algebra. New York, 1923.
High school algebra, second course. New York, 1926.
RUSK, W. J. See McCLENON, R. B.

SANDERS, S. T. See Section I of this Catalog, under LOUISIANA. UNIVERSITY.
SANDERSON, M. Formal modular invariants with application to binary modular covariants. Diss., Chicago; New York, 1913.
SANDWICK, R. L. How to study. New York, 1915.
SANFORD, V. A short history of mathematics. Cambridge, Mass., 1930.
SARANTOPOULOS, M. S. Analyse Supérieure: Les fonctions croissantes et la théorie des fonctions entières. Algèbre supériere: Sur le calcul numerique d'une catégorie de racines des équations algèbriques. Diss., Strasbourg; Paris, 1923.
SCHLAUCH, W. S. See WENTWORTH, G.
SCHLESINGER, L. Automorphe Funktionen. Berlin, 1924.
SCHLESINGER, L. and PLESSNER, A. Lebesguesche Integrale und Fouriersche Reihen. Berlin, 1926.
SCHORLING, R. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.
SCHORLING, R. and CLARK, J. R. Modern mathematics, 7th & 8th grades. New York, 1924.
Modern algebra, 9th grade. New York, 1924.
Modern mathematics, 7th school year, New ed. 1929.
Modern mathematics, 8th school year, New ed. 1929.
Modern algebra, First course. Yonkers, 1929.
SCHORLING, R., CLARK, J. R., and LINDELL, S. A. Modern algebra, Second course. Yonkers, 1929.
SCHROEDER, R. Differential- und Integralrechnung. Leipzig, 1905.
SCHWARTZ, A. J. See YOUNG, J. W.
SCHWERDT, H. Nomographie. Berlin, 1927.
SEYMOUR, F. E. Plane geometry. New York, 1925.
SHAPLEY, H. and HOWARTH, H. E. A source book in astronomy. New York, 1929.
SHAW, J. B. Freshman algebra. New York, 1929.
SHORT, R. L. and ELSON, W. H. Introduction to mathematics. Boston, 1916.

SIDDONS, A. W. See GODFREY, C.
SILBERSTEIN, L. Elements of vector algebra. London, 1919.
SIMON, M. Analytische Geometrie der Ebene. Leipzig, 1900.
SIMPSON, T. McN., Jr. See CALHOUN, J. C.
SISAM, C. H. See Section I of this Catalog, under COLORADO COLLEGE.
SMAIL, L. L. History and synopsis of the theory of summable infinite processes. Eugene, Ore., 1925.
Mathematics of finance. New York, 1925.
Plane trigonometry. New York, 1926.
College algebra. New York, 1931.
SMITH, D. E. See LING, G. H.; also MULLINS, G. W.; and WENTWORTH, G.
See also Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA, and U.S. BUREAU OF EDUCATION.
SOMMERFELD, A. Three lectures on atomic physics. New York, 1927.
SOPER, H. E. Frequency arrays. Cambridge, 1922.
SPEISER, A. Theorie der Gruppen von endlicher Ordnung. Berlin, 1927.
SPILLMAN, W. J. and LANG, E. Law of diminishing returns. New York, 1924.
STAGER, H. W. A Sylow factor table of the first twelve thousand numbers. Washington, 1916.
See also Section I of this Catalog, under CALIFORNIA. UNIVERSITY.
STEWARD, G. C. The symmetrical optical system. Cambridge, 1928. (Cambridge tracts, No. 25)
STONE, J. C. The teaching of arithmetic. Norwood, Mass., 1918.
STRACHAN, R. C. Table of hyperbolic radians. (Mimeographed.) New York, 1928.
STROHAL, R. Die Grundbegriffe der reinen Geometrie in ihrem Verhältnis zur Anschauung. Berlin, 1925.
SULLIVAN, J. W. H. The history of mathematics in Europe. Oxford, 1925.
SUTTON, C. W. and LENNES, N. J. Brief business arithmetic. Norwood, 1918.
Business arithmetic. Norwood, 1918.

TAYLOR, E. H. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
THOMSON, J. J. Beyond the electron. Cambridge, 1928.
THORNDYKE, L. History of magic and experimental science. 2 vols. New York, 1923.
TONER, J. V. See GOGGIN, W. J.
TOUTON, F. C. See HAWKES, H. E.; also MINER, G. W.
TURNER, B. M. Plane cubics with a given quadrangle of inflexions. Diss., Bryn Mawr; Baltimore, 1922.
TUTTLE, L. Introduction to laboratory physics. Philadelphia, 1915.
TZSCHIRNER, B. Wetterkarte und Wettervorhersage. Berlin, 1927.

UPTON, C. B. See Section I of this Catalog, under MATHEMATICAL ASSOCIATION OF AMERICA.

DE LA VALLÉE POUSSIN, C. Lecons, sur l'approximation des fonctions. Paris, 1919.
VAN TUYL, G. H. Modern business arithmetic. New York, 1923.
New complete business arithmetic. New York, 1924.
VAN VLECK, E. B. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
VEBLEN, O. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
VERRIJP, D. P. A. Vierdecimalige Tafels. Groningen, 1926.
Groote Tafel in Vijfdecimalen. Groningen, 1924.
VIVANTI, G. Analytische Funktionen. Leipzig, 1906.

WALKER, B. M. On the resolution of higher singularities of algebraic curves into ordinary nodes. Diss. Chicago, 1906.
WALLACE, R. R. See KINGSBURY, H. B.
WALMSLEY, C. Introductory course in mathematical analysis. Cambridge, 1926.
WALSH, C. B. See AUERBACH, M.
WARREN, L. A. H. See WILSON, N. R.
WASHBURNE, C. and others. Washburne individual arithmetic, Books 1–5, with correction book, test book, key to test book, and teacher's manual. Yonkers, 1927.
WATERBURY, L. A. Handbook of mathematics for engineers. New York, 1918.
WATKEYS, C. W. See GALE, A. S.
WATSON, G. N. See WHITTAKER, E. T.
WEAVER, J. H. See CARMICHAEL, R. D.
WELLS, W. and HART, W. W. New high school arithmetic. Boston, 1919.
Modern plane and solid geometry. New York, 1927.
Modern algebra, third semester course. Boston, 1929.
WENTWORTH, G. and SMITH, D. E. Trigonometric and logarithmic tables. Boston, 1914.
See also LING, G. H.
WENTWORTH, G., SMITH, D. E., and SCHLAUCH, W. S. Commercial algebra. Book I. Boston, 1917.
WESTCOTT, G. J. B. See MILNE, W. J.
WHIPPLE, F. J. W. See JACKSON, C. S.
WHITE, C. E. Theory of irreducible cases of equations. Part II. Lancaster, 1913.
WHITE, C. J. and BLACKBURN, P. P. Elements of theoretical and descriptive astronomy. New York, 1920.

WHITE, E. V. See CALHOUN, J. W.
WHITE, H. S. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
WHITTAKER, E. T. See Section I of this Catalog, under EDINBURGH UNIVERSITY.
WHITTAKER, E. T. and WATSON, G. N. Modern analysis. Cambridge, 1920.
WIELEITNER, H. Die Geburt der modernen Mathematik. I. Analytische Geometrie. Karlsruhe, 1924.
  Rechnen und Algebra. Berlin, 1927.
WIJDENES, P. Nieuwe school-algebra. Deels 1, 2 & 3. Groningen, 1924.
  Hoogere algebra. Groningen, 1924.
  Tien Jaargangen. Deel 2. Groningen, 1924.
  See also MOLENBROEK, P.
WILCZYNSKI, E. J. See Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
WILLIAMS, D. R. Thèses: 1re, Compléments au théorème de M. Julia; 2me, Propositions données par la Faculté. Palerme, 1928. (Présentées, Univ. de Strasbourg.)
WILLIAMS, E. H. See Section I of this Catalog, under ILLINOIS. UNIVERSITY.
WILLIAMS, J. E. See Section I of this Catalog, under VIRGINIA POLYTECHNIC INSTITUTE.
WILLIAMS, K. P. The dynamics of the airplane. New York, 1921.
WILLIS, C. A. Plane geometry. Philadelphia, 1922.
WILSON, E. B. Aeronautics. New York, 1920.
WILSON, N. R. and WARREN, L. A. H. College algebra. New York, 1928.
WOLFF, G. See KLEIN, F.
WOODS, F. S. Higher geometry. Boston, 1922.
  See also Section I of this Catalog, under AMERICAN MATHEMATICAL SOCIETY.
WOODS, F. S. and BAILEY, F. H. Elementary calculus. Boston, 1922.
  Elementary calculus. Rev. ed. Boston, 1928.

YEATON, C. H. Surfaces characterized by certain special properties of their directrix congruences. Diss. Chicago; Bologna, 1916.
YOUNG, J. W. See JOHNSON, R. A.
  See also Section I of this Catalog, under CARUS MATHEMATICAL MONOGRAPHS.
YOUNG, J. W. and MORGAN, F. M. Plane trigonometry. New York, 1919.
YOUNG, J. W. and SCHWARTZ, A. J. Plane geometry Norwood, 1915.
YOUNG, J. W. A. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.
YOUNG, J. W. A. and JACKSON, L. L. High school algebra. New York, 1913.
  Plane geometry. New York, 1916.
YOUNG, J. W. A. and LINEBARGER, C. E. Elements of calculus. New York, 1920.

ZOOK, G. F. See Section I of this Catalog, under U. S. BUREAU OF EDUCATION.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE CO-OPERATION OF

ELIZABETH CARLSON, N. A. COURT, OTTO DUNKEL, H. S. EVERETT, B. F. FINKEL, R. E. GILMAN, R. A. JOHNSON, H. W. KUHN, J. R. MUSSELMAN, H. L. OLSON, D. E. SMITH, F. M. WEIDA

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 9, NOVEMBER
PART I

The November number is issued in two parts of which Part II is the Register of officers and members. Price 50¢ to non-members.

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

[illegible] 3. C. N. Moore; 4. [illegible] 7. W. E. [illegible] E. [illegible] Hedrick; 11. H. E. Slaught; [illegible] E. Landau; 15. [illegible] Smith; 16. [illegible] 19. Don [illegible] H. Dala[illegible] Mrs. [illegible] E. [illegible] J. H. H[illegible] Couwens; [illegible] 42. Mrs. P. H. [illegible] C. W. Watkeys; 44. C. [illegible] Mrs. Guy [illegible] Ingraham; [illegible] Ingraham; 52. [illegible] Mrs. W. F. Cheney; [illegible] H. [illegible] Ayres; 60. [illegible] Graves; [illegible] Bliss; 62. [illegible] L. Rietz; 63. Elizabeth [illegible]; 64. Julia T. Colpitts; [illegible] 67. H. S. Vandiver; 68. G. C. Evans; 69. D. L. [illegible]oll; 70. P. G. Robinson; 71. W. E. Roth; 72. Mrs. F. W. Owens; 73. W. C. Brenke; 74. Dunham Jackson; 75. Mrs. Dunham Jackson; 76. [illegible]; 77. [illegible] G. Boyce; 78. L. M. Graves; 79. J. W. Young; 80. Helen Smith; 81. W. H. McEwen; 82. I. J. Schoenberg; 83. Isabelle Gillilland; 84. C. F. Craig; 85. R. W. Brink; 86. L. LaPaz; 87. T. Rado; 88. R. G. Sanger; 89. W. L. Hart; 90. F. W. Owens; 91. Max Scherberg; 92. J. B. Meyer; 93. W. B. Carver; 94. E. B. Skinner; 95. J. V. McKelvey; 96. Mrs. J. V. McKelvey; 97. Guy Clements; 98. Rachael B. Adams; 99. Lennie Copeland; 100. Mrs. E. P. Lane; 101. Louis Ingold; 102. Mrs. H. E. Buchanan; 103. Mrs. C. G. Latimer; 104. Gladys Gibbens; 105. A. B. Coble; 106. H. P. Pettit; 107. C. C. MacDuffee; 108. A. J. Maria; 109. W. H. Roever; 110. Mrs. W. H. Roever; 111. W. T. Reid; 112. J. A. Bullard; 113. B. W. Jones; 114. E. A. Ross; 115. Mrs. J. B. Meyer; 116. H. P. Evans; 117. Warren Weaver; 118. H. W. March; 119. G. R. Clements; 120. Eugene Taylor; 121. C. R. Adams; 122. Clara Smith; 123. Deane Montgomery; 124. P. R. Rider; 125. H. E. Buchanan; 126. Nola Lee Anderson; 127. C. G. Latimer; 128. Mrs. A. B. Coble; 129. Mrs. A. Cohen; 130. C. F. Bowles; 131. C. M. Jensen; 132. A. Cohen; 133. C. O. Oakley; 134. Myron Rosskopf; 135. Paul Eberhardt; 136. Marie Hove; 137. Gladys Banes; 138. D. S. Morse; 139. Mrs. D. S. Morse; 140. Lois W. Griffiths; 141. W. L. Moore; 142. Mrs. Hille; 143. Einar Hille; 144. W. P. Webber; 145. E. W. Miller; 146. C. I. Lubin; 147. E. B. Stouffer; 148. J. W. Boyce; 149. A. T. Craig; 150. Philip Franklin; 151. Miss M. B. White; 152. Mrs. J. V. Uspensky; 153. Clara Buchanan; 154. Ma're Weiss; 155. H. S. Everett; 156. Theo. Palmer; 157. M. L. Hartung; 158. Mrs. M. L. Hartung; 159. Mabel Schmeiser; 160. E. E. Moots; 161. E. H. Cutler; 162. Norman Miller; 164. N. H. McCoy; 165. E. H. Hildebrandt; 166. J. V. Uspensky.

GROUP PHOTOGRAPH TAKEN AT THE MINNEAPOLIS MEETING, 1931

GROUP PHOTOGRAPH TAKEN AT THE MINNEAPOLIS MEETING, 1931

THE FIFTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The fifteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of Minnesota, Minneapolis, Minn., on Friday, Monday and Tuesday, September 4, 7 and 8, 1931, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred fifty-nine were present at the meetings, including the following one hundred forty-eight members of the Association.

H. M. ACKLEY, State Teachers College, Kalamazoo, Mich.
C. R. ADAMS, Brown Unive sity
L. K. ADKINS, State Teachers College, La Crosse, Wis.
NOLA LEE ANDERSON, Sophie Newcomb College
R. C. ARCHIBALD, Brown University
W. L. AYRES, University of Michigan

GLADYS L. BANES, Butler University
R. W. BARNARD, University of Chicago
W. J. BERRY, Polytechnic Institute of Brooklyn
G. A. BLISS, University of Chicago
C. F. BOWLES, South Dakota State School of Mines
M. G. BOYCE, Western Reserve University
W. C. BRENKE, University of Nebraska
R. W. BRINK, University of Minnesota
W. E. BROOKE, University of Minnesota
H. E. BUCHANAN, Tulane University
J. A. BULLARD, University of Vermont
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
J. W. CAMPBELL, University of Alberta
S. ELIZABETH CARLSON, University of Minnesota
W. B. CARVER, Cornell University
W. F. CHENEY, Connecticut Agricultural College
E. W. CHITTENDEN, University of Iowa
G. R. CLEMENTS, U. S. Naval Academy
A. B. COBLE, University of Illinois
C. J. COE, University of Michigan
ABRAHAM COHEN, Johns Hopkins University
TERESA COHEN, Pennsylvania State College
JULIA T. COLPITTS, Iowa State College
R. P. CONKLING, Newark Technical School, Newark, N. J.
JUNE F. CONSTANTINE, University of Iowa
LENNIE P. COPELAND, Wellesley College
W. H. H. COWLES, Pratt Institute
E. H. CUTLER, Lehigh University

H. H. DALAKER, University of Minnesota
MARIAN E. DANIELLS, Iowa State College
L. L. DINES, University of Saskatchewan
B. F. DOSTAL, University of Florida
LUCY T. DOUGHERTY, Junior College, Kansas City, Kans.

J. M. EARL, Municipal University of Omaha
Mrs. R. B. EIDE, River Falls, Wis.
G. C. EVANS, The Rice Institute
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

SARAH FAHNESTOCK, Marymount College
ANNIE W. FLEMING, Iowa State College
W. B. FORD, University of Michigan
PHILIP FRANKLIN, Massachusetts Institute of Technology
T. C. FRY, Bell Telephone Laboratories, New York, N. Y.

H. J. GAY, Worcester Polytechnic Institute
L. O. GHORMLEY, University of Tennessee
GLADYS GIBBENS, University of Minnesota
CORNELIUS GOUWENS, Iowa State College
L. M. GRAVES, University of Chicago
LOIS W. GRIFFITHS, Northwestern University
H. R. GRUMMANN, Washington University
SOLOMON GUTTMAN, Minneapolis, Minn.

LAURENCE HAMPTON, Alabama Polytechnic Institute
CLARA L. HANCOCK, Junior College, Virginia, Minn.
W. L. HART, University of Minnesota
M. L. HARTUNG, Wisconsin High School, Madison, Wis.

E. R. HEDRICK, University of California at Los Angeles
EINAR HILLE, Princeton University
H. M. HOSFORD, University of Arkansas
E. V. HUNTINGTON, Harvard University
J. W. HURST, Montana State College

LOUIS INGOLD, University of Missouri
M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
C. M. JENSEN, Macalester College
B. W. JONES, Cornell University

W. H. KIRCHNER, University of Minnesota

E. P. LANE, University of Chicago
R. E. LANGER, University of Wisconsin
LINCOLN LA PAZ, Ohio State University
C. G. LATIMER, University of Kentucky
J. D. LEITH, University of North Dakota
ANNA D. LEWIS, Lake Erie College
C. I. LUBIN, University of Cincinnati

C. C. MACDUFFEE, Ohio State University
N. B. MACLEAN, McGill University
N. H. MCCOY, Smith College
W. H. MCEWEN, Mount Allison University
J. V. MCKELVEY, Iowa State College
Mrs. J. V. MCKELVEY, Iowa State College
A. S. MCMASTER, University of Colorado
T. E. MASON, Purdue University
J. B. MEYER, State Teachers College, Valley City, N. D.
I. L. MILLER, South Dakota State College
NORMAN MILLER, Queen's University
W. L. MISER, Vanderbilt University
C. N. MOORE, University of Cincinnati
W. L. MOORE, University of Louisville
E. E. MOOTS, Cornell College
D. S. MORSE, Union College
MARSTON MORSE, Harvard University
THIRZA A. MOSSMAN, Kansas State Agricultural College
PAUL MUEHLMAN, St. Louis University

MARIE M. NESS, University of Minnesota

C. O. OAKLEY, Brown University
LOUIS O'SHAUGHNESSY, Virginia Polytechnic Institute
A. L. O'TOOLE, University of Minnesota
F. W. OWENS, Pennsylvania State College
Mrs. F. W. OWENS, State College, Pa.

H. P. PETTIT, Marquette University
G. C. PRIESTER, University of Minnesota

TIBOR RADÓ, Ohio State University
RUTH RASMUSEN, South Dakota State College
S. E. RASOR, Ohio State University
W. T. REID, University of Chicago
B. L. REMICK, Kansas State Agricultural College
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
H. L. RIETZ, University of Iowa
W. C. RISSELMAN, Northern Arizona Teachers College
H. M. ROBERTS, Minneapolis, Minn.
W. H. ROEVER, Washington University
M. F. ROSSKOPF, Brown University
W. E. ROTH, University of Wisconsin
INEZ RUNDSTROM, Gustavus Adolphus College

MABEL F. SCHMEISER, State Teachers College, Wayne, Nebr.
E. W. SHELDON, University of Alberta
R. R. SHUMWAY, University of Minnesota
L. P. SICELOFF, Columbia University
E. B. SKINNER, University of Wisconsin
H. E. SLAUGHT, University of Chicago
CLARA E. SMITH, Wellesley College
E. R. SMITH, Iowa State College
G. W. SMITH, University of Kansas
I. W. SMITH, North Dakota Agricultural College
VIRGIL SNYDER, Cornell University
E. B. STOUFFER, University of Kansas

EUGENE TAYLOR, University of Idaho
C. F. THOMAS, Case School of Applied Science
ELLA THORP, University of Minnesota
H. G. TITT, Huron College

H. S. VANDIVER, University of Texas

L. E. WARD, University of Iowa
C. W. WATKEYS, University of Rochester
WARREN WEAVER, University of Wisconsin
W. P. WEBBER, Louisiana State University
MARIE J. WEISS, Sophie Newcomb College
MARION B. WHITE, Carleton College
ELIZABETH, W. WILSON, Cambridge, Mass.
G. L. WINKELMANN, St. John's University
FREDRICK WOOD, Hamline University

J. W. YOUNG, Dartmouth College

The meetings were held on the campus of the University of Minnesota. The mathematicians were comfortably housed in Sanford Hall and had their dining room and social rooms there. Such provisions, which have become increasingly frequent in recent years, promoted very noticeably the pleasant character of the meetings because of the more personal contacts which are otherwise not so largely possible. The committee on arrangements under the chairmanship of Professor Brink planned carefully in advance for the convenience of the visitors, and as a result the matters of registration and of service throughout the week were carried out very smoothly.

About one hundred and twenty-five were present at the joint meeting on Friday evening, the majority remaining over the week-end and being reinforced by many new comers on Sunday and Monday. Coming just before the close of the Summer Session of the Society for the Promotion of Engineering Education, the Friday evening meeting formed an important link between that session and the meetings of the two mathematical organizations. Numerous parties drove about the beautiful boulevards of Minneapolis and vicinity on Saturday and Sunday.

A reception was held on Monday evening, Dean Ford of the Graduate School and Mrs. Ford welcoming the visiting mathematicians. A delightful luncheon was given for the visiting ladies on Tuesday noon at the Lafayette Club on Lake Minnetonka, automobiles being provided for all the ladies who attended.

The joint dinner was held on Wednesday evening at the Minneapolis Automobile Club. A cordial welcome was given in a speech by President L. D. Coffman. Professor Hedrick gave information as to the nature of the summer meetings at Los Angeles in September, 1932 and the incidental delights which may be found in connection with the meetings. He offered a resolution which was adopted by rising vote, expressing to the University authorities, and to the committee on local arrangements, our appreciation of their provision of the excellent and comfortable facilities afforded during the meetings. Brief speeches were made by Miss Weiss, who graciously invited the mathematicians to the New Orleans meetings in December and spoke of the many attractions in and about New Orleans; by Professor MacLean, who spoke in a very fitting manner of the cordial relations between the Dominion of Canada and the United States and called for definite efforts to promote active relations between the two countries; and by Professor Weaver, who commented very wittily on humorous incidents of the week and on the proposed plan for publicity in matters mathematical.

The American Mathematical Society held its thirty-seventh summer meeting and fifteenth colloquium from Tuesday to Friday, the colloquium lectures being given by Professor Marston Morse of Harvard University on "The calculus of variations." On Wednesday morning Professor Edmund Landau of the University of Göttingen gave an address by invitation on "Schnirelmann's theorem," and on Thursday morning Professor C. C. MacDuffee of Ohio State University gave an address by invitation on "Ideals in linear algebras." Sessions

for the reading of papers were held on Tuesday and Wednesday afternoons and Friday morning.

The Mathematical Association held sessions on Friday evening, Monday afternoon and Tuesday morning, Professor H. P. Hammond presiding at the first session and Vice-president C. N. Moore at the other two. The Association is under obligations to the committee consisting of Professor Warren Weaver (chairman), W. E. Brooke, Jewell C. Hughes, and J. H. Taylor for the preparation of an excellent program. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

Joint Session of the Association with the Society for the Promotion of Engineering Education

(1) "Some great engineer-mathematicians" by Dean C. S. Slichter, University of Wisconsin.

(2) "Some frequently overlooked mathematical principles of descriptive geometry" by Professor W. H. Roever, Washington University.

1. Professor Slichter pointed out the broad character of the learning of the mathematician of old, in particular the double field of Archimedes in pure mathematics and engineering. After giving a summary of the work of Leonardo, Euler, the Bernoullis and many others in other fields than mathematics alone, he contrasted Lord Kelvin and Heaviside. Even as an undergraduate Kelvin, as William Thomson, discovered the method of electrical images. Within one year of graduation he was called to the chair of natural philosophy at Glasgow and held this position for more than fifty years, combining throughout his career remarkable achievements in pure mathematics and applications to the telegraph, the submarine telegraph, electrical measurement, and so on, winning fame and wealth by his attainments. Heaviside, who came from a family of engineers, held office with a telegraph company for several years during which he showed that quadruplex telegraphy was possible and contributed to scientific journals papers of much ability and originality. But because of increasing deafness he spent all his life after the age of twenty-four as a scientific hermit at Torquay. His mathematical studies were profound and difficult to understand. They were often cut through the requirements of the editors, so that his work has been difficult to follow and to appraise. His inventions in telephonic developments are worth millions to all the world; but he did not take out patents on these and was not conspicuous or wealthy as compared with Lord Kelvin. Happily he was suitably honored in his last years. It is a cause for scientific hopefulness in this day of over-specialization that men like these two do appear at culminating epochs in the affairs of men.

2. The purpose of Professor Roever's paper is to call attention to some of the mathematical principles of Descriptive Geometry which are either overlooked or understressed, due to the fact that the subject is now taught almost exclusively by teachers of drawing rather than by mathematicians.

After tracing the evolution of Descriptive Geometry from the art of stone-cutting, the problem of this subject was precisely stated as that of representing the objects of space by figures in a plane and of solving the problems of space by constructions which can be executed in the plane. It was then pointed out that the sight-process leads to the methods of projection as a means of representation and that this method while sufficient for the artist has to be augmented in order to provide a means for obtaining an unambiguous correspondence between space and the plane so that the problems of space may be solved by constructions in the plane. Thus one is led to methods of double projection such, for instance, as the Mongean and axonometric methods as well as to methods of double trace, such as the method of free perspective and the clinographical method (including the topographical method). It was also shown that certain methods of double projection would have no significance without such fundamental theorems of Descriptive Geometry as the theorem of Schwarz (or the theorem of Gauss) for orthographic projection, and the theorem of Pohlke for oblique projection. Several of the problems of space were solved by each of the methods above mentioned. Attention was also called to methods of constructing the bounding curves of projections of surfaces.

First Session of the Association

(3) "The present situation in projective differential geometry" by Professor E. P. Lane, University of Chicago.

(4) "Business and statistics" by Professor H. C. Carver, University of Michigan.

3. Professor Lane first called attention to the past, or history, of Projective Differential Geometry. Then he described the present situation as reflected by the literature. Finally, he pointed out some of the directions which the development of this science might follow in the future.

The history of projective differential geometry is comprised in two distinct periods; first, that of the discovery of isolated results now recognized to be of a projective differential nature; second, that of the conscious study of projective differential geometry and of the organization of theories of this subject. The earlier period extended over approximately the two and one-quarter centuries ending in 1875. During this time results later incorporated in the science of projective differential geometry were announced by such men as Dupin, Chasles, P. Serret, Moutard, Hermite, Lie, and Darboux. Since 1875 projective differential geometry has been consciously studied first by Halphen and later by Wilczynski, Fubini, and others.

The present situation in projective differential geometry may be considered as reflected by the literature, and more particularly, by the books which have been published or have been announced on this subject. Of the books actually published there are five (one in two volumes) by Wilczynski, Tzitzéica, Fubini, and Cech. There are two books announced but not yet published, one by Bompiani and one by Lane.

Any predictions as to the course of future development of a science are necessarily somewhat uncertain but may be interesting nevertheless. One unsolved problem is to construct a projective differential theory of a curved variety of k dimensions immersed in a linear space of n dimensions, with $1<k<n-1$ and $n>4$, analogous to Wilczynski's or Fubini's theory of a surface in ordinary space. Sets of varieties which are the loci of linear spaces in correspondence are being studied by means of certain systems of linear differential equations of the first order. Another topic proposed for study is projective differential geometry *in the large*. And it seems quite likely that sufficient attention has not yet been given to the projective differential study of singularities of configurations.

4. Professor Carver called attention to the fact that the trend of business statistics is, and necessarily must be, toward the mathematical end of the statistics spectrum, and stressed the fact that mathematical statistics—essentially a combination of probability and pure mathematics—is sadly in need of a rigor which only specialists in pure mathematics can supply. After defining briefly several statistical concepts such as *average*, *standard deviation*, *frequency distribution*, etc., and describing briefly the theory of sampling, he went on to describe the theory of inverse sampling, referring to the work recently accomplished by R. A. Fisher and to the excellent survey by Professor P. R. Rider in the *Annals of Mathematics* for October, 1930. He concluded with a discussion of business statistics, expressing regret that that phase of business statistics which deals with the analysis and projection of time series rests inexcusably on an exceedingly unstable foundation. This work is being done by economists and "professional forecasters" who are far more interested in making predictions than they are in estimating the probability that the actual occurrences will differ from their forecasts by more than a specified per cent. Professor Carver insisted that with very few exceptions estimates of the future terms in time series should be expressed numerically and in precisely the same terms that will subsequently be employed in recording the corresponding facts for historical purposes. When this is done, we can at least obtain empirical approximations for the probable errors of these estimates and we may then expect that competition among forecasters will result both in an increase in the reliability and in a corresponding decrease in the number of forecasters. It is highly desirable that our students of business administration possess a sufficient mathematical background to enable them to know at what point in their work they should ask the aid of mathematically trained statisticians. Mathematical statisticians are not at the present time capable of solving all the problems that might reasonably be submitted to them. For instance, the entire theory of sampling will probably require fundamental alterations in order that it may be applied properly to time series, since each sample contains ordered variates with respect to time. Professor Carver's plea was that mathematicians should realize the importance of this branch of applied mathematics, since the business world will soon be insisting on a better cooperation of mathematics and economics.

Second Session of the Association

(5) "Some mathematical aspects of the new physics" by Professor J. H. Van Vleck, University of Wisconsin, by invitation.

(6) "Functions of the Mathematical Association of America," a retiring presidential address by Professor J. W. Young, Dartmouth College.

5. This paper will appear in this Monthly early in 1932.
6. This paper will appear in this Monthly early in 1932.

Meeting of the Board of Trustees

Eight trustees were present at the meeting on Monday evening.

The following thirty-five persons and one institution were elected to membership on applications duly certified:

To Individual Membership

H. M. Ackley, A.M. (Olivet). Prof., Western State Teachers Coll., Kalamazoo, Mich.

Sister Alice Irene, A.M. (St. Canisius). Prof., Coll. of St. Rose, Albany, N. Y.

N. B. Allison, A.M. (Kentucky). Instr., Univ. of Kentucky, Lexington, Ky.

Max Astrachan, A.M. (Brown). Instr., Brown University, Providence, R. I.

N. A. Banks, M.S. (Wilberforce). Prof., Tillotson Coll., Austin, Tex.

J. I. Bohnert, Jr. Senior, Carnegie Inst. of Tech., Pittsburgh, Pa.

G. M. Brown, M.Sc. (Univ. of Leeds, England) Tiffin, Ohio

C. E. Buell, A.B. (Oberlin), Grad. Student, Ohio State Univ., Columbus, Ohio

W. H. H. Cowles, A.M. (Columbia). Prof., Head of Dept., Math and English, Pratt Inst., Brooklyn, N. Y.

J. H. Edmonston, A.B. (George Washington). 1441 Fairmont St. N. W., Washington, D.C.

J. D. Elder, Ph.D. (Calif. Inst. of Tech.). Instr., Univ. of Michigan, Ann Arbor, Mich.

Donald Faulkner, A.B. (Stetson). Prof., J. B. Stetson Univ., DeLand, Fla.

Virginia I. Felder, M.S. (Tulane). Head of Dept., Copiah-Lincoln Junior Coll., Wesson Miss.

George Fisanick, A.B. (Pennsylvania State). Grad. student, Univ. of Michigan, Ann Arbor, Mich.

J. M. Gleason, A.M. (California). Instr., State Teachers Coll., San Diego, Calif.

H. R. Grummann, A.M. (Minnesota). Asst. Prof., Washington Univ., St. Louis, Mo.

M. L. Hartung, Ph.D. (Wisconsin). Instr., Teaching of Math., Wisconsin High School, Madison, Wis.

A. E. Heins, Student, Massachusetts Inst. of Tech., Cambridge, Mass.

A. S. Householder, A.M. (Cornell). Asst. Prof., Washburn Coll., Topeka, Kans.

D. V. Lanham. Student, New River State Coll., Montgomery, W. Va.

W. S. Litterick, M.S. (Brown). Instr., Peddie School, Hightstown, N. J.

N. B. MacLean, Ph.D. (Chicago). Prof., Appl. Math., Joint Chmn, Dept. of Math., McGill Univ., Montreal, Canada

A. W. Martin, Ph.D. (Chicago). Prof., Coll. of Puget Sound, Tacoma, Wash.

Mary E. Meade, A.M. (Virginia). Acting Dean of Women, Martin Coll., Pulaski, Tenn.

H. J. Miles, Ph. D. (California). Instr., Univ. of Illinois, Urbana, Ill.

S. S. Morris, M.S. (West Virginia). Prof., Broaddus Coll., Philippi, W. Va.

Brother Norbert, M.S. (Notre Dame). Instr., St. Edwards Univ., Austin, Tex.

Eunice C. Orr, A.M. (Indiana). Teacher, High School, Arcadia, Ind.

A. L. O'Toole, Ph.D. (Michigan). National Research Fellow, Univ. of Minnesota, Minneapolis, Minn.

W. D. Peck, A.B. (Lebanon Valley). Head of Dept., Asheville School for Boys, Asheville, N. C.

Fred Robertson, A.M. (Indiana). Instr., Iowa State Coll., Ames, Iowa

MABEL F. SCHMEISER, Ph.D. (Ohio State). Head of Dept., State Teachers Coll., Wayne, Nebr.

S. J. SMITH, A.M. (Pittsburgh). Instr. State Teachers Coll., Lock Haven, Pa.

RUSSELL SULLIVAN, A.B. (Yale). 1431 N. Meridian St., Indianapolis, Ind.

F. G. WEIMER, M.S. (West Virginia). Grad. Asst., West Virginia Univ., Morgantown, W. Va.

To Institutional Membership

PENNSYLVANIA COLLEGE FOR WOMEN, Pittsburgh, Pa.

The Trustees voted to approve the dates of August 29–September 2, 1932, for the mathematical meetings at Los Angeles. They also authorized the president to appoint a committee on publicity in mathematics with himself as chairman, and they adopted a resolution expressing their appreciation of the action of the officers of the Society for the Promotion of Engineering Education in organizing and administering a summer school for teachers of mathematics to engineering students, recognizing also the very efficient work of Professor H. P. Hammond, director of the summer sessions, and of Dean O. M. Leland, the chairman of the session at Minneapolis and, as well, the part of the University of Minnesota in the contribution of its equipment and its moral and financial support.

W. D. CAIRNS, *Secretary*

THE S.P.E.E. SUMMER SESSION FOR TEACHERS OF MATHEMATICS TO ENGINEERING STUDENTS

By H. P. HAMMOND, Polytechnic Institute of Brooklyn

In cooperation with the Mathematical Association of America and the American Mathematical Society, the Society for the Promotion of Engineering Education held at Minneapolis a session of its Summer School for Engineering Teachers, devoted to the study and discussion of methods of teaching undergraduate courses in mathematics to engineering students. The session was held just prior to the annual meetings of the mathematical societies, beginning on August 24 and concluding September 5.

These Summer School sessions, which were begun in 1927, are held each year in various institutions throughout the country and are devoted in turn to different divisions of the engineering curriculum. The session of this summer was the first in the field of mathematics. Its success is evidenced by the fact that the attendance was the largest of any session which the S.P.E.E. has held. Eighty-nine teachers, representing the various teaching ranks in almost equal proportion, were present during the session. This group represented sixty different institutions in thirty-three states and provinces of Canada. Such widely separated localities as Massachusetts and Vermont at one extreme, and California at the

other, were included. In comparison with groups at other sessions, the mathematics conference was marked, to an unusual degree, by the maturity and responsible positions held by those who attended. For the first time, a number of women members of engineering and other faculties were in attendance.

The various periods of the session were conducted chiefly as lectures presented by members of the staff, followed by open discussions from the floor. The program was arranged in five principal divisions:

1. A series of lectures on educational principles—learning and teaching—by Professor Francis T. Spaulding, of the Graduate School of Education at Harvard University.

2. The content and method of presenting undergraduate courses in mathematics for engineering students. This was the principal division of the program and consisted of a series of lectures beginning with trigonometry and extending through differential equations. Professors E. R. Hedrick, Louis O'Shaughnessy, W. E. Brooke, W. J. Berry, E. V. Huntington and Warren Weaver were the staff members who conducted this portion of the program.

3. Advanced mathematics—a few lectures devoted to various phases of mathematics extending beyond the customary undergraduate subjects, by Professors Dunham Jackson, H. L. Rietz, Warren Weaver, C. N. Moore and Leigh Page.

4. Certain applications of mathematics in the field of engineering practices, by Professor Stephen Timoshenko and Dr. T. C. Fry.

5. History and miscellaneous. Two lectures were delivered by Professor R. C. Archibald on the history of mathematics, and a number of other miscellaneous topics were discussed by Dean O. M. Leland, Dean M. E. Haggerty and Dean C. S. Slichter.

Throughout the two weeks of the session the entire group was housed at Sanford Hall, a dormitory of the University of Minnesota, and the meetings were conducted in buildings of the College of Engineering and Architecture of the University.

In addition to the cooperation of the two mathematical societies, the session was sponsored by the Engineering Foundation and was held in cooperation with the University of Minnesota. Dean O. M. Leland, of the University of Minnesota, served as the Local Director, and Professor C. A. Herrick as the Secretary.

The S.P.E.E. will publish, during the coming year, a series of bulletins or monographs giving the substance of many of the lectures presented. These will appear in the Journal of Engineering Education and as separately bound booklets, which will be available for circulation and can be obtained by applying to the Director of Summer Schools, Society for the Promotion of Engineering Education, 99 Livingston Street, Brooklyn, N. Y.

THEOREMS RELATING TO THE PRE-GRECIAN MATHEMATICS

By G. A. MILLER, University of Illinois

In a previous number of this Monthly[1] the present writer directed attention to a few theorems relating to the *Rhind Mathematical Papyrus*. The present article is devoted to the noting of several other theorems relating to the pre-Grecian mathematical developments. The fact that theorems which were not even formulated at the time may nevertheless have exerted a powerful influence on the development of our subject is forcibly illustrated by the history of elementary geometry. It is well known that the properties which form the subject matter of our modern elementary geometry are the invariants under a group of transformations which was called the principal group by F. Klein. On the other hand, the ancients developed many of the theorems of this elementary geometry without formulating the concept of group of transformations and without knowing that they were studying invariants under such a group.

According to H. Poincaré the concept of group was one of the earliest mathematical concepts to be acquired by the human race. If this is correct it would seem that our general histories of mathematics should exhibit evidences of this concept in some of the earliest mathematical developments. Probably one reason why this has not been done up to the present time is that it requires a keen mathematical insight to comprehend some of these evidences, especially since there is not a single instance on record where the ancient people formulated explicitly a group theory theorem. It is however true that if we study some of the early mathematical developments in the light of such theorems we can now discover clarifying and guiding principles which otherwise would not present themselves and hence a knowledge of these theorems is useful to the modern mathematical historian.

One of the most important and most elementary groups of all mathematics is the one formed by the totality of the positive rational numbers when they are combined by multiplication. While it should not be assumed that the pre-Grecian mathematicians formulated the theorem that these numbers constitute an *abelian* group when they are thus combined it is interesting to note evidences of their recognition of its commutative property. Even in the *Rhind Mathematical Papyrus* the multiplier and multiplicand are sometimes interchanged, and in the seventh book of Euclid's *Elements* it is noted explicitly that the product of two positive real numbers is independent of their order, but it was not until the early part of the nineteenth century that a special term *commutative* was introduced for this fundamental property by F. Servois in volume 5 (1814) of *Annales de Mathématiques*, page 98. It is very interesting to note in the development of mathematics evidences of such distantly related steps towards the proper recognition of such fundamental concepts.

In the introduction to the second edition (1927) of his *Theorie der Gruppen*

[1] Vol. 38 (1931), p. 194.

von endlicher Ordnung A. Speiser speaks of a prehistoric group theory extending back at least as far as 1500 B.C., and of a revival thereof in about 1600 A.D. which failed to command much interest soon thereafter. These conclusions were based largely on the symmetry involved in ancient ornaments and on the study of regular figures in ancient times. These figures naturally suggest the movements of space which transform them as entireties into themselves but interchange their parts. In particular, the eight movements of space which transform into itself a square constitute an interesting non-abelian group now known as the octic group. Since this group is related to a unit of surface measurement employed in very ancient times one might be inclined to think that its properties were known to some of the pre-Grecian mathematicians, but there is no historical evidence which proves that even the ancient Greeks had become familiar with these properties notwithstanding their very elementary character.

Among the symmetrical geometric figures the circle and the sphere are outstanding and it has recently been stated that the ancient Egyptians used the theorem that the area of a hemisphere is equal to that of two great circles on the corresponding sphere.[2] It is however not quite accurate to say that they found the exact area of a hemisphere since their rule for finding the area of a circle was quite inaccurate from the modern point of view. While the circle naturally appealed to the ancient peoples on account of its symmetry and the fact that it admits a group of rotations of infinite order yet we have no evidence that a single pre-Grecian mathematician developed a method by means of which the value of the ratio of its circumference to its diameter can be found to any desired degree of accuracy. Such negative facts in the history of pre-Grecian mathematics are as important as the positive ones for the purpose of securing a correct understanding of this history.

Similarly we should emphasize negative facts relating to the influence of the concept of group on early mathematical developments in order to secure a correct notion of this influence. One such negative fact is furnished by the late general adoption of a number system which forms a group with respect to the operation of addition. The most fundamental such system is furnished by the positive and negative integers, including zero. It is well known that the use of negative numbers was not fully established until about the beginning of the nineteenth century and that such noted mathematicians as H. Cardan and R. Descartes expressed very erroneous views relating thereto. Moreover, according to the *Rhind Mathematical Papyrus* by A. B. Chace, volume 1, page 6, "the Egyptians could not solve directly the problem of finding the multiplicand when the multiplier and product were given." This seems to imply that the group concept did not fully dominate their operation of multiplication.

Our modern mathematics involves many arbitrary divisions which do not circumscribe the developments relating to our subject. That is, these develop-

[2] W. W. Struve, *Quellen und Studien zur Geschichte der Mathematik*,—Quellen, vol. 1 (1930), p. 172.

ments penetrate these lines of division and exhibit mathematics as a unit. In particular, we now frequently speak of the four fundamental rational operations and the student is sometimes led to think that these operations are distinct instead of interpenetrating. It is somewhat misleading to say that the operations of addition, subtraction, multiplication, and division appear explicitly in the *Rhind Mathematical Papyrus*. According to J. Tropfke's *Geschichte der Elementar-Mathematik*, volume 1 (1930), page 73, the ancient Egyptians were not conscious of actual calculating operations. Even the ancient Greeks used many different words for the same fundamental rational operation and thus exhibit that in their times there was still lacking a well established custom as regards the use of these operations and the value of a drill therein so as to replace the need of penetrating insight by a formal method.

With respect to the group concept our four fundamental rational operators reduce to two, viz., addition and multiplication, since the inverse operations belong to the group which involves the direct ones. The fact that we persist in speaking of four fundamental operations notwithstanding this obvious theorem seems to imply that the group concept fails to dominate here. On page 70 of the volume on the history of mathematics noted in the preceding paragraph J. Tropfke remarks that one might assert that the number of the different operations formerly used was inversely proportional to the mathematical advancement. This evidently does not apply to the pre-Grecian mathematics and it may be too early to observe any tendency to reduce our four fundamental operations to two on account of an increasing familiarity with the group concept. At any rate it is interesting to note the connection between the group concept and these operations and the possibility of such a reduction.

A rational number may be defined as the infinite set of numbers which are equivalent to a fixed common fraction.[3] An operation like $a+b=c$ may therefore be regarded as a certain combination of the infinite sets represented by the fixed numbers a and b respectively into a single set represented by c. The fact that there is an infinite set of common fractions which are separately equal to the common fraction 2/1, for instance, and that when we perform the ordinary rational operations with respect to fixed numbers we may also regard them as operations with infinite sets of equivalent numbers was probably not explicitly noted by any pre-Grecian mathematicians but they certainly did recognize the equivalence of various common fractions and hence may have been vaguely inspired by this broader view of the fundamental operations of arithmetic. The statement that 2 times 3 is 6 becomes much more significant if we think of the infinitudes of equivalent numbers involved therein.

The ancient Sumerians and the ancient Babylonians seem to have been the only pre-Grecian peoples who made even a partial use of a positional numerical notation. It should be noted that while every positive rational integer can be represented positionally by means of k distinct integer symbols with respect to

[3] Cf. E. Landau, Grundlagen der Analysis, 1930, p. 35.

the arbitrary natural number $k>1$ as a base it is not possible to select this base so that every positive common fraction can be represented positionally in a finite form by means of a starting symbol and these k integer symbols. In fact, this can obviously be done only when the denominator of the common fraction involves only prime factors which divide k. This marked difference as regards the possible positional notation of positive integers and of positive common fractions is fundamental in the history of elementary mathematics from a higher standpoint and hence should be emphasized therein.

By using 60 for k the ancient Babylonians were able to represent the common fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ by the integers 30, 20, 15, 12, and 10 respectively. Just as the *Rhind Mathematical Papyrus* seems to exhibit the predominating interest of the ancient Egyptians in the use of common fractions so the large base 60 may exhibit the predominating interest of the ancient Babylonians in the use of these fractions. Recently Professor O. Neugebauer stated that the extensive so-called multiplication tables of the ancient Babylonians had for their object the conversion of common fractions into sexagesimal ones, *Quellen und Studien zur Geschichte der Mathematik*,—Studien, volume 1 (1930), page 193. This seems to support the view just noted, and these tables must have furnished many illustrations of the difficulties involved in representing common fractions positionally.

An invariant under similarity transformations which was recognized very early is the constant ratio between the circumference and the diameter of a circle as the size of the circle is varied. The fact that the circumferences of all of these circles are now commonly supposed to make the same angle with their diameters but when all of them are tangent to a given line at the same point thereof they may be so arranged and ordered that any one of these angles lies within all of those which precede it exhibits the need of postulates in elementary geometry and may help to explain why this need seems to have been first recognized in connection with this subject. Notwithstanding this obvious need of postulates there is no evidence tending to prove that any pre-Grecian mathematicians made explicit use of postulates. They did however verify many of their results and in some cases these verifications constitute proofs. The most general theorems used by the pre-Grecian mathematicians were however not proved by them in any extant work and this represents the widest difference between the pre-Grecian mathematics and that of the ancient Greeks.

It has recently been discovered that the fundamental theorem commonly known as the Pythagorean theorem was probably used by the ancient Babylonians but no evidence of a proof by them has yet been found. Since this theorem relates to an infinite number of triangles it cannot be proved by verifications alone while the theorem that four things can be permuted in 24 distinct ways can obviously be proved in this way. The table of unit fractions with which the *Rhind Mathematical Papyrus* opens also furnishes special theorems which can be thus proved. The fact that the ancient Egyptians gave so many verifications exhibits a tendency on their part towards proving mathematical results instead

of merely accepting them. An unduly unfavorable impression in regard to ancient Egyptian mathematics has been made during recent years by various writers who asserted that the area of a triangle was found therein by multiplying one-half the base by a side instead of by the altitude. Recently W. W. Struve seems to have proved conclusively that the ancient Egyptians used a correct theorem in this connection; *Quellen und Studien zur Geschichte der Mathematik*,—Quellen, volume 1 (1930), page 172.

ON THE DERIVATIVES OF $(w/\sin w)^k$ AT $w=0$

By CARROLL V. NEWSOM, University of New Mexico

Recently the writer had occasion to investigate the higher derivatives of the function, $f(w)=(w/\sin w)^k$. This investigation led to the formulation of the following theorem which appears to be new and noteworthy:

THEOREM:

$$\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin\ w}\right)^k\right]_{w=0}=\frac{\sum\alpha_1\alpha_2\cdots\alpha_r}{{}_{k-1}C_r}. \tag{1}$$

in which k is any given positive integer[1] $\geqq 2$, $1\leqq r\leqq k-1$; *and where* $\sum\alpha_1\alpha_2\cdots\alpha_r$ *denotes the sum of the* $\binom{k-1}{r}$ *products of r factors each formed by taking the possible combinations of the* $(k-1)$ *quantities;* $\pm(k-2)i$, $\pm(k-4)i$, $\cdots$, $\left\{\begin{smallmatrix}\pm i\\0\end{smallmatrix}\right\}$ *r at a time; i having the usual interpretation,* $i=\sqrt{(-1)}$, *and where* $\left\{\begin{smallmatrix}\pm i\\0\end{smallmatrix}\right\}$ *is understood as* $\pm i$ *or* 0 *according as k is odd or even.*[2]

Proof: To establish this theorem it is sufficient, after defining the right member of (1) to be unity in case $r=0$, to verify the following identity in x where x is regarded as an auxiliary variable:

$$\begin{aligned}&\sum_{r=0}^{k-1}\left\{\frac{(k-1)!}{r!(k-r-1)!}\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}x^{k-r-1}\right\}\\ &\equiv\sum_{r=0}^{k-1}\left[\sum\alpha_1\alpha_2\cdots\alpha_r\right]x^{k-r-1}.\end{aligned}\tag{2}$$

This is equivalent to (1) because the coefficients of like powers of x on the right and on the left sides of (2) will necessarily be equal if (2) is an identity, and the equating of these coefficients gives precisely (1) wherein r ranges over the prescribed range; namely, $1\leqq r\leqq k-1$.

Inasmuch as (2) is readily found by trial to be true in case k is small, as $k=2$ and $k=3$, the inductive process of proof will be employed.

Assuming then the correctness of (2) for any particular k, we proceed to show that it holds true also when k is replaced by $k+2$. To do this let us first

---

[1] Evidently a further definition of the symbols employed is necessary when $k=1$.

[2] Contrast the result here given with the result obtained, for example, by Schwatt, *Introduction to Operations with Series*, University of Pennsylvania Press, (1924), page 74.

multiply both members of (2) by $(x+ki)(x-ki)$; that is, by x^2+k^2. The product of the left member by k^2 is evidently

$$\sum_{r=0}^{k-1}\left\{\frac{k^2(k-1)!}{r!(k-r-1)!}\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}\right\}x^{k-r-1}. \tag{3}$$

Before multiplying the same member of (2) also by x^2 it is desirable to write it in the form

$$\begin{aligned} &x^{k-1}+\frac{(k-1)!}{(k-2)!}\left[\frac{d}{dw}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}x^{k-2} \\ &+\sum_{r=0}^{k-1}\left\{\frac{(k-1)!(k-r-2)(k-r-1)}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}x^{k-r-3}\right\}. \end{aligned} \tag{4}$$

Since $[d/dw(w/\sin w)^k]_{w=0}=0$, the product of (4) and x^2 becomes

$$x^{k+1}+\sum_{r=0}^{k-1}\left\{\frac{(k-1)!(k-r-2)(k-r-1)}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}x^{k-r-1}\right\}. \tag{5}$$

In considering the product of the right member of (2) and x^2+k^2, it is necessary to recall the interpretation of $\sum\alpha_1\alpha_2\cdots\alpha_r$ as originally given in the statement of the theorem. If we employ the root-coefficient relations of the elementary theory of equations the right member of (2) is seen to be

$$[x+(k-2)i][x-(k-2)i][x+(k-4)i][x-(k-4)i]\cdots\left\{\begin{matrix}[x+i][x-i]\\ x\end{matrix}\right\}.$$

Hence it becomes evident that the product of this right member by x^2+k^2 is

$$[x+ki][x-ki][x+(k-2)i][x-(k-2)i][x+(x-4)i][x-(k-4)i]\cdots\left\{\begin{matrix}[x+i][x-i]\\ x\end{matrix}\right\}. \tag{6}$$

This product may be expressed in the form

$$x^{k+1}+\sum_{r=0}^{k-1}\left[\sum\alpha_1\alpha_2\cdots\alpha_{r+2}\right]x^{k-r-1}, \tag{7}$$

where the groups of factors employed in $\sum\alpha_1\alpha_2\cdots\alpha_{r+2}$ are now all the possible combinations of the quantities $\pm ki$, $\pm(k-2)i,\cdots,\left\{\begin{smallmatrix}\pm i\\ 0\end{smallmatrix}\right\}$ taken $r+2$ at a time. Hence, having assumed the truth of (2) for any given value of k, we have the following identity in x:

$$\begin{aligned} &x^{k+1}+\sum_{r=0}^{k-1}\left\{\frac{k^2(k-1)!}{r!(k-r-1)!}\left[\frac{d^r}{dw^r}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}\right. \\ &\left.+\frac{(k-1)!(k-r-2)(k-r-1)}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^k\right]_{w=0}\right\}x^{k-r-1} \end{aligned} \tag{8}$$

$$\equiv x^{k+1} + \sum_{r=0}^{k-1}\left[\sum \alpha_1\alpha_2 \cdots \alpha_{r+2}\right] x^{k-r-1}.$$

But we may now show that the left member of (8) is identically equal to

$$x^{k+1} + \sum_{r=0}^{k-1}\left\{\frac{(k+1)!}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^{k+2}\right]_{w=0} x^{k-r-1}\right\}, \tag{9}$$

which statement is equivalent to showing that (2) holds true, as desired, when k is replaced by $k+2$ inasmuch as (8) then will assume the form

$$\sum_{r=0}^{k+1}\left\{\frac{(k+1)!}{r!(k-r+1)!}\left[\frac{d^{r}}{dw^{r}}\left(\frac{w}{\sin w}\right)^{k+2}\right]_{w=0} x^{k-r+1}\right\}$$
$$\equiv \sum_{r=0}^{k+1}\left[\sum \alpha_1\alpha_2 \cdots \alpha_{r}\right] x^{k-r+1}$$

which is what results from (2) when k is replaced by $k+2$. To do this, it suffices to show that the corresponding coefficients of the like powers of x in (9) and in the left member of (8) are equal. In other words, it is sufficient to show that

$$\begin{aligned} &\frac{k^2(k-1)!}{r!(k-r-1)!}\left[\frac{d^{r}}{dw^{r}}\left(\frac{w}{\sin w}\right)^{k}\right]_{w=0} \\ &+\frac{(k-1)!(k-r-2)(k-r-1)}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^{k}\right]_{w=0} \\ &=\frac{(k+1)!}{(r+2)!(k-r-1)!}\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^{k+2}\right\}_{w=0}. \end{aligned} \tag{10}$$

Now, let us assume for the moment the correctness of (10). If we multiply both its members by

$$\frac{(r+2)!(k-r-1)!}{(k-1)!},$$

we obtain the equivalent relation

$$\begin{aligned} &(k-r-2)(k-r-1)\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^{k}\right]_{w=0} \\ &\qquad + k^2(r+1)(r+2)\left[\frac{d^{r}}{dw^{r}}\left(\frac{w}{\sin w}\right)^{k}\right]_{w=0} \\ &\qquad\qquad - k(k+1)\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{w}{\sin w}\right)^{k+2}\right]_{w=0} = 0. \end{aligned} \tag{11}$$

For convenience, let us now write k in the form $k=-p$, $p\leqq -2$. Then we may rewrite (11) in the form

$$(12)\quad (p+r+2)(p+r+1)\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{\sin w}{w}\right)^p\right]_{w=0} + p^2(r+1)(r+2)\left[\frac{d^r}{dw^r}\left(\frac{\sin w}{w}\right)^p\right]_{w=0} - p(p-1)\left[\frac{d^{r+2}}{dw^{r+2}}\left(\frac{\sin w}{w}\right)^{p-2}\right]_{w=0} = 0.$$

If p now be treated temporarily as a variable, it is a matter of observation to see that the left member of (12) is a rational polynomial in p of degree $r+4$. Hence, if it can be shown that the algebraic equation (12) is satisfied by every positive intéger p, its identical nature for all real values of p is established and hence the identical nature of (11) is established for all real values of k; in particular, for the given positive integral value with which we are concerned. In order to do this, it is desirable at this point to recall that

$$(13)\qquad \left[\frac{d^r}{dw^r}\left(\frac{\sin w}{w}\right)^p\right]_{w=0} = \frac{r!}{(r+p)!}\left[\frac{d^{r+p}}{dw^{r+p}}\sin^p w\right]_{w=0};$$

where p is any positive integer. This follows immediately from a consideration of

$$\frac{d^{r+p}}{dw^{r+p}}\left[w^p\left(\frac{\sin w}{w}\right)^p\right]_{w=0}.$$

In fact, if we employ the theorem of Leibnitz upon the derivative of the product within the bracket, it is seen that the substitution of $w=0$ causes all derivatives of w^p to vanish except the pth. The term corresponding to this derivative is

$$\frac{(r+p)!}{r!}\left[\frac{d^r}{dw^r}\left(\frac{\sin w}{w}\right)^p\right]_{w=0}.$$

Relation (13) then follows immediately.

Employing (13) upon each term of (12) gives

$$(14)\qquad \left[\frac{d^{r+p+2}}{d^{r+p+2}}\sin^p w\right]_{w=0} + p^2\left[\frac{d^{r+p}}{dw^{r+p}}\sin^p w\right]_{w=0} - p(p-1)\left[\frac{d^{r+p}}{dw^{r+p}}\sin^{p-2} w\right]_{w=0} = 0.$$

Now the first term of (14) may be written

$$\left[\frac{d^{r+p}}{dw^{r+p}}\left(\frac{d^2}{dw^2}\sin^p w\right)\right]_{w=0}, \tag{15}$$

which becomes, after actually obtaining the second derivative involved,

$$p(p-1)\left[\frac{d^{r+p}}{dw^{r+p}}\sin^{p-2}w\right]_{w=0} - p^2\left[\frac{d^{r+p}}{dw^{r+p}}\sin^p w\right]_{w=0}. \tag{16}$$

The substitution of this result in (14) causes it to vanish identically. It is now readily observed that it is possible to retrace our steps from this identity to establish the validity of (10).

In summarizing, we may say that we have thus established the equality of the corresponding coefficients of like powers of x in (9) and in the left member of (8). Thus (2) also holds true when k is replaced by $k+2$. Hence the induction is complete and the theorem is established.

ON THE REPRESENTATION OF A LORENTZ TRANSFORMATION BY MEANS OF TWO-ROWED MATRICES

By FRANCIS D. MURNAGHAN, Johns Hopkins University

The connection between quaternions and two rowed matrices is very well known[1] and an application of this connection to questions arising, in modern physical theory, from consideration of the spinning electron has been given in a very interesting recent paper by van der Waerden.[2] It is the purpose of the present note to treat in a very elementary way the ideas of van der Waerden and to give some results not mentioned in his paper.

If we have a two-rowed square matrix

$$\eta = \begin{Vmatrix} \eta_1^1 & \eta_2^1 \\ \eta_1^2 & \eta_2^2 \end{Vmatrix}$$

we may set $\eta_1^1 = ct - x$; $\eta_2^2 = ct + x$; $\eta_2^1 = y + iz$; $\eta_1^2 = y - iz$ and the determinant of the matrix then appears in the form $c^2t^2 - x^2 - y^2 - z^2$. If the four quantities (x, y, z, ct) are real the elements η_1^1, η_2^2 of the matrix η must be real whilst the elements η_2^1, η_1^2 must be conjugate imaginaries: $\eta_1^2 = \bar{\eta}_2^1$ (in technical terms the matrix η must be *Hermitian*). Let us now transform the matrix η into another matrix ζ by means of two non-singular matrices A and B as follows:

$$\zeta = A\eta B \tag{A}$$

and ask the conditions on the transforming matrices A and B that when (x, y, z, ct) are real the quantities (x', y', z', ct') derived from ζ by means of the equations

---

[1] Cf. L. E. Dickson, *Modern Algebraic Theories*, pp. 46, 110.

[2] B. L. van der Waerden, *Spinoranalyse*, Göttinger Nachrichten (1929), pp. 100–109.

$$\zeta_1^{\,1} = ct' - x'; \quad \zeta_2^{\,2} = ct' + x'; \quad \zeta_2^{\,1} = y' + iz'; \quad \zeta_1^{\,2} = y' - iz'$$

should also be real. We shall require also that the determinant of the matrix $\boldsymbol{B}$ be the reciprocal of the determinant of $\boldsymbol{A}$ so that the determinant of ζ will be equal to the determinant of η:

$$(B) \qquad c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2.$$

It is at once apparent that ζ is unaltered if the matrix $\boldsymbol{A}$ is multiplied by a scalar m provided $\boldsymbol{B}$ is multiplied by the reciprocal $1/m$ of the scalar so there is no lack of generality in taking $\boldsymbol{A}$ and $\boldsymbol{B}$ to be matrices whose determinants are unity and we shall suppose this agreed to. The equation (A) is equivalent to the four equations:

$$(C) \qquad \begin{aligned} \zeta_1^{\,1} &= a_1^{\,1}b_1^{\,1}\eta_1^{\,1} + a_1^{\,1}b_1^{\,2}\eta_2^{\,1} + a_2^{\,1}b_1^{\,1}\eta_1^{\,2} + a_2^{\,1}b_1^{\,2}\eta_2^{\,2} \\ \zeta_2^{\,1} &= a_1^{\,1}b_2^{\,1}\eta_1^{\,1} + a_1^{\,1}b_2^{\,2}\eta_2^{\,1} + a_2^{\,1}b_2^{\,1}\eta_1^{\,2} + a_2^{\,1}b_2^{\,2}\eta_2^{\,2} \\ \zeta_1^{\,2} &= a_1^{\,2}b_1^{\,1}\eta_1^{\,1} + a_1^{\,2}b_1^{\,2}\eta_2^{\,1} + a_2^{\,2}b_1^{\,1}\eta_1^{\,2} + a_2^{\,2}b_1^{\,2}\eta_2^{\,2} \\ \zeta_2^{\,2} &= a_1^{\,2}b_2^{\,1}\eta_1^{\,1} + a_1^{\,2}b_2^{\,2}\eta_2^{\,1} + a_2^{\,2}b_2^{\,1}\eta_1^{\,2} + a_2^{\,2}b_2^{\,2}\eta_2^{\,2} \end{aligned}$$

and if $\zeta_1^{\,1}$, $\zeta_2^{\,2}$ are to be real when $\eta_1^{\,1}$, $\eta_2^{\,2}$ are real whilst $\eta_2^{\,1}$, $\eta_1^{\,2}$ are conjugate imaginaries we must have

$$\begin{aligned} \bar{a}_1^{\,1}\bar{b}_1^{\,1} &= a_1^{\,1}b_1^{\,1}; \quad \bar{a}_1^{\,1}\bar{b}_1^{\,2} = a_2^{\,1}b_1^{\,1}; \quad \bar{a}_2^{\,1}\bar{b}_1^{\,2} = a_2^{\,1}b_1^{\,2} \\ \bar{a}_1^{\,2}\bar{b}_2^{\,1} &= a_1^{\,2}b_2^{\,1}; \quad \bar{a}_1^{\,2}\bar{b}_2^{\,2} = a_2^{\,2}b_2^{\,1}; \quad \bar{a}_2^{\,2}\bar{b}_2^{\,2} = a_2^{\,1}b_2^{\,2} \end{aligned}$$

which imply

$$b_1^{\,1} = \lambda\bar{a}_1^{\,1}; \quad b_2^{\,1} = \mu\bar{a}_1^{\,2}; \quad b_1^{\,2} = \lambda\bar{a}_2^{\,1}; \quad b_2^{\,2} = \mu\bar{a}_2^{\,2}; \quad \lambda \text{ and } \mu \textit{ real}.$$

In order that $\zeta_1^{\,2}$ may at the same time be the conjugate imaginary to $\zeta_2^{\,1}$ it is necessary and sufficient that $\mu=\lambda$ and then the fact that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are of determinant unity gives

$$\lambda^2 = 1 \quad \text{i.e.} \quad \lambda = \pm 1.$$

We may consider in detail merely the case where $\lambda=1$, the case $\lambda=-1$ being equivalent to this followed by a change of sign of ζ i.e. of (x', y', z', ct'). Hence our transformation (A) is of the type

$$(D) \qquad \zeta = A\eta\bar{A}'$$

where the prime attached to a matrix signifies the transposed matrix. It follows that to the two matrices $\pm A$ correspond the same transformation from η to ζ. The transformation from (x, y, z, ct) to (x', y', z', ct') induced by (D) is readily found from (C) on writing $b_1^{\,1}=\bar{a}_1^{\,1}$; $b_2^{\,1}=\bar{a}_1^{\,2}$; $b_1^{\,2}=\bar{a}_2^{\,1}$; $b_2^{\,2}=\bar{a}_2^{\,2}$ and using the relations

$$2x' = \zeta_2^{\,2} - \zeta_1^{\,1}; \quad 2ct' = \zeta_2^{\,2} + \zeta_1^{\,1}; \quad 2y' = \zeta_2^{\,1} + \zeta_1^{\,2}; \quad 2iz' = \zeta_2^{\,1} - \zeta_1^{\,2}.$$

It may be conveniently given by means of the following table:

Table I.

$\downarrow\!\!\rightarrow$	x	y	z	ct
x'	$\frac{1}{2}(a_2^2\bar{a}_2^2 - a_2^1\bar{a}_2^1 - a_1^2\bar{a}_1^2 + a_1^1\bar{a}_1^1)$	$\frac{1}{2}(a_1^2\bar{a}_2^2 - a_1^1\bar{a}_2^1 + a_2^2\bar{a}_1^2 - a_2^1\bar{a}_1^1)$	$\frac{1}{2i}(a_1^1\bar{a}_2^1 - a_1^2\bar{a}_2^2 - a_2^1\bar{a}_1^1 + a_2^2\bar{a}_1^2)$	$\frac{1}{2}(a_2^2\bar{a}_2^2 - a_2^1\bar{a}_2^1 + a_1^2\bar{a}_1^2 - a_1^1\bar{a}_1^1)$
y'	$\frac{1}{2}(a_2^2\bar{a}_2^1 + a_2^1\bar{a}_2^2 - a_1^2\bar{a}_1^1 - a_1^1\bar{a}_1^2)$	$\frac{1}{2}(a_1^2\bar{a}_2^1 + a_1'\bar{a}_2^2 + a_2^2\bar{a}_1^1 + a_2^1\bar{a}_1^2)$	$\frac{1}{2i}(a_2^2\bar{a}_1^1 - a_1^1\bar{a}_2^2 + a_2^1\bar{a}_1^2 - a_1^2\bar{a}_2^1)$	$\frac{1}{2}(a_2^2\bar{a}_2^1 + a_2^1\bar{a}_2^2 + a_1^2\bar{a}_1^1 + a_1^1\bar{a}_1^2)$
z'	$\frac{1}{2i}(a_2^1\bar{a}_2^2 - a_2^2\bar{a}_2^1 + a_1^2\bar{a}_1^1 - a_1^1\bar{a}_1^2)$	$\frac{1}{2i}(a_1^1\bar{a}_2^2 - a_1^2\bar{a}_2^1 - a_2^2\bar{a}_1^1 + a_2^1\bar{a}_1^2)$	$\frac{1}{2}(a_1^1\bar{a}_2^2 - a_1^2\bar{a}_2^1 + a_2^2\bar{a}_1^1 - a_2^1\bar{a}_1^2)$	$\frac{1}{2i}(a_2^1\bar{a}_2^2 - a_2^2\bar{a}_2^1 - a_1^2\bar{a}_1^1 + a_1^1\bar{a}_1^2)$
ct'	$\frac{1}{2}(a_2^2\bar{a}_2^2 + a_2^1\bar{a}_2^1 - a_1^2\bar{a}_1^2 - a_1^1\bar{a}_1^1)$	$\frac{1}{2}(a_1^2\bar{a}_2^2 + a_1^1\bar{a}_2^1 + a_2^2\bar{a}_1^2 + a_2^1\bar{a}_1^1)$	$\frac{1}{2i}(a_2^2\bar{a}_1^2 - a_1^2\bar{a}_2^2 + a_2^1\bar{a}_1^1 - a_1^1\bar{a}_2^1)$	$\frac{1}{2}(a_2^2\bar{a}_2^2 + a_2^1\bar{a}_2^1 + a_1^2\bar{a}_1^2 + a_1^1\bar{a}_1^1)$

This table furnishes us with a parametric representation of the sixteen coefficients of those real homogeneous linear transformations from (x, y, z, ct) to (x', y', z', ct') for which $c^2t'^2 - x'^2 - y'^2 - z'^2$ is equal to $c^2t^2 - x^2 - y^2 - z^2$. These are called Lorentz transformations and the representation is in terms of 4 complex parameters $(a_1^1, a_2^1, a_1^2, a_2^2)$ which are connected by the relation $a_1^1a_2^2 - a_2^1a_1^2 = 1$; in other words three of the complex parameters are arbitrary so that six real independent parameters are involved. A particular case of interest arises when the transforming matrix A, of determinant unity, is such that $\bar{A}' = A^{-1}$. In this case A is said to be *unitary*, and we shall denote it by $\boldsymbol{U}$. Since

$$A^{-1} = \begin{Vmatrix} a_2^2 & -a_2^1 \\ -a_1^2 & a_1^1 \end{Vmatrix}$$

it follows that $u_2^2 = \bar{u}_1^1$; $u_1^2 = -\bar{u}_2^1$ and our table reduces to

Table II.

$\downarrow\!\!\rightarrow$	x	y	z	ct
x'	$(u_1^1\bar{u}_1^1 - u_2^1\bar{u}_2^1)$	$-(u_1^1\bar{u}_2^1 + \bar{u}_1^1u_2^1)$	$i(u_2^1\bar{u}_1^1 - u_1^1\bar{u}_2^1)$	0
y'	$(u_1^1u_2^1 + \bar{u}_1^1\bar{u}_2^1)$	$\frac{1}{2}(u_1^{12} + \bar{u}_1^{12} - u_2^{12} - \bar{u}_2^{12})$	$\frac{1}{2i}(\bar{u}_1^{12} - u_1^{12} + \bar{u}_2^{12} - u_2^{12})$	0
z'	$i(\bar{u}_1^1\bar{u}_2^1 - u_1^1u_2^1)$	$\frac{1}{2i}(u_1^{12} - \bar{u}_1^{12} + \bar{u}_2^{12} - u_2^{12})$	$\frac{1}{2}(\bar{u}_2^{12} + u_2^{12} + u_1^{12} + \bar{u}_1^{12})$	0
ct'	0	0	0	1

(The fact that $ct' = ct$ is a consequence of the transformation

$$(E) \qquad \zeta = \boldsymbol{U}\eta\boldsymbol{U}^{-1}$$

is obvious without calculation since under such a transformation the first invariant or trace $\eta_\alpha^\alpha \equiv \eta_1^1 + \eta_2^2$ is invariant, i.e.

$$\zeta^{\alpha}_{\alpha} = u^{\alpha}_{\beta}\eta^{\beta}_{\delta}u^{-1\delta}_{\alpha} = (u^{\alpha}_{\beta}u^{-1\delta}_{\alpha})\eta^{\beta}_{\delta} = \eta^{\beta}_{\beta}).$$

This table gives a parametric representation of the nine coefficients of those real homogeneous linear transformations from (x, y, z) to (x', y', z') for which $x'^2+y'^2+z'^2=x^2+y^2+z^2$ i.e. of the real orthogonal transformations on three variables. The representation is in terms of two complex parameters (u_1^1, u_2^1) connected by the relation $u_1^1\bar{u}_1^1+u_2^1\bar{u}_2^1=1$ so that three real parameters are involved. On writing $u_1^1=\rho+i\lambda$; $u_2^1=-\nu+i\mu$ we secure the usual parametric representation of Rodrigues.[3]

If we write out explicitly the equation (D) we have

$$\zeta^{p}_{q} = a^{p}_{\alpha}\eta^{\alpha}_{\beta}\bar{a}'^{\beta}_{q}$$

and we may, if we wish, regard η^p_q as a (new kind of) mixed tensor of the second rank. If A is a real unitary matrix the definition is equivalent to the usual definition of elementary vector analysis. The general law of transformation of the new class of tensors is put sufficiently in evidence by the example:

$$\zeta^{pq}_{rst} = a^{p}_{\alpha}a^{q}_{\beta}\eta^{\alpha\beta}_{\gamma\delta\epsilon}\bar{a}'^{\gamma}_{r}\,\bar{a}'^{\delta}_{s}\,\bar{a}'^{\epsilon}_{t}$$

which is the law of transformation for a tensor which is contravariant of rank two and covariant of rank three. The tensors of rank one (or vectors) are now pairs of complex numbers which are transformed according to the formulae

$$\zeta^{p} = a^{p}_{\alpha}\eta^{\alpha};\ \zeta_{p} = \bar{a}'^{\alpha}_{p}\eta_{\alpha}$$

and these vectors are called spin-vectors. As we have seen, the ordinary four-vector, (x, y, z, ct) is derivable from a mixed spin-tensor of the second rank.

Canonical Form of the Lorentz Transformation

It is evident from Table I that when $a_2^1=0$, $a_1^2=0$ the transformation from (x, y, z, ct) to the variables (x', y', z', ct') takes a very simple form. Since the determinant of A is unity $a_2^2=1/a_1^1$ and if we set $a_1^1\bar{a}_1^1=\bar{e}^{\theta}$; $\bar{a}_1^1/a_1^1=e^{i\phi}$ (so that θ and ϕ are real) we find

$$(F)\qquad \begin{aligned} x' &= \cosh\theta\ x + \sinh\theta\ ct; & y' &= \cos\phi\ y + \sin\phi\ z \\ ct' &= \sinh\theta\ x + \cosh\theta\ ct; & z' &= -\sin\phi\ y + \cos\phi\ z. \end{aligned}$$

This is a transformation of coordinates due to a rotation of the (y, z) axes through an angle ϕ in their plane together with a (pseudo) rotation of the (x, ct) axes through an angle θ in their plane.

The general transformation (D) may be reduced to the simple case just discussed as follows. From $\zeta=A\eta\bar{A}'$ we have $B\zeta\bar{B}'=BAB^{-1}.B\eta\bar{B}'.\bar{B}'^{-1}A'\bar{B}'$ or

[3] Cf. Ames and Murnaghan, *Theoretical Mechanics*, p. 79 or L. E. Dickson, *Modern Algebraic Theories*, p. 100.

$$(D') \qquad \zeta_* = A_* \eta_* A_*'$$

where $\eta_* = B\eta\bar{B}'$; $\zeta_* = B\zeta\bar{B}'$; $A_* = BAB^{-1}$ and B has its determinant unity. The transformations from η to η_* and from ζ to ζ_* are the same Lorentz Transformation and (D') may be interpreted as the same transformation as (D) referred to new axes of coordinates. It is now merely necessary to show that a matrix B can, save in one exceptional case, be found so that $a_{*2}{}^1 = 0$, $a_{*1}{}^2 = 0$. Instead of calling upon the general theory of similar matrices[4] it is more useful for us to actually construct the transforming matrix B. Since the determinant of B is unity we have

$$B^{-1} = \begin{Vmatrix} b_2{}^2 & -b_2{}^1 \\ -b_1{}^2 & b_1{}^1 \end{Vmatrix}$$

and a glance at the equations (C) tells us that in order that $A_* = BAB^{-1}$ may have its elements $a_{*2}{}^1$, $a_{*1}{}^2$ equal to zero the ratio of $b_1{}^1$ to $b_2{}^1$ and of $b_1{}^2$ to $b_2{}^2$ must be roots of the quadratic equation

$$a_2{}^1\xi^2 + (a_2{}^2 - a_1{}^1)\xi - a_1{}^2 = 0.$$

Remembering that $a_1{}^1a_2{}^2 - a_2{}^1a_1{}^2 = 1$ the roots of this equation appear as

$$(\xi_1, \xi_2) = [(a_1{}^1 - a_2{}^2) \pm (I^2 - 4)^{1/2}]/2a_2{}^1,$$

where $I = a_1{}^1 + a_2{}^2$. Since the determinant of B is to be unity these two roots must be distinct so that, save for the exceptional cases where $I = \pm 2$, A can be reduced to the form A_*, with $a_{*2}{}^1 = 0$, $a_{*1}{}^2 = 0$ by means of the matrix B:

$$B = \begin{Vmatrix} \xi_1 b_2{}^1 & b_2{}^1 \\ \xi_2 b_2{}^2 & b_2{}^2 \end{Vmatrix}$$

where $(\xi_1 - \xi_2)b_2{}^1 b_2{}^2 = 1$. An easy calculation gives

$$a_{*1}{}^1 + a_{*2}{}^2 = I; \quad a_{*1}{}^1 a_{*2}{}^2 = 1$$

(results which are evident without calculation since both I and the determinant of A are invariant under the transformation $A_* = BAB^{-1}$). Hence $a_{*1}{}^1$ and $a_{*2}{}^2$ are the two roots of the quadratic equation $w^2 - Iw + 1 = 0$.

The four quantities $a_{*1}{}^1\bar{a}_{*1}{}^1$, $a_{*1}{}^1\bar{a}_{*2}{}^2$, $\bar{a}_{*1}{}^1 a_{*2}{}^2$, $a_{*2}{}^2\bar{a}_{*2}{}^2$ satisfy, therefore, the quartic equation whose roots are the four products of a root of $w^2 - Iw + 1 = 0$ by a root of $v^2 - \bar{I}v + 1 = 0$. Eliminating w and v between these two equations and the equation $\lambda = wv$ we obtain

$$(G) \qquad \lambda^4 - \lambda^3(I\bar{I}) + \lambda^2(I^2 + \bar{I}^2 - 2) - \lambda(I\bar{I}) + 1 = 0.$$

The four roots of this reciprocal equation are readily obtained by setting $\lambda + \lambda^{-1} = 2\mu$ when we find

---

[4] Cf. L. E. Dickson, *Modern Algebraic Theories*, p. 104.

$$(H) \qquad (\mu_1, \mu_2) = \tfrac{1}{4}[I\bar{I} \pm (I^2 - 4)^{1/2}(\bar{I}^2 - 4)^{1/2}]$$

so that μ_1 and μ_2 are real. If we mark in an auxiliary complex plane the points $(0, I^2, 4)$ and denote the lengths of the sides of the resulting triangle by (a, b, c) we have $(\mu_1, \mu_2) = (c \pm a)/b$ so that one of the values $\mu_1 \geqq 1$ whilst μ_2 has its numerical value less than or equal to 1. We may, therefore, set $\mu_1 = \cosh\theta$ $\mu_2 = \cos\phi(\theta, \phi$ real) and the four values of λ are

$$\lambda_1 = e^{-\theta}, \ \lambda_2 = e^{+\theta}; \ \lambda_3 = e^{-i\phi}, \ \lambda_4 = e^{+i\phi}.$$

(When I is real two of these values coincide with the common value unity; when the absolute value of I is less than 2 we have $\lambda_1 = \lambda_2 = 1$, i.e., $\theta = 0$ and when the absolute value of I is greater than 2 we have $\lambda_3 = \lambda_4 = 1$, i.e. $\phi = 0$). It follows that the Lorentz transformation appears in the canonical form (F) with (x_*, y_*, z_*, ct_*) written for (x, y, z, ct) and $(x_*', y_*', z_*', ct_*')$ for (x', y', z', ct').

The equation (G) arises in another connection which throws considerable light on the results already obtained. Let us ask if there exists a matrix η which is transformed by a given matrix A (in accordance with the law of transformation D) into a scalar multiple of itself i.e. $\zeta = \lambda\eta$ or

$$(I) \qquad \lambda\eta = A\eta\bar{A}'.$$

The four equivalent homogeneous equations in the quantities $(\eta_1^1, \eta_2^1, \eta_1^2, \eta_2^2)$ must have the determinant of their coefficients zero and so we obtain what may be called the *proper* or characteristic equation of the matrix A:

$$(J) \qquad \begin{vmatrix} a_1^1\bar{a}_1^1 - \lambda & a_1^1\bar{a}_2^1 & a_2^1\bar{a}_1^1 & a_2^1\bar{a}_2^1 \\ a_1^1\bar{a}_1^2 & a_1^1\bar{a}_2^2 - \lambda & a_2^1\bar{a}_1^2 & a_2^1\bar{a}_2^2 \\ a_1^2\bar{a}_1^1 & a_1^2\bar{a}_2^1 & a_2^2\bar{a}_1^1 - \lambda & a_2^2\bar{a}_2^1 \\ a_1^2\bar{a}_1^2 & a_1^2\bar{a}_2^2 & a_2^2\bar{a}_1^2 & a_2^2\bar{a}_2^2 - \lambda \end{vmatrix} = 0.$$

Denoting as before by I the first invariant or trace of the matrix A

$$I = a_1^1 + a_2^2$$

the proper equation[5] is (on remembering that the determinant of A is unity)

$$\lambda^4 - \lambda^3(I\bar{I}) + \lambda^2(I^2 + \bar{I}^2 - 2) - \lambda I\bar{I} + 1 = 0$$

It follows at once from (I) that $\lambda\eta_* = A_*\eta_*\bar{A}'_*$ where $A_* = \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1}$, $\eta_* = \boldsymbol{B}\eta\bar{\boldsymbol{B}}'$ so that λ is also a proper number of the *similar* matrix A_*, the corresponding proper matrix being obtained from η by a transformation of type (D). If the transforming matrix $\boldsymbol{B}$ is so chosen that $a_{*2}^{\ 1} = 0$, $a_{*1}^{\ 2} = 0$ (which can always be

[5] This definition of the proper-values (eigenwerte) of a matrix A is interesting because it is given without leaving the field of square matrices. The usual definition involves the introduction of vectors i.e. matrices of one column. Thus instead of (A) we have $Ax = y$ (explicitly, $a_\alpha^r x^\alpha = y^r$) and the proper numbers are those for which $y = \lambda x$.

done by a matrix $\boldsymbol{B}$ of determinant unity proved the invariant I of $\boldsymbol{A}$ is different from ± 2) the equation $\lambda\eta_* = A_*\eta_*\bar{A}_*'$ is equivalent to the equations

$$\lambda\eta_{*1}{}^1 = a_{*1}{}^1\bar{a}_{*1}{}^1\eta_{*1}{}^1;\ \lambda\eta_{*2}{}^1 = a_{*1}{}^1\bar{a}_{*2}{}^2\eta_{*2}{}^1;\ \lambda\eta_{*1}{}^2 = a_{*2}{}^2\bar{a}_{*1}{}^1\eta_{*1}{}^2;\ \lambda\eta_{*2}{}^2 = a_{*2}{}^2\bar{a}_{*2}{}^2\eta_{*2}{}^2$$

so that the four proper numbers are $\lambda_1 = a_{*1}{}^1\,\bar{a}_{*1}{}^1 = \bar{e}^{\theta}$; $\lambda_2 = a_{*2}{}^2\,\bar{a}_{*2}{}^2 = e^{\theta}$; $\lambda_3 = a_{*2}{}^2\,\bar{a}_{*1}{}^1 = e^{i\phi}$; $\lambda_4 = a_{*1}{}^1\,\bar{a}^*{}_2{}^2 = e^{-i\phi}$ (since the determinant of A_* is unity $a_{*2}{}^2 = 1/a_{*1}$). The corresponding proper matrices are

$$\eta_*{}^1 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix};\ \eta_*{}^2 = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix};\ \eta_*{}^3 = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix};\ \eta_*{}^4 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$$

An arbitrary matrix

$$\eta_* = \begin{Vmatrix} \eta_{*1}{}^1 & \eta_{*2}{}^1 \\ \eta_{*1}{}^2 & \eta_{*2}{}^2 \end{Vmatrix}$$

may be written in the form

$$\eta_* = \eta_{*1}{}^1\eta_*{}^1 + \eta_{*2}{}^2\eta_*{}^2 + \eta_{*1}{}^2\eta_*{}^3 + \eta_{*2}{}^1\eta_*{}^4$$

and when this is transformed by the matrix $\boldsymbol{A}$, in accordance with (D), we have

$$\zeta_* = \lambda_1\eta_{*1}{}^1\eta_*{}^1 + \lambda_2\eta_{*2}{}^2\eta_*{}^2 + \lambda_3\eta_{*1}{}^3\eta_*{}^2 + \lambda_4\eta_{*2}{}^1\eta_*{}^4$$

or explicitly,

$$\zeta_{*1}{}^1 = \lambda_1\eta_{*1}{}^1\ ;\ \zeta_{*2}{}^2 = \lambda_2\eta_{*2}{}^2\ ;\ \zeta_{*2}{}^1 = \lambda_4\eta_{*2}{}^1\ ;\ \zeta_{*1}{}^2 = \lambda_3\eta_{*1}{}^2$$

These equations are equivalent to the equations (F) with (x_*, y_*, z_*, ct_*) written for (x, y, z, ct) and (x_*', y_*', z_*', ct') for (x', y', z', ct').

The Special Lorentz Transformations for which $I = \pm 2$.

When the invariant $I = a_1{}^1 + a_2{}^2$ of the matrix $\boldsymbol{A}$ has either of the values ± 2 it is not possible to find a matrix $\boldsymbol{B}$ of determinant unity such that the matrix $A_* = \boldsymbol{BAB}^{-1}$ has both the elements $a_{*2}{}^1$, $a_{*1}{}^2$ equal to zero. Assuming that $a_2{}^1$ is not zero it is merely necessary to have $b_1{}^1 : b_2{}^1 = a_1{}^1 - a_2{}^2 : 2a_2{}^1$ in order that $a_{*2}{}^1 = 0$. Since $a_{*1}{}^1 + a_{*2}{}^2 = \pm 2$ and $a_{*1}{}^1 a_{*2}{}^2 = 1$ we have $a_{*1}{}^1 = \pm 1$; $a_{*2}{}^2 = \pm 1$ (it is sufficient to consider the case $I = 2$ since the same Lorentz transformation is furnished by the two matrices $\pm \boldsymbol{A}$). Hence $a_{*1}{}^1 = 1$; $a_{*2}{}^2 = 1$ and if $a_{*1}{}^2$ were zero $\boldsymbol{A}$ would be the unit matrix $\boldsymbol{E}$ so that we would have $\boldsymbol{BAB}^{-1} = \boldsymbol{E}$ i.e. $\boldsymbol{BA} = \boldsymbol{B}$ necessitating $\boldsymbol{B} = \boldsymbol{E}$. Hence the Lorentz transformations for which $I = \pm 2$ cannot be put in the canonical form (F) unless the transformation in question is the identity. When $\boldsymbol{A} = \boldsymbol{U}$ is unitary, so that the Lorentz transformation leaves t invariant, and $I = 2$ it follows from $u_2{}^2 = \bar{u}_1{}^1$; $u_1{}^2 = -\bar{u}_2{}^1$, $u_2{}^2 = 2 - u_1{}^1$, $u_1{}^1u_2{}^2 - u_2{}^1u_1{}^2 = 1$ that $\boldsymbol{U} = \boldsymbol{E}$ (for $u_2{}^1\bar{u}_2{}^1 = (1 - u_1{}^1)^2$ so that $u_1{}^1$ is real and therefore $= u_2{}^2$ so that $u_1{}^1 = u_2{}^2 = 1$ and $u_2{}^1 = 0 = u_1{}^2$). Hence there are no such unitary matrices, with determinant unity, other than the identity. This is not, however, the case for non-unitary matrices of order two. It is possible,

when $I=2$, to find axes of coordinates for which $a_{*2}^{1}=0$ and $a_{*1}^{2}=\epsilon$ is of arbitrarily small modulus but nevertheless it is impossible to make $\epsilon=0$. The following canonical form for the special Lorentz transformations, other than the identity, for which $I=2$ corresponds to $a_{*2}^{1}=0$, $a_{*1}^{2}=\epsilon$

$$(K)\qquad \begin{aligned} x' &= x + \tfrac{1}{2}(\epsilon+\bar{\epsilon})y + \tfrac{1}{2}(\epsilon-\bar{\epsilon})iz + \tfrac{1}{2}\epsilon\bar{\epsilon}(ct-x)\\ y' &= y + \tfrac{1}{2}(\epsilon+\bar{\epsilon})(ct-x)\\ z' &= z + \tfrac{1}{2}i(\epsilon-\bar{\epsilon})(ct-x)\\ ct' &= \tfrac{1}{2}\epsilon\bar{\epsilon}(ct-x) + \tfrac{1}{2}(\epsilon+\epsilon)y + \tfrac{1}{2}(\epsilon-\bar{\epsilon})iz + ct \end{aligned}$$

It is clear that $ct'-x'=ct-x$; $(\epsilon+\bar{\epsilon})z'-i(\epsilon-\bar{\epsilon})y'=(\epsilon+\bar{\epsilon})z-i(\epsilon-\bar{\epsilon})y$. The four characteristic numbers λ are all unity and the characteristic matrices are of the type

$$\begin{Vmatrix} 0 & -\bar{\epsilon} \\ \epsilon & \eta_{*2}^{2} \end{Vmatrix}$$

where η_{*2}^{2} is arbitrary.

SMALL OSCILLATIONS OF THE NEUTRAL HELIUM ATOM NEAR THE EQUILATERAL TRIANGLE POSITIONS

By H. E. BUCHANAN, Tulane University

Introduction. Various papers[1] have been published in recent years on the neutral helium atom considered from the standpoint of the classical mechanics. In all of them, except in Professor Crudeli's paper, the masses of the negatively charged electrons are considered to be infinitesimal compared to the mass of the nucleus. Professor Crudeli neglects the mutual attraction due to gravitation and in the last few paragraphs of his work he makes the mass of the central nucleus approach infinity in order to obtain a simplified form of his characteristic equation. His treatment of the complex algebra is very remarkable.

This paper treats the same problem that Professor Crudeli treated. It obtains all of his and the following new results:

1. The center of gravity integrals are used to reduce the order of the differential equations and the resulting degree of the characteristic equation is lowered from the eighteenth to the twelfth.

2. Certain checks are provided which insure the accuracy of the algebraic manipulations. These checks were lacking in Professor Crudeli's paper but his results are correct.

---

[1] [Rawles, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 618 to 630; Van Vleck, Bulletin of the National Research Council, vol. 10, part 4, (1926), Chapter 7; D. Buchanan, Transactions of the Royal Society of Canada, Series 3, vol. 23 (1929), pp. 227 to 245; U. Crudeli, Rendiconti Del Circolo Matematico Di Palermo, vol. 51 (1926), pp. 1 to 20; Langmuir, Physical Review, vol. 17 (1921), pp. 339 to 353.]

3. The characteristic exponents are found for finite values of the masses, a complete discussion of the nature of the small oscillations is given and certain new theorems proved.

The Differential Equations. Let the mass of the nucleus be m_1 and the masses of the electrons be m_2 and m_3 respectively. We assume a positive charge of e_1 on m_1 and negative charges of e_2 and e_3 on m_2 and m_3. We neglect the gravitational attraction between the masses. Assuming Coulomb's law and using axes rotating uniformly in the $\xi\eta$ plane we have

$$\tag{1}\begin{aligned}&\frac{d^2\xi_i}{dt^2}-2\omega\frac{d\eta_i}{dt}-\omega^2\xi_i+\frac{k^2}{m_i}\frac{\partial U}{\partial\xi_i}=0,\\ &\frac{d^2\eta_i}{dt^2}+2\omega\frac{d\xi_i}{dt}-\omega^2\eta_i+\frac{k^2}{m_i}\frac{\partial U}{\partial\eta_i}=0,\\ &\frac{d^2\zeta_i}{dt^2}+\frac{k^2}{m_i}\frac{\partial U}{\partial\zeta_i}=0,\quad i=1,2,3,\end{aligned}$$

where ω is the angular velocity and where

$$U=\frac{e_1e_2}{r_{12}}+\frac{e_1e_3}{r_{13}}-\frac{e_2e_3}{r_{23}}\,.$$

The Equilateral Triangle Solutions. We seek for a solution of equations (1) in which the three bodies remain at the vertices of an equilateral triangle and rotate in circles. In this event ξ_i, η_i and ζ_i are constants and their derivatives are zero. We choose $r_{ij}=1$. Such solutions exist if we can satisfy simultaneously the equations:

$$\tag{2}\begin{cases}-m_1\omega^2\xi_1+k^2e_1e_2(\xi_1-\xi_2)+k^2e_1e_3(\xi_1-\xi_3)=0,\\ -m_2\omega^2\xi_2+k^2e_1e_2(\xi_2-\xi_1)-k^2e_2e_3(\xi_2-\xi_3)=0,\\ -m_3\omega^2\xi_3+k^2e_1e_3(\xi_3-\xi_1)-k^2e_2e_3(\xi_3-\xi_2)=0,\end{cases}$$

$$\tag{3}\begin{cases}-m_1\omega^2\eta_1+k^2e_1e_2(\eta_1-\eta_2)+k^2e_1e_3(\eta_1-\eta_3)=0,\\ -m_2\omega^2\eta_2+k^2e_1e_2(\eta_2-\eta_1)-k^2e_2e_3(\eta_2-\eta_3)=0,\\ -m_3\omega^2\eta_3+k^2e_1e_3(\eta_3-\eta_1)-k^2e_2e_3(\eta_3-\eta_2)=0,\end{cases}$$

$$\tag{4}\begin{aligned}&e_2(\zeta_1-\zeta_2)+e_3(\zeta_1-\zeta_3)=0,\\ &e_1(\zeta_2-\zeta_1)-e_3(\zeta_2-\zeta_3)=0,\\ &e_1(\zeta_3-\zeta_1)-e_2(\zeta_3-\zeta_2)=0.\end{aligned}$$

Consider first the equations (2). One solution is $\xi_1=\xi_2=\xi_3=0$. In order that there may be any other solution we must have

$$\begin{vmatrix} -m_1\omega^2+k^2(e_1e_2+e_1e_3), & -k^2e_1e_2, & -k^2e_1e_3 \\ -k^2e_1e_2, & -m_2\omega^2+k^2(e_1e_2-e_2e_3), & k^2e_2e_3 \\ -k^2e_1e_3, & k^2e_2e_3, & -m_3\omega^2+k^2(e_1e_3-e_2e_3) \end{vmatrix} = 0.$$

Now if we make $m_2=m_3=m$ and $2e=e_1=2e_2=2e_3$, replace the first column by the sum of all the columns, and remove the common factor ω^2 we shall have

$$\begin{vmatrix} m_1 & 2k^2e^2 & 2k^2e^2 \\ m & m\omega^2-k^2e^2 & -k^2e^2 \\ m & -k^2e^2 & m\omega^2-k^2e^2 \end{vmatrix} = 0.$$

The solution of this equation is

$$\omega^2 = 2k^2e^2(2m+m_1)/mm_1.$$

The values of ξ_1 and ξ_2 in terms of ξ_3 may be found from any two of equations (2). They are

$$\xi_2 = \xi_3 \text{ and } \xi_1 = -2m\xi_3/\eta_1.$$

Equations (3) can be solved in exactly the same way as equations (2) and the result is obviously the same. It is simpler however to satisfy equations (3), by $\eta_1=\eta_2=\eta_3=0$. The ζ_i equations can be satisfied by

$$\zeta_1 = 0, \quad \zeta_2 = -\zeta_3 = \tfrac{1}{2}.$$

Thus it appears that in order that the circular equilateral triangle solutions exist it is necessary that the plane of the triangle be the $\xi\zeta$ plane and that the rotation be around the ζ axis. This is equivalent to saying that the electrons and the nucleus move in parallel circles, and the line joining the centers of these circles is perpendicular to their planes.

The Integrals of Equations. (1). The usual methods of celestial mechanics may be applied to equations (1) since U is unchanged by shifting or by rotating the axes. Hence the ten integrals of the celestial mechanics exist. In particular the center of gravity integrals become

$$m_1\xi_1 + m(\xi_2+\xi_3) = 0, \text{ or } \xi_2 = -\frac{m_1\xi_1}{m} - \xi_3,$$

$$m_1\eta_1 + m(\eta_2+\eta_3) = 0, \text{ or } \eta_2 = -\frac{m_1\eta_1}{m} - \eta_3,$$

$$m_1\zeta_1 + m(\zeta_2+\zeta_3) = 0, \text{ or } \zeta_2 = -\frac{m_1\zeta_1}{m} - \zeta_3.$$

The Equations for Small Oscillations. We call the coordinates in the equilateral triangle positions $(\xi_i^{(0)}, 0, \zeta_i^{(0)})$, ζ_1^0 being zero, and make the transformation

$$\xi_i = \xi_i^{(0)} + x_i, \quad \eta_i = y_i, \quad \zeta_i = \zeta_i^{(0)} + z_i.$$

We eliminate x_2, y_2, z_2 wherever they occur by the center of gravity equations. Then equations (1) become:

$$\frac{d^2x_1}{dt^2} - 2\omega\frac{dy_1}{dt} = \omega^2(\xi_1^{(0)} + x_1) - \frac{k^2e^2}{m_1}\left\{\frac{2(\xi_1^{(0)} + x_1 - \xi_2^{(0)} - x_2)}{r_{12}} + \frac{2(\xi_1^{(0)} + x_1 - \xi_3^{(0)} - x_3)}{r_{13}^3}\right\},$$

$$\frac{d^2x_3}{dt^2} - 2\omega\frac{dy_3}{dt} = \omega^2(\xi_3^{(0)} + x_3) - \frac{k^2e^2}{m}\left\{\frac{2(\xi_3^{(0)} + x_3 - \xi_1^{(0)} - x_1)}{r_{13}^3} - \frac{(\xi_3^{(0)} + x_3 - \xi_2^{(0)} - x_2)}{r_2^3}\right\},$$

$$\text{(5)}\quad \frac{d^2y_1}{dt^2} + 2\omega\frac{dx_1}{dt} = \omega^2 y_1 - \frac{2k^2e^2}{m_1}\left\{\frac{y_1 - y_2}{r_{12}^3} + \frac{y_1 - y_3}{r_{13}^3}\right\},$$

$$\frac{d^2y_3}{dt^2} + 2\omega\frac{dx_3}{dt} = \omega^2 y_3 - \frac{k^2e^2}{m}\left\{\frac{2(y_3 - y_1)}{r_{13}^3} - \frac{y_3 - y_2}{r_{23}^3}\right\},$$

$$\frac{d^2z_1}{dt^2} = -\frac{2k^2e^2}{m_1}\left\{\frac{z_1 - \zeta_2^{(0)} - z_2}{r_{12}^3} + \frac{z_1 - \zeta_3^{(0)} - z_3}{r_{13}^3}\right\},$$

$$\frac{d^2z_3}{dt^2} = -\frac{k^2e^2}{m}\left\{\frac{2(\zeta_3^{(0)} + z_3 - z_1)}{r_{13}^3} - \frac{\zeta_3^{(0)} + z_3 - \zeta_2^{(0)} - z_2}{r_{23}^3}\right\}.$$

We expand the right members into power series in x_i, y_i, z_i, $i=1, 3$. When only the linear terms in these series are considered we have the following system:

$$\text{(6)}\quad
\begin{aligned}
\frac{d^2x_1}{dt^2} - 2\omega\frac{dy_1}{dt} &= A_{11}x_1 + A_{13}x_3 + B_{11}y_1 + B_{13}y_3 + C_{11}z_1 + C_{13}z_3,\\
\frac{d^2x_3}{dt^2} - 2\omega\frac{dy_3}{dt} &= A_{21}x_1 + A_{23}x_3 + B_{21}y_1 + B_{23}y_3 + C_{21}z_1 + C_{23}z_3,\\
\frac{d^2y_1}{dt^2} + 2\omega\frac{dx_1}{dt} &= A_{31}x_1 + A_{33}x_3 + B_{31}y_1 + B_{33}y_3 + C_{31}z_1 + C_{33}z_3,\\
\frac{d^2y_3}{dt^2} + 2\omega\frac{dx_3}{dt} &= A_{41}x_1 + A_{43}x_3 + B_{41}y_1 + B_{43}y_3 + C_{41}z_1 + C_{43}z_3,\\
\frac{d^2z_1}{dt^2} &= A_{51}x_1 + A_{53}x_3 + B_{51}y_1 + B_{53}y_3 + C_{51}z_1 + C_{53}z_3,\\
\frac{d^2z_3}{dt^2} &= A_{61}x_1 + A_{63}x_3 + B_{61}y_1 + B_{63}y_3 + C_{61}z_1 + C_{63}z_3,
\end{aligned}$$

where

$$A_{11} = \frac{9}{4}\omega^2, \quad A_{13} = 0, \quad B_{11} = 0, \quad B_{13} = 0,$$

$$C_{11} = \frac{-3\sqrt{3}\cdot k^2e^2}{2m}, \quad C_{13} = -\frac{3\sqrt{3}\cdot k^2e^2}{m_1},$$

$$A_{21} = -\frac{k^2e^2(5m - 2m_1)}{2m^2}, \quad A_{23} = \omega^2 + \frac{9k^2e^2}{2m},$$

$$B_{21} = 0, \quad B_{23} = 0, \quad C_{21} = \frac{-3\sqrt{3}\cdot k^2e^2}{2m}, \quad C_{23} = \frac{3\sqrt{3}\cdot k^2e^2}{2m},$$

$$A_{31} = 0, \quad A_{33} = 0, \quad B_{31} = 0, \quad B_{33} = 0, \quad C_{31} = 0, \quad C_{33} = 0,$$

$$A_{41} = 0, \quad A_{43} = 0, \quad B_{41} = \frac{m_1\omega^2}{2m}, \quad B_{43} = \omega^2, \quad C_{41} = 0, \quad C_{43} = 0,$$

$$A_{51} = \frac{-3\sqrt{3}\cdot k^2e^2}{2m}, \quad A_{53} = \frac{-3\sqrt{3}\cdot k^2e^2}{m_1}, \quad B_{51} = 0, \quad B_{53} = 0,$$

$$C_{51} = -\tfrac{1}{4}\omega^2, \quad C_{53} = 0, \quad A_{61} = \frac{3\sqrt{3}\cdot k^2e^2}{2m}, \quad A_{63} = \frac{-3\sqrt{3}\cdot k^2e^2}{2m}$$

$$B_{61} = 0, \quad B_{63} = 0, \quad C_{61} = \frac{-k^2e^2(4m_1 - m)}{2m^2}, \quad C_{63} = \frac{-9k^2e^2}{2m}.$$

The Characteristic Equation. The solution of the system (6) is found by substituting

$$x_i = K_i e^{\lambda t}, \quad y_i = L_i e^{\lambda t}, \quad z_i = M_i e^{\lambda t}, \quad i = 1, 3.$$

In order that there shall be any solution different from

$$K_i = L_i = M_i = 0, \quad i = 1, 3,$$

it is necessary that equations (7) shall hold.

$$(7)\quad \begin{vmatrix} -\lambda^2 + \frac{9\omega^2}{4} & 0 & 2\omega\lambda & 0 & \frac{-3\sqrt{3}\cdot k^2e^2}{2m} & \frac{-3\sqrt{3}\cdot k^2e^2}{m_1} \\ \frac{-k^2e^2(5m-2m_1)}{2m^2} & -\lambda^2 + \omega^2 + \frac{9k^2e^2}{2m} & 0 & 2\omega\lambda & \frac{-3\sqrt{3}\cdot k^2e^2}{2m} & \frac{3\sqrt{3}\cdot k^2e^2}{2m} \\ -2\omega\lambda & 0 & -\lambda^2 & 0 & 0 & 0 \\ 0 & -2\omega\lambda & \frac{m_1\omega^2}{m} & -\lambda^2+\omega^2 & 0 & 0 \\ \frac{-3\sqrt{3}\cdot k^2e^2}{2m} & \frac{-3\sqrt{3}\cdot k^2e^2}{m_1} & 0 & 0 & -\lambda^2 - \frac{1}{4}\omega^2 & 0 \\ \frac{-3\sqrt{3}\cdot k^2e^2}{2m} & \frac{3\sqrt{3}\cdot k^2e^2}{2m} & 0 & 0 & \frac{-k^2e^2(4m_1-m)}{2m^2} & -\lambda^2 - \frac{9k^2e^2}{2m} \end{vmatrix} = 0.$$

The radical $\sqrt{3}$ can be removed by multiplying the last two columns by $\sqrt{3}$ and then dividing the last two rows by $\sqrt{3}$. Let

$$\omega^2 = k^2e^2u^2, \quad \lambda = Kex, \quad \text{then} \quad u^2 = 2(2m + m_1/mm_1.$$

After removing the factor K^2e^2, and reducing by the factor λ, which appears in the third row, and replacing the first column by the difference of the first and $2\omega/\lambda$ times the third we find

$$(8) \qquad \begin{vmatrix} -x^2-\dfrac{7u^2}{4} & 0 & 2ux & 0 & -\dfrac{9}{2m} & -\dfrac{9}{m_1} \\ \dfrac{2m_1-5m}{2m^2} & -x^2+u^2+\dfrac{9}{2m} & 0 & 2ux, & -\dfrac{9}{2m} & \dfrac{9}{2m} \\ 0 & 0 & -x & 0 & 0 & 0 \\ \dfrac{-2m_1u^3}{mx.} & -2ux & \dfrac{m_1u^2}{m} & -x^2+u^2 & 0 & 0 \\ -\dfrac{3}{2m} & -\dfrac{3}{m_1} & 0 & 0 & -x^2-\dfrac{u^2}{4} & 0 \\ -\dfrac{3}{2m} & \dfrac{3}{2m} & 0 & 0 & \dfrac{m-4m_1}{2m^2} & -x^2-\dfrac{9}{2m} \end{vmatrix} = 0.$$

Equation (8) shows another root $x=0$. Hence (7) has the double root 0.

Expanding (8) by the minors of the third row and multiplying and dividing by x we obtain the fifth order determinant

$$(9) \qquad \begin{vmatrix} -x^2-\dfrac{7u^2}{4} & 0 & 0 & -\dfrac{9}{2m} & -\dfrac{9}{m_1} \\ \dfrac{2m_1-5m}{2m^2} & -x^2+u^2+\dfrac{9}{2m} & 2u & -\dfrac{9}{2m} & \dfrac{9}{2m} \\ \dfrac{-2m_1u^3}{m} & -2ux^2 & -x^2+u^2 & 0 & 0 \\ -\dfrac{3}{2m} & -\dfrac{3}{m_1} & 0 & -x^2-\frac{1}{4}u^2 & 0 \\ -\dfrac{3}{2m} & \dfrac{3}{2m} & 0 & \dfrac{m-4m_1}{2m^2} & -x^2-\dfrac{9}{2m} \end{vmatrix} = 0.$$

The mechanics of the situation leads us to suspect that equation (9) should have the root $x^2 = -u^2$. Making this substitution, replacing the second row by the difference between the second and third rows and removing certain common factors the left member of (9) becomes

$$
\begin{vmatrix}
-\dfrac{3u^2}{4} & 0 & -\dfrac{9}{2m} & -\dfrac{9}{m_1} \\
\dfrac{11m+10m_1}{m} & 9 & -9 & 9 \\
-\dfrac{1}{2m} & -\dfrac{1}{m_1} & \frac{1}{4}u^2 & 0 \\
-\dfrac{3}{2m} & \dfrac{3}{2m} & \dfrac{m-4m_1}{2m^2} & u^2-\dfrac{9}{2m}
\end{vmatrix}.
$$

Replace the third column by the difference of the third and fourth.

Now multiplying the third column by $4/(m_1u)$ and subtracting from the fourth for a new fourth column it appears that the fourth and second columns are identical. Hence the determinant vanishes and equation (7) has the roots $\lambda = \pm\iota\omega$. We shall use this fact as a check on our expansion of (9) later. As a check on the correctness of the operations on equation (7) the author has substituted $\lambda = \pm\iota\omega$ in that determinant and found that it is satisfied.

We now expand (9) by the elements of the third column and each of the resulting fourth order determinants by the minors of the elements of the last column. This gives after some reductions

$$
-\frac{36u^2}{m_1}\Delta_1 + 4u^2\left(x^2+\frac{9}{2m}\right)\Delta_2 + (x^2-u^2)\left(x^2+\frac{7u^2}{4}\right)\Delta_3 + \frac{27u^2}{4}(x^2-u^2)\Delta_4 = 0,
$$

where

$$
\Delta_1 = \begin{vmatrix}
x^2+\dfrac{m_1u^2}{m} & x^2 & 0 \\
\dfrac{3u^2}{4} & \dfrac{3}{m_1} & x^2+\frac{1}{4}u^2 \\
0 & \dfrac{3}{2m} & -x^2-\dfrac{m_1u^2}{m}
\end{vmatrix}, \quad
\Delta_2 = \begin{vmatrix}
x^2+\dfrac{7u^2}{4} & 0 & \dfrac{9u^2}{4} \\
\dfrac{m_1u^2}{m} & x^2 & 0 \\
\dfrac{3}{2m} & \dfrac{3}{m_1} & x^2+\dfrac{u^2}{4}
\end{vmatrix},
$$

$$
\Delta_3 = \begin{vmatrix}
-x^2+u^2+\dfrac{9}{2m} & 0 & -\dfrac{9}{2m} \\
\dfrac{3}{m_1} & x^2+\frac{1}{4}u^2 & 0 \\
-\dfrac{3}{2m} & x^2+\dfrac{m_1u^2}{m} & x^2+\dfrac{9}{2m}
\end{vmatrix},
$$

$$\Delta_4 = \begin{vmatrix} -x^2+\dfrac{m_1u^4}{4} & -x^2+u^2+\dfrac{9}{2m} & +\dfrac{9}{2m} \\ \tfrac{1}{4}u^2 & \dfrac{1}{m_1} & 0 \\ 0 & \dfrac{3}{2m} & x^2+\dfrac{9}{2m} \end{vmatrix} = 0.$$

Expanding these determinants and collecting according to powers of x we find the following equation of the 5th degree in x^2:

$$\begin{aligned}(10)\quad x^{10} + 4u^2x^8 &+ \left(\frac{87u^4}{16} + \frac{81u^2}{8m} - \frac{27}{2mm_1} - \frac{27}{m^2}\right)x^6 \\ &+ \left(-\frac{31u^6}{8} + \frac{279u^4}{8m} + \frac{27u^4}{m_1} - \frac{81u^2}{4m^2}\right)x^4 \\ &+ \left(\frac{7u^8}{16} + \frac{369u^6}{32m} + \frac{135u^4}{8mm_1} + \frac{459u^4}{16m^2}\right)x^2 + \frac{9u^8}{8m} + \frac{81u^6}{4m^2} = 0.\end{aligned}$$

This equation should have the root $x^2 = -u^2$. Substituting and remembering that $u^2 = 2(2m+m_1)/mm_1$ it is fairly easy to verify that $x^2 = -u^2$ is a root of (10). Thus the degree of equation (10) can be reduced by dividing the left member by x^2+u^2. There results the following equation of the 4th degree.

$$(11)\quad \begin{aligned} x^8 + 3u^2x^6 &+ \left(\frac{15u^2}{m} + \frac{39u^2}{4m_1} - \frac{27}{2mm_1} - \frac{27}{m^2}\right)x^4 \\ &+ \left(\frac{31u^4}{2m} + \frac{7u^4}{4m_1}\right)x^2 + \frac{9u^6}{4m} + \frac{81u^4}{4m^2} = 0.\end{aligned}$$

If we choose the unit of mass so that $m=1$ and replace $1/m_1$ by y, u^2 becomes $2(2y+1)$ and equation (11) reduces to

$$\begin{aligned}(12)\quad x^8 + 6(2y+1)x^6 &+ 3(13y^2+22y+1)x^4 \\ &+ (2y+1)^2(7y+62)x^2 + 18(2y+1)^2(y+5) = 0.\end{aligned}$$

For $y=0$ (12) becomes

$$(13)\qquad x^8 + 6x^6 + 3x^4 + 62x^2 + 90 = 0,$$

which factors readily into

$$(x^4 - 2x^2 + 10)(x^4 + 8x^2 + 9) = 0.$$

It follows that the roots of (12) for $y=0$ are

$$x^2 = 1 \pm 3\iota,\ x^2 = -4 \pm \sqrt{7},\ \iota = \sqrt{(-1)}$$

The Characteristic Exponents, $y \neq 0$. The parameter y is the ratio of the mass of one of the electrons to the mass of the nucleus. For our purposes it may be taken as .00014. We seek a solution of (12) in the form $x^2 = A + By \cdots$, the smallness of y making it unnecessary to go further than the first power. Direct substitution and equation to zero the coefficient of y gives

$$B = -\frac{3(4A^3 + 22A^2 + 85A + 126)}{2(2A^3 + 9A^2 + 3A + 31)}.$$

Hence

$$x^2 = 1 + 3\iota + \frac{-5 + 9\iota}{2} y, \quad x^2 = -4 + \sqrt{7} - \left(\frac{7}{2} - \frac{5}{7}\sqrt{7}\right) y,$$

$$x^2 = 1 - 3\iota + \frac{-5 - 9\iota}{2} y, \quad x^2 = -4 - \sqrt{7} - \left(\frac{7}{2} + \frac{5}{7}\sqrt{7}\right) y.$$

When y is given the numerical value above and the square root of the result is taken we find

$$x = \pm(\alpha + \beta\iota), \quad x = \pm \nu\iota,$$
$$x = \pm(\alpha - \beta\iota), \quad x = \pm \rho\iota,$$

where the numerical values of α, β, ν and ρ can be easily found.

It is clear from the above computations that the characteristic exponents for $y \neq 0$ differ very slightly from their values for $y = 0$.

The solutions of equations (7) are now all known. We group them together here for reference

$$\lambda = 0,\ 0,\ \pm \omega\iota,\ \pm Ke(\alpha + \iota\beta),\ \pm Ke(\alpha - \iota\beta),\ \pm Ke\nu\iota,\ \pm Ke\rho\iota.$$

The Solution of Equations (6). From the above values of the characteristic exponents we can write the general solutions of equations (6).

$$\begin{aligned} x_i &= Ki_1 + Ki_2 t + Ki_3 \epsilon^{+\omega\iota t} + Ki_4 \epsilon^{-\omega\iota t} + Ki_5 \epsilon^{Ke\nu\iota t} + Ki_6 \epsilon^{-Ke\nu\iota t} + Ki_7 \epsilon^{Ke\rho\iota t} \\ &\quad + Ki_8 \epsilon^{-Ke\rho\iota t} + \epsilon^{Ke\alpha t}(Ki_9 \epsilon^{Ke\beta\iota t} + Ki_{10} \epsilon^{-Ke\beta\iota t}) + \epsilon^{-Ke\alpha t}(Ki_{11} \epsilon^{Ke\beta\iota t} + Ki_{12} \epsilon^{-Ke\beta\iota t}). \\ y_i &= Li_1 + Li_2 t + Li_3 \epsilon^{\omega\iota t} + Li_4 \epsilon^{-\omega\iota t} + Li_5 \epsilon^{Ke\nu\iota t} + Li_6 \epsilon^{-Ke\nu\iota t} + Li_7 \epsilon^{Ke\rho\iota t} \\ (15) &\quad + Li_8 \epsilon^{-Ke\rho\iota t} + \epsilon^{Ke\alpha t}(Li_9 \epsilon^{Ke\beta\iota t} + Li_{10} \epsilon^{-Ke\beta\iota t}) + \epsilon^{-Ke\alpha t}(Li_{11} \epsilon^{Ke\beta\iota t} + Li_{12} \epsilon^{-Ke\beta\iota t}). \\ z_i &= Mi_1 + Mi_2 t + Mi_3 \epsilon^{\omega\iota t} + Mi_4 \epsilon^{-\omega\iota t} + Mi_5 \epsilon^{Ke\nu\iota t} + Mi_6 \epsilon^{-Ke\nu\iota t} + Mi_7 \epsilon^{Ke\rho\iota t} \\ &\quad + Mi_8 \epsilon^{-Ke\beta\iota t} + \epsilon^{Ke\alpha t}(Mi_9 \epsilon^{Ke\beta\iota t} + Mi_{10} \epsilon^{-Ke\beta\iota t}) + \epsilon^{-Ke\alpha t}(Mi_{11} \epsilon^{Ke\beta\iota t} + Mi_{12} \epsilon^{-Ke\beta\iota t}). \end{aligned}$$

where ϵ is the Naperian base and $i = 1, 3$.

All of the terms of these solutions are periodic except the linear term in t and those terms involving $\epsilon^{\pm ke\alpha t}$. If the system is disturbed in any way it will, in general, break up because of the non-periodic terms. For general initial con-

ditions there will appear, in addition to the terms linear in t, oscillations of the following types:

1. Periodic, period $2\pi/\omega = 2\pi/(keu)$,
2. Periodic, period $2\pi/(kev)$,
3. Periodic, period $2\pi/(ke\rho)$,
4. Non-periodic damped vibrations arising from the terms $\epsilon^{-ke(\alpha\pm\iota\beta)t}$,
5. Non-periodic vibrations arising from the terms $\epsilon^{ke(\alpha\pm\iota\beta)t}$.

From the above results the following theorem is apparent:

In small oscillations of the neutral helium atom, near the equilateral triangle positions, the periods of the periodic terms are inversely proportional to the charge on each electron.

This theorem applies equally as well to the periodic portions of the non-periodic terms, excepting, of course, the linear terms in t.

There appear 72 arbitrary constants in the solutions (15). By well known methods we may choose twelve of them as the constants of integration and the remaining ones can be found as linear functions of the twelve. We choose L_{1j}, $j=1 \cdots 12$ arbitrarily.

The determination of K_{12}, K_{32}, L_{32}, M_{12}, and M_{32} in terms of L_{12} is of particular interest. To effect this we substitute in equations (6):

$$x_i = Ki_1 + Ki_2t, \quad y_i = Li_1 + Li_2t, \quad z_i = Mi_1 + Mi_2t, \quad i = 1, 3.$$

Since these must satisfy (6) identically in t, the coefficient of t and the constant terms must vanish. We find

$$K_{12} = K_{32} = M_{12} = M_{32} = 0, \quad L_{32} = (-m_1/2m)L_{12}.$$

Hence if the disturbance which produces the oscillations has no component away from the plane of the equilateral triangle L_{12} and L_{32} will be zero and there will be no linear term in t.

Let the initial condition be

$$x_i = a_i, \quad x_i' = b_i, \quad y_i = c_i, \quad y_i' = d_i, \quad z_i = b_i, \quad z_i' = g_i \text{ at } t = 0, \quad i = 1, 3.$$

Applying these conditions to equations (15) it is apparent that the twelve arbitrary constants L_{1j}, $j=1 \cdots 12$ are linear functions of the initial constants. Therefore in order that each term of the solutions be periodic it is necessary to impose five linear conditions on the twelve initial constants. Seven of the twelve initial constants may be chosen arbitrarily and the other five may be determined uniquely by the conditions

$$L_{1,2} = 0, \quad L_{1,9} = 0, \quad L_{1,10} = 0, \quad L_{1,11} = 0, \quad L_{1,12} = 0.$$

If these conditions are satisfied each term of the solutions will be periodic with the periods

$$2\pi/\omega, \quad 2\pi/(kev), \quad 2\pi/(Ke\rho).$$

By imposing the three linear conditions on the initial constants, $L_{1,2}=0$, $L_{1,9}=0$, $L_{1,10}=0$ the solutions may be made to contain only periodic terms and damped vibrations. *Therefore nine of the initial constants may be chosen arbitrarily and the system consisting of the nucleus and two electrons will have only periodic oscillations or damped vibrations near the equilateral triangle positions.*

A NOTE PERTAINING TO KEPLER'S THIRD LAW

By D. H. RICHERT, Bethel College

The purpose of this note is to present in a simple way the derivation of Kepler's third law as corrected by Newton.

It is assumed that the orbit of a planet is circular so that the acceleration is along the normal.

Let M be the mass of the sun, m_1 the mass of a planet, T_1 its period and a_1 the mean distance between the centers of mass of sun and planet.

If a_1 is the radius of the circular orbit and T_1 the period, then the acceleration imparted by the sun to a unit mass of the planet is given by

$$GM/a_1^2, \tag{1}$$

where G is the constant of gravitation. Since this acceleration is along the normal it is equal to

$$(2\pi/T_1)^2\cdot a_1, \quad \text{for} \quad v^2/a_1 = (2\pi a_1/T_1)^2(1/a_1). \tag{2}$$

Setting (1) equal to (2) gives

$$GM/a_1^2 = (2\pi/T_1)^2\cdot a_1. \tag{3}$$

Similarly for a second planet of mass m_2, period T_2 and mean distance a_2 the relation

$$GM/a_2^2 = (2\pi/T_2)^2\cdot a_2 \tag{4}$$

holds. From (3) and (4) follows Kepler's third law:

$$T_1^2 : T_2^2 = a_1^3 : a_2^3\,. \tag{5}$$

A planet, however, describes its orbit not around the center of the sun but around the common center of gravity of sun and planet. If r_1 is the distance of the first planet from this common center of gravity, we must replace (2) by

$$(2\pi/T_1)^2\cdot r_1. \tag{6}$$

Now $m_1 r_1 = M(a_1 - r_1)$, hence

$$r_1 = [M/(M + m_1)]a_1.$$

Using this value of r_1 (6) reduces to

$$(2\pi/T_1)^2[M/(M + m_1)]a_1. \tag{7}$$

Setting (1) equal to (7) gives

(8) $$GM/a_1^2 = (2\pi/T_1)[M/(M+m_1)]a_1,$$

or

(9) $$G(M+m_1)/a_1^3 = (2\pi/T_1)^2.$$

Similarly for the second planet the relation

(10) $$G(M+m_2)/a_2^3 = (2\pi/T_2)^2.$$

From (9) and (10) follows

(11) $$T_1^2(M+m_1):T_2^2(M+m_2) = a_1^3:a_2^3,$$

which is Kepler's third law as corrected by Newton.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A MNEMONIC FOR EULER'S CONSTANT

By MORGAN WARD, California Institute of Technology

The first ten digits for Euler's constant may be remembered by aid of the following mnemonic:

"These numbers proceed to a limit Euler's subtle mind discerned."
5 7 7 2 1 5 6 6 4 9

Since the next significant figure is a zero, the sentence comes to a very natural stopping place.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Contributions to the Calculus of Variations. The University of Chicago Press, 1930. Lithoprinted. 350 pages, $3.00.

A collection of five dissertations in the field of Calculus of Variations, reproduced directly from type-written copy.

Fourier'sche Reihen, mit Aufgaben. Von J. Wolff, Utrecht. Groningen, P. Noordhoff N.V., 1931. 60 pages.

Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integrodifferentialgleichungen. Von Leon Lichtenstein, Leipzig. Berlin, Julius Springer, 1931. x+164 pages. RM 16.80.

Analytical Mechanics, for Students of Physics and Engineering. By H. M. Dadourian. Third Edition, Revised. New York, D. Van Nostrand Company, 1931. xiv+428 pages. $4.00.

Fundamentals of Electricity and Magnetism. By Leonard B. Loeb. New York, John Wiley & Sons, 1931. xx+432 pages. $4.00.

Craftsmanship in the Teaching of Elementary Mathematics. By F. W. Westaway. London, Blackie and Son, 1931. xvi+666 pages.

Intermediate Calculus. By Percey F. Smith and William R. Longley. New York, Ginn and Company, 1931. xiv+458 pages.

Applications of the Absolute Differential Calculus. By A. J. McConnell. London, Blackie and Son, 1931. xii+318 pages. 20 s.

Geschichte der Elementar-Mathematik. Von J. Tropfke. Erster Band, Rechnen. Dritte, verbesserte, und vermehrte Auflage. Berlin, Walter de Gruyter & Co., 1930. x+224 pages.

Algebraische Theorie der Körper. Von Ernst Steinitz. Neu herausgegebenen, mit Erlauterungen und einem Anhang: Abriss der Galoisschen Theorie versehen, von R. Baer und Helmut Hasse. Berlin, Walter de Gruyter & Co., 1930. 150 pages.

Thermodynamics. By A. W. Porter. New York, E. P. Dutton, 1931. viii+96 pages. $1.10.

Einführung in die Theorie der Kontinuierliche Gruppen. By G. Kowalewksi. Leipzig, Akd. Verl. M.B.H., 1931. x+396 pages.

Strahlenoptik. By M. Herzberger. Berlin, Julius Springer, 1931. xiv+196 pages. Paper, 18 marks.

The Cultural Significance of Accounts. By D. R. Scott. New York, Henry Holt and Company, 1931. x+316 pages.

Relativitätstheorie als Verstandliche Wissenschaft. By L. Hopf. Berlin, Julius Springer, 1931. viii+148 pages.

REVIEWS

Differentialgleichungen Reeller Funktionen. By E. Kamke. Mathematik und ihre Anwendungen in Monographien und Lehrbüchern. Band VII. Akademische Verlags-Gesellschaft m.b.H., Leipzig, 1930, xiv+436 pages, with 43 figures.

The outstanding feature of this excellent text on differential equations in the real domain is the exhaustive treatment of existence theorems for the ordinary

case. Methods of solution are developed in ample manner but more stress is laid on the properties of the solutions.

For the second order linear system the polar coordinate form is used to develop the usual oscillation and other properties associated with the work of Sturm. In this method, the central idea is to make use of the angle function and to exhibit directly the two solutions of the system as sine and cosine of this angle. This device was first published by Prüfer in 1926 in the "Mathematische Zeitschrift."

The treatment of partial differential equations is confined to equations of the first and second order. A brief discussion of the vibrating string is found in the last section.

At the close of the book are found the solutions of the forty-four splendid problems scattered throughout the text. More such problems would be desirable. Finally references are given to articles in the Encyclopädie der Mathematischen Wissenschaften and a list of text books is included.

H. J. ETTLINGER

The Principles of Quantum Mechanics. By P. A. B. Dirac. The International Series of Monographs on Physics. Oxford University Press, Oxford, 1930. x+257 pages. $6.00.

Dirac's contributions to the development of quantum mechanics have been of great importance. The method used in this book is to postulate two kinds of numbers; the first kind are *complex* and the second kind are vectors representing states. Laws of operation are assumed and this elementary skeleton is gradually clothed with more and more flesh until it pulses with the life blood of modern quantum mechanics.

As the hand of the author proceeds to erect the mathematical framework, addition, multiplication, conjugates, functions, expansion theorems, *Eigenvalues*, probability, differentials, contact transformations, Poisson brackets, matrices make their successive appearance. From these abstract beginnings emerge the foundations stones of quantum mechanics, the physical stuff such as electrons, protons, photons, radiation, polarization, emission, absorption, Schroedinger's wave equation, de Broglie's group velocities, Heisenberg's matrices, electron spin, Zeeman effect, Heisenberg's principle of indeterminacy, Einstein's law of radiation not in historical order but in convincing form.

Dirac's present book is a very valuable and clear contribution, in a field where mathematical complexities threatened to overwhelm the average interested scientist.

H. J. ETTLINGER

Advanced Calculus. By George A. Gibson. MacMillan and Company, London, 1931. xviii+510 pages. $6.50.

Someone has said that a course in advanced calculus can be most anything. We grant this provided the "most anything" includes: (1) a careful discussion

of the ideas of continuity, derivative, and differential, as applied to a function of a single variable, with examples of the various types of discontinuous functions; (2) a rigorous study of infinite sequences and series with special emphasis on the concepts of uniform and non-uniform convergence; (3) a study of partial derivatives, and differentials of a function of more than one variable, with an extensive drill in the formal processes involved in partial differentiation, change of variable, and the various applications of these to line, surface, and space integrals; (4) the existence theorems for implicit functions; (5) a thorough, but not too extended, treatment of integration based on Riemann's definition.

How does the book under review "measure up" on these points? In regard to the first, the ground is well covered, and the illustrations are happily chosen. It is, perhaps, desirable to have a few more examples of the various types of discontinuities. The notation $f'(x)$ is used to designate the derivative. The symbol dy/dx is not introduced until the differential is defined, which is as it should be. The geometric interpretation of the differential is omitted.

On the question of series and sequences, the author begins pretty much *in medias res*. It is taken for granted that the reader is thoroughly familiar with the concepts of convergence and divergence, and with the simpler tests for each. This is certainly not true of the rank and file of students on this continent who enter advanced calculus classes. Some previous contact with the ideas of uniform and non-uniform convergence also seem to be assumed. The significance of uniform convergence is stated clearly enough, and some examples are given of series which converge uniformly. On the other hand, not a single example comes to our notice which illustrates what uniform convergence is not. This is a defect. It would almost appear that the author had read neither Osgood's classical paper in the American Journal of Mathematics, vol. 19, nor his discussion of sequences and series in his *Funktionentheorie*. We greatly fear that he who gets his first orientation to the idea of uniform convergence from Professor Gibson's book will be in a position with the student who said to his teacher: "You have told me three times what uniform convergence is, and now I don't know it again."

Among the other topics coming under the head of series are: substitution of a series in a series; Abel's theorem; revision of series; the rearrangement of terms of a series; the more delicate convergence tests of Kummer, Gauss, Raabe, Abel, and Dirichlet. These are discussed in a very satisfactory manner. There is also a study of the remainder in Taylor's theorem, which is based on algebraic forms. While the treatment is excellent, it is not, perhaps, the best method of approach for the student who is taking up this topic for the first time.

The treatment of partial derivatives and of total differentials is adequate and concise. It is followed by the usual discussion of change of variables, in which a few general problems are dealt with. It usually happens that the student who first assimilates the methods of these general problems is liable to try to apply the formulas there developed to each particular situation that arises. This can be avoided by going from a careful study of partial and total

derivatives directly to exercise involving change of variable. The student will, for the most part, got his results by devious and clumsy methods. He will, however, acquire power and confidence in performing this type of work, which, after all, is the important thing. Professor Gibson's book lends itself admirably to such a procedure for the reason that the list of exercises is copious and varied.

The chapter on implicit functions follows the usual lines. The conditions under which the various theorems are proved are more drastic than is necessary, but this is an advantage if the book is to be used in introductory courses. Another commendable feature is the careful and enlightening way in which the reader is shown just what the theorems are all about.

In his presentation of the theory of integration the author has adhered to the definition of Riemann, and has accomplished the utmost possible without recourse to the modern point set theory. Several of the principal theorems are stated in more than one way. The technique is distinctive and pleasing, and the emphasis, for the most part, properly placed. Especially to be commended is the discussion of the change of order of integration for improper double integrals. In all seven chapters are devoted to the various phases of integration. One is surprised to find that no mention is made of the usual theorems on integration of series and sequences.

In these days the working tool of the mathematician is the modern theory of integration based on measurable sets and measurable functions. All the theorems concerning Riemann integrals are special cases of theorems in the modern theory and, for the most part, more elegantly obtained by means of the modern theory. This being the case, one wonders if Professor Gibson is justified in devoting so much space to Riemann integrals.

Other topics are: A good introductory exposition of the real number system; a chapter on sets, sequences, and limiting points, which includes the Bolzano-Weierstrass theorem and Cauchy's necessary and sufficient conditions for the existence of a limit; derivatives of determinants; Wronskians; linear dependence of functions; complex functions of a real variable. In connection with the latter we find the remark: "It is fairly evident that

$$\left|\int_a^b w\,dx\right| \leqq \int_a^b |w|\,dx,\text{''}\quad w = u + iv.\text{''}$$

Possibly it is evident for Riemann integrals, but I think the reader will agree that it needs proof in the case of Lebesgue integration.

There is also included a brief but good discussion of the Gamma function, both as the limit of a product and as an integral, and a chapter on infinite products. The treatment of infinite products assumes no previous knowledge on the part of the reader. The definitions and first principles are clearly and effectively stated. Of such theorems as are given, the proofs are concise but contain sufficient detail to make easy reading for the novice.

It is our opinion that Professor Gibson's book is a valuable contribution to the literature in the field which it covers. Only the occasional teacher will find

it suitable for class room use, but it should be a part of the working equipment of every student and instructor of advanced calculus.

R. L. JEFFERY

Geometry of Four Dimensions. By A. R. Forsyth, Cambridge University Press, 1930, vol. 1, xxix+468 pages, vol. 2, xi+520 pages.

This compendious two volume work (about 1000 pages) deals with its subject matter in an avowedly old-fashioned spirit: no tensors, little abridged notation; for instance, the coordinates are denoted by x, y, z, v rather than x_1, x_2, x_3, x_4. The presentation makes a point of explicitness and everything is written out *in extenso*. Forsyth, with his encyclopedic outlook, does this sort of thing very well, and has here performed a valuable service by taking the trouble to carry out to the end and exhibit in full detail what most geometers would be content to indicate in principle, seeming to follow Schopenhauer's dictum that everything that can be done should be done so that the history of the world may be complete.

The work divides itself into five parts:

1°. Linear sub-manifolds, one, two, three-dimensional; the fundamental four-space is itself assumed linear (flat, homaloidal).

2°. Skew curves in the four-space; there is also a section on skew curves in n-space.

3°. Surfaces as existing freely in the four-space, without reference to three-dimensional manifolds in which they may be contained.

4°. Regions, i.e., three-dimensional sub-manifolds of the four space.

5°. Invariants under rigid motion and under arbitrary change of parameters of curves, surfaces and regions, as embedded in one another and existing freely in the four-space.

The following samples of terminology may be of use to the prospective reader. *Homaloid*, a general term including: *line*, one-dimensional linear manifold; *plane*, two-dimensional linear manifold; *flat*, three-dimensional linear manifold; and the fundamental four-space. *Orbiculate amplitude*, a comprehensive term including: *circle*: $x^2+y^2=a^2$, $z=0$, $v=0$; *sphere*: $x^2+y^2+z^2=a^2$, $v=0$; *globe*: $x^2+y^2+z^2+v^2=a^2$.

The *principal directions* of a curve are its *tangent*, *principal normal*, *binormal* and *trinormal*: the last is perpendicular to the *osculating flat*, having four con- consecutive points in common with the curve. A curve has *curvature* $1/\rho$, *torsion* $1/\sigma$, and *tilt* $1/\tau$, this being dw/ds where dw, the *angle of tilt*, is the angle between two consecutive osculating flats. The Frenet formulas with their skew-symmetric matrix of coefficients $1/\rho$, $1/\sigma$, $1/\tau$, are set up. There is an osculating circle, sphere, and globe to the curve, each with its center and radius.

Surface means a two-dimensional manifold, *region* a three-dimensional manifold contained in the basic four-space. The geodesics of a surface are studied,

its lines of curvature, and other properties expressing its relation to the surrounding four-space.

In the theory of regions are included geodesics and minimal surfaces relative to a given region; the latter are surfaces of least area among those lying entirely in the region and bounded by a prescribed closed curve of the region. Quadric regions, defined by an equation of the second degree in the four coordinates, are studied. The general theory of surfaces as contained in a region is taken up; in particular, the geodesics of the surface are studied in their relation to the region in which the surface is embedded. The *globular representation* of a region generalizes the classic Gauss spherical representation. A region has three principal centers of curvature, whose locus is the *centro-region* (evolute). *Nul-surfaces*, of identically vanishing area, in a given region are discussed. Minimal regions are considered; these are regions of least volume with a given two-dimensional boundary.

The work ends with the theory of invariants of geometric configurations in the four-space. The fundamental group is the metric group of the coordinates combined with the total group of parameter transformations. The Lie theory is used, and a large number of invariants are set up explicitly, wherein only algebraic and differential operations are used supplemented by the integration of linear partial differential equations. Geometric interpretation is found for many of the invariants.

The reader who wishes to see, in concrete and explicit form, the actual features of four-dimensional geometry will find his wish amply met by browsing through Forsyth's book.

J. DOUGLAS

Elementare Reihenlehre. By Hans Falckenberg. Sammlung Göschen, Walter de Gruyter, 1926, 136 pages.

Komplexe Reihen. Nebst Aufgaben über reelle und komplexe Reihen. By Hans Falckenberg. Sammlung Göschen, Walter de Gruyter, 1931, 140 pages.

The first little book contains a brief treatment of the real number system and a conventional treatment of the simpler features of the classical theory of sequences and series of real variables. The pages on the elementary functions as defined by series are one of its best parts.

The second book is supplementary to the first. There are twenty-four pages treating series in the complex domain. The remainder of the volume is given to exercises and their solutions. There are many excellent and natural exercises to accompany a course on series. However, in my opinion the name *exercise* is belied when there is an accompanying solution; and inasmuch as the accumulation of sets of exercises without solutions readily available is a valuable addition to the literature of those subjects taught in our universities, I feel that this volume is not as helpful as it should be to the cause of university teaching.

TOMLINSON FORT

Funktionentheorie, Erste Teil: Grundlagen der allgemeinen Theorie der analytische Funktionen. Vierte, verbesserte Auflage. By Konrad Knopp. Sammlung Göschen, Walter de Gruyter, 1930.

There is little doubt but that this is the best monograph on functions of a complex variable yet written. The author has compressed an astonishing amount within its one hundred and thirty eight small pages and all is written with the utmost clarity and in a most interesting style. The present edition differs in detail only from earlier editions with which most teachers of the subject are already familiar and to such it needs no recommendation. To those unfamiliar with any edition the following chapter headings will give some idea of the scope of the present little book: I: "Punktmengen in der Ebene;" II: "Funktionen einer komplexen Veränderlichen;" III: "Das Integral einer stetigen Funktionen;" IV: "Der Cauchysche Integralsatz;" V: "Die Cauchyschen Integralformeln;" VI: "Reihen mit veränderlichen Gliedern;" VII: "Die Entwicklung analytische Funktionen in Potenzreihen;" VIII: "Analytische Fortsetzung und vollständige Definition der analytischen Funktionen;" IX: "Ganze transzendente Funktionen;" X: "Die Laurentsche Entwicklung;" XI: "Die verschiedenen Arten singulärer Stellen."

The monograph in question should be studied in conjunction with the second volume by the same author: *Anwendungen und Weiterfuhrung der allgemeinen Theorie.* The scope of this is partly indicated by the following chapter headings from the 1920 edition: I: "Ganze Funktionen;" II: "Merimorphe Funktionen;" III: "Periodische Funktionen;" IV: "Wurzel und Logarithmus;" V: "Algebraische Funktionen;" VI: "Das analytische Gebilde."

Accompanying each volume is a volume of exercises by Professor Knopp and also of the Sammlung Göschen. These should be in the hands of every teacher of the theory of functions of a complex variable.

TOMLINSON FORT

Niels Henrik Abel. Eine Schilderung Seines Lebens und Seiner Arbeit. By Dr. V. Bjerknes. Julius Springer, Berlin, 1930. v+136 pages.

Two years ago the world commemorated the centenary of the death of Niels Henrik Abel on April 6, 1829. This brought forth a number of books and brochures dealing with his distinguished services to mathematics and his absorbingly interesting life history. Of these many biographies and reviews probably none surpasses that of Dr. Bjerknes for keen understanding of the mathematician and man Abel, and for a lucid setting forth of the inner events of his life. It is a complement to the mathematical memoirs on Abel written by his father, C. A. Bjerknes, in 1880. Written in Norwegian in 1929, it has now been translated into German.

Abel's life reminds one of a Shakespearian tragedy, even to the swift-moving climax. From childhood on, it was the struggle of a man's mind and soul and body against the exigencies of life in a world which knows not always genius.

It was a pitifully uneven struggle for shelter and mere physical sustenance by one all afire with a consuming zeal for mathematical discovery; and an even greater struggle to get the mathematicians of his day to understand and appreciate his discoveries. Except from Holmboe and Crelle, he met only apathy and non-understanding. The cold reception of Gauss, Cauchy, and Legendre was gall to his fine, sensitive, often shrinking nature. By the efforts of Holmboe he received a stipend from the state to study abroad two years (1825–1827). In October, 1826 he submitted his now well-known memoir on transcendental functions to the French Academy. In this memoir was included his famous addition theorem. With inexcusable negligence the committee, Cauchy and Legendre, put it aside. On the results of this memoir young Abel had staked his hopes for a professorial appointment to the university at Christiania or for a renewal of his stipend. Now he received neither. Deeply in debt he had to discontinue his studies abroad. The next year, 1828, he eked out existence as a substitute lecturer in a military high school and also gave one course at the university.

Finally, even for him, better days seemed coming. His contributions in Crelle's Journal began to create attention, and through the influence of Crelle an appointment to the University of Berlin was forthcoming. In December, 1828 he made a trip to spend Christmas with his fiancee. It was a strenuous trip; and it is said that his clothing was insufficient. He caught a cold that later settled on his lungs. His body, weakened by want, was not equal to the emergency, and he died on April 6, 1829, at the early age of 27.

On his death bed, knowing his days were to be few, he re-wrote for Crelle's Journal the main points of the addition theorem that had been submitted to and pigeon-holed by the Paris committee. As he lay there looking at the two closely-written pages of the proof of his immortal theorem, which in a short period of convalescence he had salvaged for the world, he concluded optimistically: "I shall later bring out numerous applications of this theorem which will throw light on the functions in question." Says Mittag-Leffler: "January 6, 1829 is more worthy to be remembered in the history of civilization than the memorial days of kings, emperors, and countries. On this day Abel, lying in bed, wrote the greatest thought of his life for Crelle's Journal, the Abelian addition theorem, which was hailed forthwith as "monumentum aere perennius," and which even now, 100 years after Abel's birth, is looked upon as the high-point of the development of mathematics."

The irony of life! Two days after his death came the announcement of his appointment to the Berlin professorship. And a few months later Legendre read through the memoir on transcendental functions submitted three years before, and there was found the addition theorem. Both Cauchy and Legendre gave it unstinted praise. But the 24-year old author, who desperately needed that recognition in 1826, was no more.

Bjerknes gives considerable space to Abel's correspondence. Abel was a very fine letter writer, and narrates and comments with all the zest and spontaneity

of a young man on his first travels, to whom many things in life are new. We must remember that Abel never reached his 27th birthday. In these letters we learn that he seldom came to a city without visiting one or two theatres. That one luxury he allowed himself, even if the meals had to be slim. Often he was homesick, desperately homesick. His impression of Paris undoubtedly reflects the reception he had met there. "All want to instruct," he says, "no one wishes to learn. The only things a Frenchman desires from foreigners are things of practical import; as to thinking, he believes that can only be done by them."

Bjerknes has aimed to make real for us not only the scholar Abel, but the youth and the young man Niels Henrik,—his friendships, his love romance, his travels, his plans, his hopes. In this he has succeeded admirably. The reviewer has no adverse criticism of sufficient import to mention in a formal review. He only regrets that there is not an English translation for the many readers who do not read German.

M. A. Nordgaard

MATHEMATICS CLUBS

April 1931: "Elliptic integrals" by E. E. Rutgers; "A unique integration" by C. L. Hickok; "Calculus of finite differences" by M. Freedman; "The transcendental numbers π and e" by I. Kaufman; "Curves having no derivatives" by W. H. Glover.

May 1931: "An account of Dr. Michelson's life and work" by Professor A. L. Greenlees; "The atom" by J. J. Jelicks and G. A. Downsbrough; "A problem in probabilities" by I. J.Millner.

The club in cooperation with the club at the New Jersey College for Women, sponsored two lectures by visiting speakers during the year, namely, Professor R. C. Archibald of Brown University spoke on "Mathematics prior to the Greeks" and Dr. S. A. Schelkunoff of the Bell Telephone Laboratories spoke on "Telephony and the application of mathematics to some of its problems."

The year's meetings closed with the annual banquet.

ISADORE CHERTOFF, *Secretary.*

Case Mathematical Club, Case School of Applied Science.

The Case mathematics club was organized for the purpose of stimulating the study of mathematics, for the purpose of giving the students an appreciation of the unity of the field of mathematics, and for the purpose of showing to the students the beauty of non-technical and modern mathematics in a way which is impossible in the class room.

Any undergraduate or post-graduate student and any member of the faculty is eligible for membership in the club. The sole requirement is that he be interested in mathematics and attend the meetings of the club regularly. Nominal dues of $1.00 per term are required of regular members.

The officers for 1930–1931 were: Louis G. Henyey, President; John R. Parks, Vice President; Chester J. Stoemple, Secretary and Treasurer; Briggs H. Napier, Recording Secretary.

The active members during the past year numbered about 30.

The meetings and programs were as follows:

May 16, 1930: "Geometries of higher space" by Dr. Max Morris, Assistant Professor of mathematics.

October 10, 1930: "Sets of integers with equal sums of like powers" by Dr. O. E. Brown, Instructor in mathematics.

October 23, 1930: "Theorems in modern geometry" by Dr. J. R. Musselman, Professor of mathematics at Western Reserve University.

November 7, 1930: "Homogeneous coordinates and the line at infinity" by Mr. B. C. Getchell, Instructor in mathematics.

November 21, 1930: "Approximations to the value of Pi" by J. R. Parks.

December 19, 1930: "The calculus of finite differences" by C. Shepherd.

January 9, 1931: "Theory of groups and transformations" by Louis G. Henyey.

February 20, 1931: "The algebra of matrices and quaternions" by Mr. G. N. Garrison, Instructor in mathematics.

March 13, 1931: "Irrational numbers" by Mr. R. S. Burington, Instructor in mathematics.

March 27, 1931: "The mathematical solution of Cryptograms" by James Black.

April 17, 1931: "A method for determining the orbits of binary stars" by Dr. J. J. Nassau, Professor of Astronomy.

May 1, 1931: "A problem in Astronomy" by Assistant Professor E. M. Justin.

May 14, 1931: "The unity of mathematics" by Professor W. G. Simon, Western Reserve University.

The meetings are generally held at the school, but this year several were held off the campus. The meeting of May 6, 1930 was held at the home of President Wickenden; the meeting of October 23, 1930 was held in the home of Dean Focke. A meeting at the Case Astronomical Observatory was held on November 7, 1930. The final meeting of the year, a dinner meeting, was held at the Case Social Club on May 14, 1931.

BRIGGS H. NAPIER, *Secretary.*

Mathematics Club of the University of California at Los Angeles.

The aim of our mathematics club is the advancing of mathematics, the promoting of good fellowship among those interested in mathematics and the upholding of the traditions of the

University. Membership is open to all members of the faculty of the department of mathematics. Student membership is open to the students who have taken or are taking calculus. Secondary membership is open to any other student interested in mathematics. All members are entitled to vote but only primary members may hold office. Officers are elected for a period of one semester. They assume their duties at the beginning of the semester following their election.

The number of active members this year was thirty. The average attendance at the meetings, including faculty members, was about twenty.

The officers for the first semester were: Reed Lawlor, President; Alta Blackford, Vice President; Nadga Gray, Secretary; Lyle Sullivan, Treasurer; Virginia Woods, Librarian.

The officers for the second semester were Reed Lawlor, President; Alta Blackford, Vice President; Jean Robb, Secretary; Inez Hopkins, Treasurer; Norman Hinton, Librarian.

The meetings and programs were as follows:

September 24, 1930: "Horner's method" by Reed Lawlor.

November 5, 1930: "Method of determining the area of polygons" by Ernest von Seggern.

December 10, 1930: "The ideas of the calculus in every-day life" by Dr. Hedrick; "Glimpses into the lives of great mathematicians" by Miss H. Glazier; "The number of rational roots satisfying the equation" by Mr. John Hill; "Requisites for a good student are those needed in every-day life" by Dr. James.

January 14, 1931: "Measurement of stellar distances" by Melville Short.

February 18, 1931: "The relation between mathematics and philosophy" by Norman Hinton.

March 18, 1931: "Mathematical fallacies" by Dr. Raymond Garver; "Mathematical puzzles" by Miss Evelyn Fink.

April 22, 1931: "The honey-bee and the calculus" by Miss Evelyn Fink.

May 20, 1931. "Farewell address by the President." Election of officers for 1931–1932.

In November twenty members and friends hiked to Robert's Camp and spent Saturday and Sunday there. Lunch, dancing and bridge were featured.

On December 27, a bridge party was held in Newman Hall. There were about 100 people present including Professors and alumni.

On March 12, twenty members and their guests went to the Clark observatory. After visiting the observatory, a reception was held at the home of the Vice President, Alta Blackford.

On May 8, the club enjoyed a swimming party at the Deauville Beach Club. About fifteen were present.

JEAN ROBB, *Secretary.*

Mathematics Club of the Harris Teachers College.

The report covers the activities from September 1930 to January 1931. The sponsor of our club is Dr. Jesse Osborn, Professor of mathematics. The meetings are held biweekly.

The officers for the term were. Esther Berger, President; Dorothy Bahn, Vice President; Hildegard Graefe, Secretary-Treasurer; Mary Messina and Jennie Louise Waddell, representatives to college publication.

The meetings and programs were as follows:

October 6, 1930: "Astronomy" by Beth Murry; "Comets and meteors" by Bernice Rahn.

October 20, 1930: "Ancient number systems" by Elizabeth Lisy; "Men important in the history of mathematics" by Dorothy Bahn.

November 3, 1930: "Mathematics as amusement" by Iola Uhrhans; "Magic squares" by Hazel Willisan.

November 17, 1930: "The continents and oases of Mars" by Margaret Ellsperman; "Astronomy" by Anastasia Snabada.

December 1, 1930: "History of problems" by Dorothy Close; "The development of geometry" by Dorothy Bennet.

December 15, 1930: "Astronomy" by Mr. Miller, Professor of Chemistry.

January 5, 1931: "Puzzles and simple problems" by Eleanor Erskine and Candace Wesbrock.

HILDEGARDE GRAEFE, *Secretary.*

The Mathematics Club of the Cooper Union Institute of Technology.

The officers for 1930–1931 were: L. A. Kenworthy, '31, President; J. J. Murphy, '31, Vice President; J. W. Kinkel, '33, Secretary; C. H. Kropp, '33, Treasurer; E. H. Ryan, '33, Assistant Secretary.

The club has a membership of 163.

The meetings and programs were as follows:

November 6, 1930: "Transformations" by Mr. F. H. Miller, Columbia University.
December 4, 1930: "Elliptic and hyperbolic functions" by G. Kosolopoff, '32.
December 18, 1930: "Theory of the slide-rule and its uses" by G. O'Sullivan, '31.
January 28, 1931: "Harmonic ratio" by L. D. N. Green, '34.
February 26, 1931: "Algebraic charts" by W. W. Rigrod, '33.
March 19, 1931: "Trisection of an angle" by K. Itkin, '34.
April 9, 1931: "Mathematical work of the Bureau of Standards" by Mr. Irving Hartmann.
April 16, 1931: "Vector analysis" by Professor W. J. Pickett.
Election of officers.

C. H. LEHMANN, *Faculty Advisor.*

Dartmouth Mathematical Club, Dartmouth College

The Dartmouth Mathematical Club was founded on October 6, 1930 "to make possible the presentation and discussion of subjects of general mathematical interest." Although primarily an undergraduate club, membership is open to anyone interested on payment of the regular dues of one dollar per semester. There are at present twenty-two members, and frequently non-members are guests at the meetings.

Meetings are ordinarily held every two weeks while College is in session. It is customary to serve refreshments during the discussion following each meeting.

The officers for 1930–1931 were: Earle L. Morawski, '31, President; Emerson Cooley, '31, Vice President; Merit P. White, '30, Secretary-Treasurer.

The meetings and programs were as follows:

October 6, 1930: "Ovals and equiangular polygons" by Professor J. W. Young.
October 20, 1930: "Poincaré's world" by E. L. Morawski, '31.
November 3, 1930: "World maps" by Professor B. H. Brown.
November 17, 1930: "Russell's mathematical logic" by R. B. Colburn, '33; "Mathematical paradoxes" by H. Howe, '33.
January 5, 1931; "Magic squares" by E. T. Mecutchen, '31.
January 19, 1931: "The Mosaic law" by Professor C. E. Wilder.
February 11, 1931: "Mathematical curiosities" by E. Cooley, '31; E. T. Mecutchen, '31; E. L. Morawski, '31; Readings from Leacock's "Literary Lapses" by Dr. R. Robinson.
March 23, 1931: "Mathematical tricks" by E. C. Corson, '34; "Solution of the cubic equation" by Frank Engel, '34; "Solution of the quartic equation" by George Engel, '34.
April 13, 1931: "The nine-point circle" by Dr. F. W. Perkins.
April 27, 1931: "History of mathematical notation" by W. C. Johnson, '33; "History of the slide rule" by Edwin B. Thomas, '34.
May 18, 1931: "Special relativity" by E. L. Morawski, '31; Election of officers for 1931–1932.

ROBIN ROBINSON, *Chairman Departmental Committee Advisory to the Mathematical Club.*

The Mathematics Club of the University of Kansas

The officers for 1930–1931 were: Lilly Somers, President; Margaret Sturges, Vice President; Elizabeth Hyer, Secretary-Treasurer. The officers are elected the spring of the preceding year by a committee and elected at a general meeting. There are thirty-eight active members and fourteen faculty members. The purpose of the club is to stimulate interest in mathematics and to discuss

interesting topics which do not arise in the classroom. The members are selected by the faculty from the outstanding students who have ten or more hours of credit in mathematics.

The meetings and programs were as follows:

October 20, 1930: Business meeting.
November 3, 1930: "Peculiar properties of circulates" by Mr. Philip Bell.
November 17, 1930: "Determinants" by Professor E. B. Stouffer.
December 1, 1930: "Einstein's film."
December 15, 1930: "Electrical integrators and harmonic analyzers" by Mr. Billy Moore.
January 12, 1931: "Geometric method for solving a biquadratic equation" by Miss Winona Venard.
February 16, 1931: "Pi and how to get it" by Mr. Howard Wingert.
March 2, 1931: "Multiple correlation and how to predict it" by Professor Dinsmore Alter.
March 16, 1931: "The Pascal triangle" by Miss Carol Stratton.
March 30, 1931: "Reversing digits" by Professor Guy Smith.
April 13, 1931: "Simple Harmonic Motion" by Professor J. J. Quinn, St. Mary's College.
April 27, 1931: Business meeting.
May 14, 1931: Picnic.

ELIZABETH HYER, *Secretary.*

Students Mathematical Round Table of the University of Illinois.

The officers for 1930–1931 were: First Semester—C. A. Jackokes, President; Ella Marie Meythaler, Secretary; R. H. Skidmore, Treasurer. Second Semester—Wm. Ted Martin, President; Anice Seybold, Secretary; Gertrude Stith, Treasurer.

The object of the club is to interest students of mathematics in various phases of the subject not usually included in the standard mathematical courses, to stimulate the reading of current mathematical literature, and to keep its members informed in as non-technical a fashion as possible concerning the progress of mathematics and mathematical education. The members consist of graduate students and upper classmen interested in mathematics.

The meetings and programs were as follows:

October 30, 1930: "Polar reciprocation with respect to conics" by Miss Josephine Chanler.
November 13, 1930: "On metric properties of dual curves" by Mr. C. A. Jackokes.
December 11, 1930: "On the particular solutions of some differential equations" by Mr. R. H. Skidmore.
January 15, 1931: Election of officers; "On the zeros of Bessel's functions" by Mr. E. L. Welker.
February 17, 1931. "Graphical solutions of differential equations" by Miss Ella Marie Meythaler.
March 3, 1931. "The fourth dimension" by Miss Anice Seybold.
March 17, 1931. "Classification of geometries" by Professor A. B. Coble.
April 14, 1931. "Plücker line coordinates" by Mr. W. G. Warnock.

ANICE SEYBOLD, *Secretary.*

The Mathematics Club of New York University.

The mathematics club, organized a few years ago, endeavors to create within its members a desire for individual research in mathematics, and the presentation of this work before the members of the club. Also, topics which do not ordinarily come up in the course of undergraduate study have been presented in talks by various members of the faculty. The club, furthermore, has published during the past year two mathematical journals entitled "X" under the editorship of Mr. Reubin Roth. Coaching classes in Trigonometry, College Algebra and Mathematical Analysis have been conducted at regular periods during the year for those students in the University who feel weak in those subjects. Two socials, one each term, were held for the members and their friends.

The officers for 1930–1931, elected at the first meeting of the club in September 1930, were: Mayer I. Birshtein, President; Bertha Berson, Vice President; Sidney Flamm, Secretary; Louis Leitner, Treasurer; Charles K. Payne, Faculty Adviser; Reubin Roth, Editor of "X".

The meetings and programs were as follows:

October 31, 1930: "The relativity controversy" by Mr. C. K. Payne, Instructor in mathematics.
November 14, 1930: "The trisection of angles" by Max Leleiko.
November 28, 1930: "The three famous problems of antiquity" by Sol Feith.
December 5, 1930: "Astronomical calculations by geometry" by Seymour Klanfer.
December 19, 1930: "A solution of the cubic equation" by Abraham Katsh.
February 6, 1931: "Discussion of Fermat's theory of numbers" by Sidney Flamm.
February 20, 1931: "Operational calculus" by Professor Earnest J. Oglesby.
March 6, 1931: "The use of the straight line in mathematics" by Morris Baumel.
April 3, 1931: "Validity and truth in mathematics" by Morris Kline, Instructor in mathematics.
April 10, 1931: "The principle of duality in geometry" by Miss Jeanette Fox.

M. I. BIRSHTEIN, *President.*

Phi Chi Mu, Washington and Jefferson College.

The aim of this society is to promote interest and to stimulate activity in regard to studies pursued in the field of science; to raise the standard of scholarship, afford a reward for diligent work and to furnish an opportunity for the exchange of ideas among students of science at Washington and Jefferson College.

In order that a man be eligible for membership:—(1) He shall be in the process of completing a major in science, (2) he shall have completed the first semester of his junior year at college and (3) he shall have maintained an average of 2 (80%) or better throughout his college course.

Men are recommended for membership by a joint committee of the Heads of the Departments of Mathematics, Physics, Chemistry and Biology.

In order that a man, after election, become an active member of the society, he must present a paper on a topic of scientific interest. At present we have 7 active and 8 elected members.

The officers are elected at the last regular meeting of the school year to hold office during the ensuing year. Those elected May 13th last and holding office during 1930–1931 are: Robert R. Lyle, President; Walter S. Turpin, Secretary-Treasurer.

The meetings are held on the second Tuesday of the month throughout the school year. At these meetings, a paper is presented, followed by a general discussion.

The meetings and programs were as follows:

October 7, 1930: "The significance of 'Mathematics' and 'Science'" by W. S. Turpin.
November 12, 1930: "Topics of interest from 'The Universe around us'" by B. S. Gillespie.
December 9, 1930: "Biology as a science" by R. E. Masters.
January 13, 1931: "Transformations by means of reciprocal radii" and "Elements of plane cubic Cremona transformations" by Dr. H. C. Shaub.
February 10, 1931: "Some phases of mechanical integration-theory of polar planimeters" by R. R. Lyle.
March 10, 1931: "The mathematics of obtaining maximum mobility of aggregate in a concrete mixture and retaining minimum voids in the aggregate" by Dr. C. S. Atchison.
April 14, 1931: "Development in short-wave radio" by W. W. Lamb; "The chemical regulators of the body" by J. P. Proudfit.
May 12, 1931: "Common colds" by A. J. Trapuzzano.

ROBERT R. LYLE, *President,* WALTER S. TURPIN, *Secretary-Treasurer.*

Tulane Mathematics Club.

The officers for 1930–1931 were: Professor Chester G. Jaeger, Chairman; Mrs. Odessa Titsworth, Secretary. The officers are appointed for an indefinite length of time by the chairman of the department of mathematics of Tulane University

There is no regular membership. A card announcing each meeting is sent to those who would probably be interested, and the meeting is announced to advanced classes. Any one is permitted to attend. There are usually between twenty and forty present.

The time of meeting is irregular. There are from two to four meetings each year.

The aim of the club is to present papers on topics not ordinarily covered in courses at this school.

The meetings and programs were as follows:

November 20, 1930: "Elementary theorems on Groups" by Dr. Chester G. Jaeger.

April 30, 1931: "Vector treatment of some geometric problems" by Dr. Nola Lee Anderson.

CHESTER G. JAEGER, *Chairman.*

The Mathematics Club of the New Jersey State Teacher's College.

The officers for 1930–1931 were: Sarah Bogert, '32, President; Dorothy Holman, '32, Vice President; Eleanore Dooley, '32, Secretary; Richard Miller, '32, Treasurer.

The meetings and programs were as follows:

October 8, 1930: "Visualization in geometry" by Mr. A. Bakst, Columbia University, New York City.

October 22, 1930: "Trisection of an angle" by Lucille Drews.

November 12, 1930: "Squaring the circle" by Daniel Goss.

December 10, 1930: "Changing conceptions in secondary mathematics" by Mr. Carrol B. Quaintance, Head of department of mathematics of Cranford High School, Cranford, N. J.

January 14, 1931: "The High School Mathematics Club" by Dorothy Holman and Eleanore Dooley.

January 28, 1931: "Nomographic charts" by Gertrude Blachley.

February 11, 1931: Valentine social.

February 25, 1931: "Mathematics and Religion" by Mr. Harrison E. Webb, Principal of Market Street High School, Newark, N. J.

March 11, 1931: "Mathematical paradoxes" by Mr. A. Bakst.

March 25, 1931: "Peaucellier cells" by Jeanette Curley.

April 22, 1931: "Mathematics in aviation" by Mr. James C. Reddig, Chief engineer for the Grover Loening Aircraft Co., N. Y. C.

May 13, 1931: "The calculus of finite differences" by John Kirschoff.

May 27, 1931: Business meeting and the election of officers.

June 10, 1931: Supper hike.

ELEANORE DOOLEY, *Secretary.*

The Mathematics Club of the University of Akron.

The officers for 1930–1931, elected the first meeting in the fall by a majority vote of the members present, were: Ruth Creighton, President; Beatrice Currie, Vice President; Marjorie Halliwill, Secretary-Treasurer; Elmer Scharenberg, Chairman of the program committee.

The primary aim of the club is to promote, retain and add to the interest of Akron University students in mathematics by having the members, the faculty and outside speakers present special topics and problems before the club. Secondly, the club affords an opportunity for meeting with other students sharing a common interest in mathematics.

The following shall be eligible for membership in the club: (1) All mathematics majors and minors; (2) all engineering, chemistry and physics students; (3) any other students, adjudged by the club to be sufficiently interested in mathematics. To become a member, the candidate must be approved by the majority of members present at the meeting during which the candidate has made known his or her intentions of joining the club. Our present membership consists of sixteen students, three faculty advisors and guests to whom the meetings are open.

The meetings and programs were as follows:

January 8, 1931: "How to tell the day of the week of a certain date" by Dr. S. Selby: "A special problem in connection with the symmetrical functions of roots of an equation" by Marjorie Halliwill; "Fractional equations in which a solution is a zero of the common denominator" by Dr. H. A. Bender.

February 12, 1931: "Preferred numbers" by Mr. Dinger; "Vectors" by Dr. J. Jones.
April 16, 1931: "Graphing by making trigonometric transformations" by Miss Jane Bent; "Integrals in statistics" by Miss Ruth Creighton.
May 14, 1931. "Approximate trisection of angles" by Mr. H. Weisburg; "Surveying and its relation to mathematics" by Mr. Elmer Scharenberg.

The March meeting was held in the form of a trip to Case Observatory in Cleveland. Dr. J. J. Nassau lectured and demonstrated the solar system. Pictures of the sky were shown and the telescope and method of calculating observatory time were explained.

MARJORIE HALLIWILL, *Secretary.*

The Mathematics Club of Butler University.

The officers for 1930–1931, elected by written ballot in May 1930, were Miss Gladys Hawickhorst, President; Miss Willhelmina Feaster, Vice President; Miss Margaret Barker, Secretary; Miss Dione Kerlin, Treasurer.

The purpose of the club is two-fold: To consider and discuss current mathematical topics and to give opportunity to all students in the department to become acquainted.

Membership in the club is granted to all students of the department of mathematics, to those who have been students in the department, and to sponsors. The membership numbered twenty five for the year 1930–1931.

The meetings and programs were as follows:

October 9, 1930: "The evolution of the timepiece" by Ralph Urbain; "Mathematicians distinguished in other fields" by Douglass Brown.
November 6, 1930. "Mathematical tricks" by Mr. W. D. Carnahan of Shortridge High School.
December 18, 1930: Christmas party, mathematical games and puzzles.
January 8, 1931: "Music and mathematics" by Robert Schultz.
February 5, 1931: "Mathematics in nature" by Miss Sturdevant of the Arsenal Technical High School.
March 5, 1931. "The application of higher mathematics in the business world" by Mr. G. C. Elvers, C. E.; "Circle squarers" by Miss Juna M. Lutz, Sponsor.
April 9, 1931: "Non-Euclidean geometry" by Miss Marie Sangernibo of Washington High School; "Mathematics in the training for citizenship" by D. E. Smith (extracts read by Miss Anna K. Suter); "Construction of conic sections by paper folding" by Professor Bowersox.
May 7, 1931: "Egyptian mathematics" by Panoria Apostol: "Development of mathematics in America" by Madonna O'Hair; "Report of the meeting of the Indiana section of the Mathematical Association of America" by Miss Juna M. Lutz; Exhibition of the Chase edition of "The Rhind Papyrus" and of current mathematical periodicals.
June 4, 1931: Picnic.

JUNA M. LUTZ, *Assistant Professor.*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results areespecially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3516. *Proposed by Norman Miller, Queen's University.*

Prove that

$$\lim_{\substack{a_1\to a\\ a_2\to a\\ \cdots\\ a_{n+1}\to a}} \frac{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} & a_2^r \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{n-1} & a_{n+1}^r \end{vmatrix}}{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^n \end{vmatrix}} = \frac{r(r-1)\cdots(r-n+1)}{n!}\, a^{r-n}.$$

3517. *Proposed by Morgan Ward, California Institute of Technology.*

Let $\Delta_N(a, \omega)$ denote the determinant

$$\begin{vmatrix} \binom{1}{0} & 1-\omega & 0 & \cdots & 0 & 0 \\ \binom{2}{0} & \binom{2}{1} & 1-a\omega & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1-a^{N-3}\omega & 0 \\ \cdot & \cdot & \cdot & \cdots & \binom{N-1}{N-2} & 1-a^{N-2}\omega \\ \binom{N}{0} & \binom{N}{1} & \binom{N}{2} & \cdots & \binom{N}{N-2} & \binom{N}{N-1} \end{vmatrix}.$$

Establish the formula

$$\begin{aligned} a^{\frac{1}{2}N(N-1)}\Delta_N(a, a^{-N+1}) &= \Delta_N(a, a) \\ &= (1+a)(1+a+a^2)\cdots(1+a+a^2+\cdots+a^{N-1}). \end{aligned}$$

SOLUTIONS

232 [1915, 202; 1931, 228]. *Proposed by E. T. Bell.*

If $F(x)$ is any function of x which vanishes with x, and which, for $0<|x| \leqq |\xi|$, can be expanded in an absolutely convergent series of positive powers of x, show that a function $f(n)$ may be found, essentially in one way only, such that

$$\int_0^{\xi} \frac{1}{x}F(x)dx = -\log \prod_{n=1}^{\infty}(1-\xi^n)^{(1/n)f(n)},$$

and find the form of $f(n)$ explicitly in terms of the coefficients in the expansion of $F(x)$. Hence, as particular examples, expand (when possible by the method) e^x as an infinite product, and show that

$$\frac{1}{e} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)^{(1/n)\phi(n)},$$

where $\phi(n)$ is the totient of n.

Solution by Jack G. Deutsch, Columbia University.

We state the result desired by Professor Bell as follows.

Theorem: If $F(x)$ admits of the development,

$$F(x) = A_1x + A_2x^2 + \cdots + A_nx^n + \cdots, \tag{1.1}$$

which converges absolutely for $0 \leqq |x| \leqq |\xi| < 1$, then

$$\int_0^{\xi} \frac{F(x)}{x}dx = -\log \prod_{n=1}^{\infty}(1 - \xi^n)^{f(n)/n} \tag{1.2}$$

where $f(n)$ is given by the symbolic formula,

$$f(n) = A_n\left(1 - \frac{1}{A_{p_1}}\right)\left(1 - \frac{1}{A_{p_2}}\right) \cdots \left(1 - \frac{1}{A_{p_s}}\right), \tag{1.3}$$

in which $p_1, \cdots, p_s$ are the distinct primes which divide n and in which it is understood that A_uA_v represents A_{uv} and that (A_u/A_v) represents $A_{(u/v)}$.

Demonstration: We will at first assume the existence of an expansion 1.2 and that we can manipulate the right side of 1.2 in certain ways. We then show that $f(n)$ has the unique representation 1.3. For this purpose the following lemma will be useful.

Lemma: If $f(x)$ satisfies the relation,

$$f(d_1) + \cdots + f(d_r) = A_n,$$

where the A's are any sequence of numbers (not necessarily real) and $d_1, \cdots, d_r$ are the divisors of the integer n, then for integers $f(x)$ has the symbolic representation 1.3.

PROOF: If p is a prime we have

$$\sum_{\beta=0}^{\alpha-1} f(p^{\beta}) + f(p^{\alpha}) = A_{p^{\alpha}}, \text{ or } f(p^{\alpha}) = A_{p^{\alpha}} - A_{p^{\alpha-1}}.$$

Hence in symbolic form

$$f(p^{\alpha}) = A_{p^{\alpha}}\left(1 - \frac{1}{A_p}\right). \tag{1}$$

We shall now prove that

$$f(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) = f(p_1^{\alpha_1})f(p_2^{\alpha_2})\cdots f(p_s^{\alpha_s}), \tag{2}$$

where the p's are distinct primes, the α's are positive integers or zeros, and where it is understood that on the right each factor is replaced by its symbolic expression. It should be noted that $f(p_i^0)=A_1$. Assume that (2) is true for all values of $s\geqq 1$, and for all positive or zero integers α such that $\alpha_1+\alpha_2+,\cdots+\alpha_s\leqq k$. We shall show that it is also true if any α, say α_1, is increased by unity. Consider A_n where $n=p_1^{\alpha_1+1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$. Then A_n-A_{n/p_1}, when expressed as sums, contains only one term $f(n)$ for which the sum of the exponents can be greater than k. For the remaining terms we can by hypothesis apply the symbolic formula (2). We may then write

$$\begin{aligned} A_n - A_{n/p_1} = f(p_1^{\alpha_1+1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) + \sum_{\beta_2=0}^{\alpha_2}\cdots\sum_{\beta_s=0}^{\alpha_s} f(p_1^{\alpha_1+1})f(p_2^{\beta_2}\cdots p_s^{\beta_s}) \\ - f(p_1^{\alpha_1+1})f(p_2^{\alpha_2}\cdots p_s^{\alpha_s}). \end{aligned} \tag{3}$$

The sum on the right is precisely

$$A_{p_1^{\alpha_1+1}}\left(1-\frac{1}{A_{p_1}}\right) \text{ times } A_{n/p_1^{\alpha_1+1}}, \text{ or } A_n - A_{n/p_1}.$$

After cancellations we have

$$f(p_1^{\alpha_1+1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) = f(p_1^{\alpha_1+1})f(p_2^{\alpha_2}\cdots p_s^{\alpha_s}).$$

From (1) the theorem is true for $k=1$, and hence it is always true.

We now expand the right side of 1.2 into a convenient two-way series and equate this series to the expansion for the left side determined by 1.1

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{A_k}{k}\xi^k = f(1)&\left[\xi+\frac{\xi^2}{2}+\cdots+\frac{\xi^n}{n}+\cdots\right] \\ +\frac{f(2)}{2}&\left[\xi^2+\frac{\xi^4}{2}+\cdots+\frac{\xi^{2n}}{n}+\cdots\right] \\ &\quad\vdots \\ +\frac{f(t)}{t}&\left[\xi^t+\frac{\xi^{2t}}{2}+\cdots+\frac{\xi^{nt}}{n}+\cdots\right]. \\ &\quad\vdots \end{aligned} \tag{2.1}$$

We now equate coefficients of like powers. For a particular N the only rows that will contain a ξ^N-term are those for which t is a divisor of N. It is clear that we have $f(d_1)+\cdots+f(d_r)=A_N$ where the d's are the divisors of N. The lemma shows that $f(n)$ has the unique formulation 1.3 provided the expansion 1.2 and the above manipulations are valid. It is these points that we need settle to complete the proof of the theorem.

Lemma: If the series,

$$\sum_1^\infty A_n x^n$$

converges absolutely for $x=\xi(|\xi|<1)$, then the series

$$\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}|A_j|\right)x^n \tag{2.2}$$

converges for $x=|\xi|$.

Consider the absolutely convergent series,

$$\sum_1^\infty |A_n|\frac{|\xi|^n}{1-|\xi|}. \tag{2.3}$$

Expand it into the two-way series,

$$\begin{aligned}\sum_1^\infty |A_n|\frac{|\xi|^n}{1-|\xi|} = |A_1|\,[|\xi|+|\xi|^2+|\xi|^3+\cdots]&\\ +|A_2|\,[|\xi|^2+|\xi|^3+\cdots]&\\ +|A_3|\,[|\xi|^3+\cdots]&\\ +\cdots.&\end{aligned}$$

Since each row and each column is absolutely convergent the sum of the columns will yield a convergent series which is precisely 2.2 with $x=|\xi|$.

Now let $f(n)$ be given by 1.3. We have surely that

$$|f(n)|\leqq\sum_{j=1}^{n}|A_j|$$

for all n. By virtue of the preceding lemma the two-way series of 2.1 has a meaning and its sum is the right side of 1.2. We can calculate the value of the right side of 2.1 by collecting the terms according to the powers of ξ. Now, because of the formula for $f(n)$, we will have the equality given in 2.1 so that 1.2 is valid. This completes the proof of the theorem.

If we take $A_n=n$, we have that $f(n)$ is the Euler function, $\phi(n)$. In this case if we take $\xi=1/2$, we have the example given by Professor Bell, namely,

$$\frac{1}{e}=\prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)^{\phi(n)/n}.$$

A Note by Otto Dunkel. Another proof of 1.3 of page 540 is as follows:

Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$, and let C_i denote the class of all divisors of such integers as $n/p_1p_2,\cdots,p_i$ of which there are $\binom{s}{i}$. The class C_0 contains the single integer n, each of the other classes contain unity. All the divisors of n are contained in the group of classes $C_0, C_1,\cdots,C_s$, but there are repetitions. Consider the number of times the integer $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_i^{\alpha_i}p_{i+1}^{\beta_{i+1}}\cdots p_{i+r}^{\beta_{i+r}}$ occurs

in each class, where $\beta_k < \alpha_k$, $i+r \leqq s$. In C_j this integer will appear as a divisor of $n/p_{h_1}p_{h_2} \cdots p_{h_j}$, if no one of $p_1, p_2, \cdots, p_i$ is in the denominator of this fractional form. Hence it occurs $\binom{s-i}{j}$ times in C_j. Since

$$\sum_{j=1}^{s-i}(-1)^{j+1}\binom{s-i}{j} = 1,$$

we affix the sign $(-1)^{j+1}$ to C_j, and then the set of integers in

$$n + C_1 - C_2 + C_3 - \cdots + (-1)^{s+1} C_s$$

contains after cancellations each divisor of n once and only once. Therefore

$$A_n = f(n) + \sum f(x_1) - \sum f(x_2) + \cdots + (-1)^{s+1} \sum f(x_s),$$

where x_j runs through the integers in C_j. Thus

$$f(n) = A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s}.$$

Or in symbolic form

$$f(n) = A_n\left(1 - \frac{1}{A_{p_1}}\right)\left(1 - \frac{1}{A_{p_2}}\right) \cdots \left(1 - \frac{1}{A_{p_s}}\right).$$

315 [1915, 309]. *Proposed by H. S. Uhler.*

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

Solution by the Proposer.

Case 1. *Adequate friction.* When there is no slipping we may take moments about the instantaneous axis of rotation, that is, about the (horizontal) line of contact of the fixed and rolling bodies. Then

$$I\frac{d^2\phi}{dt^2} = agm \sin \theta, \tag{1}$$

where a, m, and I denote respectively the radius, the mass, and the moment of inertia (with respect to the instantaneous axis) of the rolling body. g symbolizes the acceleration of gravity. θ represents the angle made with the vertical by the plane which, at any instant, passes through the axis of the fixed cylinder and the center of mass of the rolling body. $d\phi/dt$ stands for the angular speed of the rolling body about the instantaneous axis. Then, if b denotes the radius of the fixed cylinder, we have

$$a\frac{d\phi}{dt} = -(b-a)\frac{d\theta}{dt}\cdot$$

Hence, by substituting in equation (1), we have also

$$\frac{d^2\theta}{dt^2} = -C_1^2 \sin\theta \tag{2}$$

where, for brevity,

$$C_1^2 \equiv \frac{a^2gm}{I(b-a)}\cdot$$

Integrating differential equation (2) once, under the condition that the movable body starts from rest when $\theta=\theta_0$, we find

$$\frac{d\theta}{dt} = C_1[2(\cos\theta - \cos\theta_0)]^{1/2}.$$

Therefore,

$$t_1 = \frac{1}{\sqrt{2}\cdot C_1}\int_0^{\theta_0}\frac{d\theta}{(\cos\theta-\cos\theta_0)^{1/2}}, \tag{3}$$

where t_1 symbolizes the time to roll from $\theta=\theta_0$ to $\theta=0$.

Case 2. *Zero friction.* It will be found at once that

$$t_2 = \frac{1}{\sqrt{2}\cdot C_2}\int_0^{\theta_0}\frac{d\theta}{(\cos\theta-\cos\theta_0)^{1/2}}, \tag{4}$$

where $C_2^2 \equiv g/(b-a)$.

By hypothesis $t_1=t_2$; hence, expressions (3) and (4) now give

$$C_1 = C_2,$$

that is,

$$I = a^2m. \tag{5}$$

Let k denote the radius of gyration of the moving body with respect to a horizontal line through the center of mass of this body then, by the theorem of parallel axes, we have

$$I = m(a^2 + k^2). \tag{6}$$

Combining relations (5) and (6), we get $k=0$ which means that the radius of the moving body must be zero. In other words, the conditions specified can not be fulfilled by bodies of finite radii. [If my memory is trustworthy I obtained a finite answer by the same processes just before this problem was submitted for publication. Hence I must have made a mistake in algebra at that time. Nevertheless, the chief point was to get the same integral in both cases and then to eliminate it.]

Also solved by H. M. Dadourian, William Hoover, J. B. Reynolds, and F. L. Wilmer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Through the courtesy of Professor W. F. Shenton of the American University, the Library of the Mathematical Association now possesses a complete set of the *Mathematical Visitor* published by Artemas Martin during the years 1877–1894, and of the *Mathematical Magazine* published by him during the years 1882–1910. This includes a copy of the very rare first edition of Volume 1, Number 1 of the *Mathematical Visitor.* It will be of interest to the members of the Association to know that the American University possesses the files of the *Visitor* and the *Mathematical Magazine* and that colleges or individuals who wish to have these magazines in their libraries, or who wish to fill out incomplete sets, may purchase them through Professor Shenton.

Princeton University announces that it has now made provision for work towards advanced degrees in mathematical physics. During the fall term of 1931–32, graduate courses in this field will be offered by Professors E. P. Adams, J. von Neumann, E. P. Wigner, E. U. Condon, and H. P. Robertson. Dr. P. A. M. Dirac, of Cambridge, will lecture on the relativistic formulation of quantum mechanics during the first part of this term.

Dr. Clifford Bell has been promoted to an assistant professorship of mathematics at the University of California at Los Angeles.

Dr. L. W. Cohen has been appointed assistant professor of mathematics at the University of Kentucky.

Assistant Professor T. F. Cope has been promoted to a professorship of mathematics at Marietta College.

Dr. F. W. Doermann, of New York University, has been promoted to an assistant professorship of physics.

Professor W. W. Elliott, of Duke University, has been granted sabbatical leave for the first semester of 1931–1932. He will spend most of this time at Brown University.

Dr. D. A. Flanders, of New York University, has been promoted to an assistant professorship.

Assistant Professor Raymond Garver, of the University of California at Los Angeles, has been promoted to an associate professorship of mathematics.

Professor Alfred Hume, of Southwestern University, Memphis, has been appointed president of Branham and Hughes Military Academy, Spring Hill, Tennessee.

Professor J. A. Hurry, of Western State College, Gunnison, Colorado, has been appointed head of the department of physics at San Antonio Junior College.

Dr. N. H. McCoy has been appointed to an assistant professorship at Smith College.

Associate Professor W. M. Miller, of Marquette University, has been appointed to an assistant professorship at Tufts College.

Dr. D. C. Morrow has been promoted to an assistant professorship at the College of the City of Detroit.

Dr. Gordon Pall has been appointed lecturer in mathematics at McGill University.

Dr. Saul Pollock, of the University of California, has been appointed to an assistant professorship of mathematics at the Indiana State Teachers College at Terre Haute.

Professor G. E. Raynor, of the University of Oklahoma, has been appointed assistant professor of mathematics at Lehigh University.

Associate Professor O. H. Rechard, of the University of Wyoming, has been promoted to a professorship of mathematics.

Associate Professor C. N. Reynolds, of West Virginia University, has been promoted to a professorship of mathematics.

Dr. John H. Roberts, of the University of Texas, has been appointed to an assistant professorship of mathematics at Duke University.

Associate Professor F. L. Wren has been promoted to a professorship of the teaching of mathematics at the George Peabody College for Teachers.

Dr. E. Kathryn Wyant has been appointed professor of mathematics at the Northeastern State Teachers College, Tahlequah, Oklahoma.

The following appointments to instructorships in mathematics are announced:

Duke University, Dr. Ruth W. Stokes.
Johns Hopkins University, Mr. G. R. Trott.
Lehigh University, Mr. S. S. Cairns.
Princeton University, Mr. Jack Levine.
Temple University, Mr. J. A. Clark.

Professor G. H. Cresse, of the University of Arizona, died on May 3, 1931.

THE CHAUVENET PRIZE

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932 for the period of 1929-1931.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. H. Bussey, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Annual Meeting of the Association, New Orleans, Louisiana, Dec. 30-31, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

Illinois, Peoria, May 1-2.
Indiana, Muncie, May 1-2.
Iowa, Davenport, May 1-2.
Kansas, Topeka, Jan. 24.
Kentucky, Lexington, May 9.
Louisiana-Mississippi, Natchitoches, La., March 13-14.
Maryland-District of Columbia-Virginia, Richmond, Va., May 9.
Michigan, Ann Arbor, March 21.
Minnesota, St. John's University, Collegeville, May 16.
Missouri, St. Louis, November.
Nebraska, Lincoln, May 8.
Ohio, Columbus, April 2.
Philadelphia, Philadelphia, Nov. 28.
Rocky Mountain, Boulder, Colo., April 17-18.
Southeastern, Auburn, Ala., April 24-25.
Southern California, Occidental College, Los Angeles, March 21.
Texas, Fort Worth, Jan. 31.

Affiliated Organizations: The New England Association of Teachers of Mathematics, The National Council of Teachers of Mathematics.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

VOLUME XXXVIII, 1931

NUMBER 9, NOVEMBER SUPPLEMENT

PART II

PRICE 50 CENTS

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

The Chauvenet Prize

The ASSOCIATION in 1926 established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the ASSOCIATION. In 1928 Professor W. B. Ford, then president of the ASSOCIATION, gave $500 the income from which was to supplement the original fund so that the prize could be awarded every three years.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics*. The second award covering the five years preceding 1930, was made to THEOPHIL HENRY HILDEBRANDT for his paper on "The Borel Theorem and its Generalizations" published in the *Bulletin* of the American Mathematical Society in 1926. Hereafter the prize is to be awarded every three years.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION,—one more of the many good reasons for membership.

THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

FOR THE ACADEMIC YEAR

1931-1932

MENASHA, WIS., AND MINNEAPOLIS, MINN.
PUBLISHED BY THE ASSOCIATION
1931

BOARD OF TRUSTEES FOR THE YEAR 1931

E. T. Bell, California Institute of Technology, *President* (*1931–1932*)
Arnold Dresden, Swarthmore College, *Vice-President*
C. N. Moore, University of Cincinnati, *Vice-President*
W. D. Cairns, Oberlin College, *Secretary-Treasurer*
R. B. McClenon, Grinnell College, *Librarian*
C. H. Yeaton, Oberlin College, *Assistant Secretary**

Appointed Members of the Board—*Committee on the Official Journal*

A. J. Kempner, University of Colorado
H. E. Slaught, University of Chicago, *Manager*
W. H. Bussey, University of Minnesota, *Editor-in-Chief for 1931*
W. B. Carver, Cornell University, *Editor-in-Chief elect for 1932*

Elected Members of the Board

To serve until January 1932

J. L. Coolidge, Harvard University
W. B. Ford, University of Michigan
Lao G. Simons, Hunter College
Oswald Veblen, Princeton University

To serve until January 1933

A. A. Bennett, Brown University
B. F. Finkel, Drury College
W. L. Hart, University of Minnesota
D. N. Lehmer, University of California

To serve until January 1934

L. L. Dines, University of Saskatchewan
T. C. Fry, Bell Telephone Laboratories
J. W. Glover, Teachers Insurance and Annuity Association
E. P. Lane, University of Chicago

For a list of all officers and Council Members or Trustees for preceding years see page 69.

STANDING COMMITTEES OF THE ASSOCIATION

Committee on Libraries

†Florian Cajori, University of California
E. S. Crawley, University of Pennsylvania
Solomon Lefschetz, Princeton University
W. R. Longley, Yale University
R. E. Root, U. S. Naval Academy
W. B. Ford, University of Michigan, *Chairman*

Editorial Committee on Carus Monographs

G. A. Bliss, University of Chicago
D. R. Curtiss, Northwestern University
A. J. Kempner, University of Colorado
H. E. Slaught, University of Chicago
J. W. Young, Dartmouth College

Committee on Mathematical Dictionary

R. C. Archibald, Brown University
H. L. Rietz, University of Iowa
H. E. Slaught, University of Chicago
D. E. Smith, Columbia University
E. R. Hedrick, University of California at Los Angeles, *Chairman*

Committee on the Arnold Buffum Chace Publication of the Rhind Papyrus

R. C. Archibald, Brown University
W. D. Cairns, Oberlin College
H. E. Slaught, University of Chicago

---

* Not a member of the Board.

†Deceased

On the Joint Committee of the Association and the National Council of Teachers of Mathematics on a one-Year Course in Plane and Solid Geometry

Dunham Jackson, University of Minnesota, *Chairman*
Ralph Beatley, Harvard University
J. O. Hassler, University of Oklahoma
C. N. Moore, University of Cincinnati
W. D. Reeve, Columbia University

Representatives on Councils of Scientific Organizations

National Research Council—R. D. Carmichael, University of Illinois, for the term ending June 11, 1932

American Association for the Advancement of Science—W. D. Cairns, Oberlin College and Dunham Jackson, University of Minnesota, for the year 1931

Associate Editors of the Official Journal for 1931

Elizabeth Carlson, University of Minnesota
N. A. Court, University of Oklahoma
Otto Dunkel, Washington University
H. S. Everett, University of Chicago
B. F. Finkel, Drury College
R. E. Gilman, Brown University
R. A. Johnson, Brooklyn College
H. W. Kuhn, Ohio State University
J. R. Musselman, Western Reserve University
H. L. Olson, Michigan State College
David Eugene Smith, Columbia University
F. M. Weida, George Washington University

Associate Editors of the Annals of Mathematics

Representing the Association 1930–1932

H. S. Vandiver, University of Texas
Mrs. Anna Pell Wheeler, Bryn Mawr College

SECTIONS OF THE ASSOCIATION

Illinois

R. W. Barnard, University of Chicago, *Chairman*
W. C. Krathwohl, Armour Institute of Technology, *Vice-Chairman*
C. N. Mills, Illinois State Normal University, *Secretary-Treasurer*
Edith I. Atkin, Illinois State Normal University, *Associate Secretary-Treasurer*

Indiana

G. E. Carscallen, Wabash College, *Chairman*
T. E. Mason, Purdue University, *Vice-Chairman*
H. T. Davis, Indiana University, *Secretary-Treasurer*

Iowa

C. W. Strom, Luther College, *Chairman*
B. D. Roberts, Parsons College, *Vice-Chairman*
J. F. Reilly, University of Iowa, *Secretary-Treasurer*

Kansas

J. J. Wheeler, University of Kansas, *Chairman*
O. J. Peterson, Emporia State Teachers College, *Vice-Chairman*
Lucy T. Dougherty, Kansas City Junior College, *Secretary-Treasurer*

Kentucky

R. S. Park, Eastern Teachers College, *Chairman*
A. R. Fehn, Centre College, *Secretary-Treasurer*

Louisiana-Mississippi

A. C. Maddox, Louisiana State Normal College, *Chairman*
J. P. Cole, Louisiana Polytechnic Institute, *Vice-Chairman*
C. D. Smith, Mississippi A. and M. College, *Vice-Chairman*
Nola L. Anderson, Sophie Newcomb College, *Secretary-Treasurer*

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

E. W. WOOLARD, George Washington University, *Chairman*
PAUL CAPRON, U. S. Naval Academy, *Secretary-Treasurer*
GILLIE A. LAREW, Randolph-Macon Woman's College, *Member Executive Committee*
OSCAR ZARISKI, Johns Hopkins University, *Member Executive Committee*

MICHIGAN

J. P. EVERETT, Western State Normal College, *Chairman*
L. A. HOPKINS, University of Michigan, *Secretary-Treasurer*
L. C. EMMONS, Michigan State College, *Member Executive Committee*

MINNESOTA

MRS. R. B. EIDE, State Teachers College, River Falls, Wis., *Chairman*
A. L. UNDERHILL, University of Minnesota, *Secretary-Treasurer*
C. M. JENSEN, Macalester College, *Member Executive Committee*
F. J. TAYLOR, College of St. Thomas, *Member Executive Committee*
MARION B. WHITE, Carleton College, *Member Executive Committee*

MISSOURI

W. H. ROEVER, Washington University, *Chairman*
A. D. PIERSON, Kansas City Junior College, *Vice-Chairman*
P. R. RIDER, Washington University, *Secretary-Treasurer*

NEBRASKA

A. K. BETTINGER, Creighton University, *Chairman*
A. L. HILL, Peru State Teachers College, *Secretary*
J. M. HOWIE, Nebraska Wesleyan University, *Treasurer*

OHIO

W. D. CAIRNS, Oberlin College, *Chairman*
RUFUS CRANE, Ohio Wesleyan University, *Secretary-Treasurer*
H. M. BEATTY, Ohio State University, *Member Executive Committee*

PHILADELPHIA SECTION

TOMLINSON FORT, Lehigh University, *Chairman*
P. A. CARIS, University of Pennsylvania, *Secretary-Treasurer*

ROCKY MOUNTAIN SECTION

O. H. RECHARD, University of Wyoming, *Chairman*
A. G. CLARK, Colorado Agricultural College, *Vice-Chairman*
A. J. LEWIS, University of Denver, *Secretary-Treasurer*

SOUTHEASTERN SECTION

W. W. RANKIN, Duke University, *Secretary-Treasurer*

SOUTHERN CALIFORNIA SECTION

H. C. VANBUSKIRK, California Institute of Technology, *Chairman*
L. D. AMES, University of Southern California, *Vice-Chairman*
P. H. DAUS, University of California at Los Angeles, *Secretary-Treasurer*

TEXAS

H. E. BRAY, Rice Institute, *Chairman*
C. R. SHERER, Texas Christian University, *Vice-Chairman*
NAT EDMONSON, JR., Texas Technological College, *Secretary-Treasurer*

INDIVIDUAL MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Closed for printing November 7, 1931.

NOTE. This list gives data for the members of the Association as taken from the register issued in December 1929, unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great effort has been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

ABERLE, F. J., B.S. (Carnegie Inst. of Tech.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *420 S. Division St.*

ACKLEY, Prof. H.M., A.M. (Olivet) Western State Teachers Coll., Kalamazoo, Mich. *602 Axtell St.*

ADAMS, B. T., A.M. (Baylor) Head of Dept., Jr. Coll., Wichita Falls, Tex.

ADAMS, Asso. Prof. C. R., Ph.D. (Harvard) Brown Univ., Providence, R. I. *260 Doyle Ave.*

ADAMS, Prof. E. P., Ph.D. (Harvard) Physics, Princeton Univ., Princeton, N. J.

ADAMS, O. S., D.Sc. (Kenyon) Senior Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

ADKINS, J. B., Ph.B. (Chicago) Teacher, Culver Milit. Acad., Culver, Ind.

ADKINS, Prof. L. K., M.S. (Chicago) Head of Dept., State Teachers Coll., La Crosse, Wis.

ADKISSON, Asst. Prof. V. W., Ph.D. (Pennsylvania) Univ. of Arkansas, Fayetteville, Ark. *505 Storer St.*

AGARD, Asso. Prof. H. L., Ph.D. (Yale) Dean, Williams Coll., Williamstown, Mass.

AGNEW, R. P., Ph.D. (Cornell) National Research Fellow, Brown Univ., Providence, R. I. *Dept. of Math.*

AITCHISON, BEATRICE, A.M. (Johns Hopkins) Grad. Student., Johns Hopkins Univ., Baltimore, Md.

AKERS, Prof. O. P., Ph.D. (Cornell) Allegheny Coll., Meadville, Pa. *360 N. Main St.*

ALBERT, Prof. O. W., Ph.D. (Washington) Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*

ALDEN, H. H., A.M. (Carnegie Inst. of Tech.) Asst., Ohio State Univ., Columbus, Ohio.

ALDERTON, Asso. Prof. NINA M., Ph.D. (California) Mills Coll., Mills College, Calif. *1918 Lakeshore Ave., Oakland, Calif.*

ALEXANDER, Prof. J. W., Ph.D. (Princeton) Princeton Univ., Princeton, N. J. *29 Cleveland Lane.*

ALEY, Pres. R. J., Ph.D. (Pennsylvania) Butler Univ., Indianapolis, Ind.

ALICE IRENE, Sister, A.M. (Columbia Coll. of St. Catherine) Instr., Columbia Coll. of St. Catherine, St. Paul, Minn.

ALICE IRENE, Sister, A.M. (St. Canisius) Prof., Coll. of St. Rose, Albany, N. Y. *979 Madison Ave.*

ALLEN, Asso. Prof. BERD R., M.S. (Chicago) Judson Coll., Marion, Ala.

ALLEN, Prof. E. E., A.M. (Park Coll.) Occidental Coll., Los Angeles, Calif.

ALLEN, E. F., B.S. (Acad. of the New Church) Instr., Acad. of the New Church, Bryn Athyn, Pa.

ALLEN, Asso. Prof. E. F., Ph.D. (Missouri) Oklahoma A. and M. Coll., Stillwater, Okla. *519 Lincoln St.*

ALLEN, FLORENCE E., Ph.D. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis. *219 Lathrop St.*

ALLEN, GERTRUDE E., B.S. (California) 122 Maple St., Reno, Nev.

ALLEN, Asso. Prof. JOSEPH, A.M. (Harvard) Coll. of the City of New York, New York, N. Y.

ALLEN, Prof. R. B., Ph.D. (Clark) Kenyon Coll., Gambier, Ohio.

ALLIOT, Prof. EUGENE, Lic. es Sc. (Paris) St. Michael's Coll., Winooski, Vt.

ALLISON, ELI, M.S. (Wesleyan) Teacher, Franklin School, New York, N. Y. *894 Riverside Drive.*

ALLISON, N. B., A.M. (Ketucky) Instr., Univ. of Kentucky, Lexington, Ky. *Dept. of Math.*

ALRICH, G. F., M.S. (Lehigh) Instr., Univ. of Maryland, College Park, Md.

AMES, Prof. L. D., Ph.D. (Harvard) Univ. of Southern California, Los Angeles, Calif.

AMICK, Prof. T. C., Ph.D. (Central Univ.) Elon College, N. C.

AMOS, C. E., B.S. (Denison) Grad. Asst., Ohio State Univ., Columbus, Ohio. *109 E. Frambes Ave.*

ANDERSON, Prof. HARRIET, B.S. (Iowa State) Grand Island Coll., Grand Island, Nebr.

ANDERSON, H. E., A.M. (Augustana) Teacher, High School, Atascadero, Calif.
ANDERSON, Prof. MARY, A.M. (Illinois) Illinois Woman's Coll., Jacksonville, Ill.
ANDERSON, Asso. Prof. NOLA L., Ph.D. (Missouri) Chm., Dept. of Math., Sophie Newcomb Coll., New Orleans, La.
ANDERSON, R. LUCILLE, A.B. (Mount Holyoke) Instr., Hunter Coll., New York, N. Y. *Dept. of Math.*
ANDERSON, Prof. W. E., Ph.D. (Pennsylvania) Miami Univ., Oxford, Ohio.
ANDERTON, Asst. Prof. ETHEL L., Ph.D. (Yale) Mount Holyoke Coll., South Hadley, Mass.
ANDREW, THURMAN, M.S. (Kentucky) Instr., Wentworth Inst., Boston, Mass. *15 Wyoming Rd., Newtonville, Mass.*
ANDREWS, ANNA H., Ph.B. (Wesleyan) Teacher, Public High School, Hartford, Conn. *167 Beacon St.*
ANDREWS, Prof. W. H., Ph.D. (Chicago) Educ., State Agric. Coll., Manhattan, Kans. *1704 Fairview Ave.*
ANNING, Asst. Prof. N. H., M.A. (Queen's) Univ. of Michigan, Ann Arbor, Mich. *1313 Henry St.*
ANSELM, Brother, A.B. (Mt. St. Joseph's Coll.) Teacher, St. Xavier High School, Louisville, Ky.
ARCHER, Prof. PETER. Canisius High School, Buffalo, N. Y., *651 Washington St.*
ARCHIBALD, Prof. R. C., Ph.D. (Strasbourg), Dr. (Univ. Padua), LL.D. (Mt. Allison) Brown Univ., Providence, R. I.
ARCHIBALD, Asst. Prof. R. G., Ph.D. (Chicago) Columbia Univ., New York, N. Y. *Hamilton Hall.*
ARMSTRONG, BEULAH M., Ph. D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*
ARMSTRONG, P. L., A.M. (Southwestern Presb. Univ.) Address unknown.
ARNAUD, J. J., B.S. (Coll. City of N. Y.) Asst. Astronomer, U. S. Naval Observatory, Washington, D. C. *2520 Hall Pl. N. W.*
ARNDT, Prof. W. F. C., Ph.D. (Göttingen) Grey Univ. Coll., Univ. of S. Africa, Bloemfontein, S. Africa.
ARNETT, ILA MAUD, A.M. (Kansas) Instr., Central Acad. and Coll., McPherson, Kans. *Central College.*
ARNOLD, Asst. Prof. H. E., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn. *9 Brainerd Ave.*
ARNOLD, Dean KATHERINE S., A.M. (Columbia) Philadelphia School for Christian Workers, Philadelphia, Pa. *1122 Spruce St.*
ARNOLD, Asst. Prof. W. C., M.S., A.M. (Chicago) DePauw Univ., Greencastle, Ind. *Box 466.*
ASHBAUGH, LAURA F. (Mrs. F. R.) A.B. (Pennsylvania) 924 W. Market St., Bethlehem, Pa.
ASHCRAFT, Prof. T. B., Ph.D. (Johns Hopkins) Colby Coll., Waterville, Me. *34 Pleasant St.*
ASHMUN, R. N., A.M. (Washington) Mathematician, Internat. Boundary Commission, Washington, D. C. *1524 Newton St. N. E.*
ASHTON, Prof. C. H., Ph.D. (Munich) Univ. of Kansas, Lawrence, Kans. *1200 Ohio St.*
ASTRACHAN, MAX, A.M. (Brown) Instr., Brown Univ., Providence, R. I.
ATCHISON, Prof. C. S., Ph.D. (Johns Hopkins) Washington and Jefferson Coll., Washington, Pa. *442 E. Bean St.*
ATKIN, Asst. Prof. EDITH I., A.M. (Columbia) Illinois State Normal Univ., Normal, Ill. *815 S. Fell Ave.*
AUDE, Prof. H. T. R., M.S. (Colgate) Colgate Univ., Hamilton, N. Y. *50 Broad St.*
AURELIUS, Brother, A.M. (Catholic Univ. of Amer.) Teacher, Keith Acad., Lowell, Mass.
AVERS, H. G., A.B. (George Washington) Geodetic Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
AYRES, FRANK, JR., M.S. (Chicago) Carlisle, Pa. *Box 53.*
AYRES, Asst. Prof. W. L., Ph.D. (Pennsylvania) Univ. of Michigan, Ann Arbor, Mich. *1225 White St.*

BABBITT, A. E., A.M. (Illinois) Secretary and Actuary, Lamar Life Ins. Co., Jackson, Miss.
BABCOCK, Dean R. W., Ph.D. (Wisconsin) Div. of General Sc., State Agric. Coll., Manhattan, Kans.
BABCOCK, Asst. Prof. WEALTHY, Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *1238 Miss. St.*
BACON, Prof. CLARA L., Ph.D. (Johns Hopkins) Goucher Coll., Baltimore, Md. *2316 N. Calvert St.*
BAGBY, Prof. L. C., A.M. (Kansas) Linsly Inst. of Tech., Wheeling, W. Va.
BAIDAFF, B. I., Dr. (Buenos Aires) Analisis, Univ. Nac., Buenos Aires, Argentina. *Belgrano 909.*

BAILEY, A. H., A.M. (Haverford) Grad. Asst., Grinnell Coll., Grinnell, Iowa.
BAILEY, Prof. E. A., M.S. (Emory) LaGrange Coll., LaGrange, Ga. *701 Vernon St.*
BAILEY, Prof. F. H., A.M. (Harvard) Massachusetts Inst. of Tech., Cambridge, Mass.
BAILEY, Asst. Prof. H. W., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *251 Math. Bldg.*
BAIRD, A. C., A.B. (Wooster) Vice-Prin., Allderdice High School, Pittsburgh, Pa. *920 East End Ave.*
BAKER, FRANCES E., M.S. (Iowa) 929 Kirkwood Ave., Iowa City, Iowa.
BAKER, Asso. Prof. IDA M., A.M. (Columbia) School of Educ., Western Reserve Univ., Cleveland, Ohio.
BAKER, Prof. R. P., Ph.D. (Chicago) Univ. of Iowa, Iowa City, Iowa. *929 Kirkwood Ave.*
BALDWIN, Asso. Prof. J. W., A.M. (Michigan) Registrar, Coll. of the City of Detroit, Detroit, Mich. *6433 Roosevelt Ave.*
BALL, N. H., B.S. (Mass. Inst. of Tech.) Instr., Massachusetts Inst. of Tech., Cambridge, Mass. *Room 2–170.*
BALLANTINE, Asso. Prof. J. P., Ph.D. (Chicago) Univ. of Washington, Seattle, Wash.
BALOF, C. A., M.S. (Iowa) Lincoln Coll., Lincoln, Ill. *529 N. Ottawa St.*
BAMFORTH, Asst. Prof. F. R., Ph.D. (Queen's) Ohio State Univ., Columbus, Ohio. *290 East 15th Ave.*
BANES, Asst. Prof. GLADYS L., Ed.M. (Harvard), A.M. (Radcliffe) Butler Univ., Indianapolis, Ind. *1556 Brookside Ave.*
BANKS, Prof. N. A., M.S. (Wilberforce) Tillotson Coll., Austin, Tex. *Box 441.*
BARBER, H. C., A.M. (Amherst) Head of Dept., Phillips Exeter Acad., Exeter, N. H. *79 Front St.*
BARBOUR, Asst. Prof. J. M., A.M. (Dickinson) Music, Wells Coll., Aurora, N. Y. *Forest Home, Ithaca, N. Y.*
BAREIS, Asst. Prof. GRACE M., Ph.D. (Ohio State) Ohio State Univ., Columbus, Ohio. *Canal Winchester, Ohio.*
BARNARD, Asso. Prof. R. W., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill. *Box 35, Eckhart Hall.*
BARNES, WALTER N., B.S. (Chicago). Address unknown.
BARNETT, Asso. Prof. I. A., Ph.D. (Chicago) Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*
BARNETT, Asst. Prof. JOSEPH, JR., A.M. (Columbia) Oklahoma A. and M. Coll., Stillwater, Okla. *226 West St.*
BARNEY, IDA, Ph.D. (Yale) Research Asst., Yale Univ. Observatory, New Haven, Conn. *110 Linden St.*
BARNHART, Prof. C. A., A.M. (Illinois) Univ. of New Mexico, Albuquerque, N. Mex. *115 S. Maple St.*
BARNHILL, Prof. J. F., A.M. (Columbia) Michigan State Normal College, Ypsilanti, Mich.
BARR, Asst. Prof. C. F., M.S. (Chicago) Univ. of Wyoming, Laramie, Wyo. *515 S. 8th St.*
BARRETT, A. J., A.B. (Arkansas) Teacher, High School, Jamaica, N. Y. *8620–162nd St.*
BARRICK, Prof. D. L., M.S. (Oklahoma) 507½ Baltimore Ave., Muskogee, Okla.
BARROW, Prof. D. F., Ph.D. (Harvard) Univ. of Georgia, Athens, Ga. *260 Cherokee Ave.*
BARTKY, Asst. Prof. WALTER, Ph.D. (Chicago) Mathl. Astr., Univ. of Chicago, Chicago, Ill.
BARTON, Prof. HELEN, Ph.D. (Johns Hopkins) Head of Dept., North Carolina Coll., Greensboro, N. C. *1027 Spring Garden St.*
BASOCO, Asst. Prof. M. A., Ph.D. (Calif. Inst. of Tech.) Univ. of Nebraska, Lincoln, Nebr. *Dept. of Math.*
BASSLER, KATHERINE R., A.M. (Bryn Mawr) Teacher, Bryn Mawr School for Girls, Baltimore, Md. *On leave 1931-32, U. S. Natl. Museum, Washington, D. C.*
BATCHELDER, Asso. Prof. P. M., Ph.D. (Harvard) Pure Math., Univ. of Texas, Austin, Tex. *808 W. 22nd St.*
BATEMAN, Prof. HARRY, Ph.D. (Johns Hopkins) California Inst. of Tech., Pasadena, Calif.
BATEN, Asst. Prof. W. D., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *914 S. State St.*
BATTIG, LEON, A.M. (Wisconsin) Instr., Extension Div., Univ. of Wisconsin, Milwaukee, Wis. *2542 N. Buffum St.*
BAUER, Prof. G. N., Ph.D. (Columbia) Statistics, Univ. of New Hampshire, Durham, N. H.
BAUER, L. A., Ph.D. (Berlin) Director Emeritus, Dept. of Terrestrial Magnetism, Carnegie Institution of Washington, Washington, D. C. *5241 Broad Branch Rd.*
BAUER, L. M., A.B. (Oakland City Coll.) Teacher, Menaul School, Albuquerque, N. Mex.
BAUR, Asso. Prof. P. E., M.S. (Michigan) Math. and Drawing, Baldwin-Wallace Coll., Berea, Ohio. *114 E. Grand St.*
BAXTER, W. E., A.M. (Kentucky) Grad. Asst., Univ. of Kentucky, Lexington, Ky. *McVey Hall.*

BAY, J. C. Librarian, The John Crerar Library, Chicago, Ill. *86 E. Randolph St.*
BAYS, Dr. SÉVÉRIN. Prof. ord., Faculté des Sciences, Univ. de Fribourg, Bethlehem, Fribourg, Switzerland.
BEAR, A. W., B.S. (Marquette) Instr., Marquette Univ., Milwaukee, Wis. *2016 W. Scott St.*
BEATLEY, Asso. Prof. RALPH, A.M. (Columbia) Harvard Grad. School of Educ., Cambridge, Mass.
BEATTY, Asst. Prof. H. M., A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *200 Tibet Rd.*
BEATTY, Prof. SAMUEL, Ph.D. (Toronto) Univ. of Toronto, Toronto, Ont., Can.
BEATY, Asso. Prof. E. B., A.M. (California) Oregon Agric. Coll., Corvallis, Ore. *21 N. 27th St.*
BEAVER, R. A., A.M. (Columbia) Instr., State Coll. for Teachers, Albany, N. Y.
BECKWITH, Prof. ETHELWYNN R. (Mrs. W. E.), Ph.D. (Radcliffe) Head of Dept., Milwaukee Downer Coll., Milwaukee, Wis.
BEENKEN, MAY M., Ph.D. (Chicago) Head of Dept., State Teachers Coll., Oshkosh, Wis.
BEETLE, Prof. R. D., Ph.D. (Princeton) Dartmouth Coll., Hanover, N. H.
BEISEL, Asst. Prof. B. R., B.S. (Allegheny) Allegheny Coll., Meadville, Pa. *345 Highland Ave.*
BELL, Asst. Prof. CLIFFORD, Ph.D. (California) Univ. of California at Los Angeles, Los Angeles, Calif. *2414 S. Burnside Ave.*
BELL, Prof. E. T., Ph.D. (Columbia) California Inst. of Tech., Pasadena, Calif.
BELL, LOIS E., A.M. (Kansas) Instr., Independence Jr. Coll., Independence, Kans. *610 N. Ninth St.*
BELL, Prof. TALMON, A.B. (Sterling) Sterling Coll., Sterling, Kans.
BELLAMY, B. C., B.S.C.E. (Wyoming) Civil Engineer, Laramie, Wyo.
BENANDER, C. A., A.B. (Harvard) 1651 Crawford Rd., Cleveland, Ohio.
BENDER, Asst. Prof. H. A., Ph.D. (Illinois) Univ. of Akron, Akron, Ohio.
BENEDICT, Pres. H. Y., Ph.D. (Harvard) Univ. of Texas, Austin, Tex.
BENEDICT, Prof. SUZAN R., Ph.D. (Michigan) Smith College, Northampton, Mass. *12 Barrett Pl.*
BENEDICTA, Sister M. (BOYLE), B.S. (Canisius) Teacher, Villa Maria Coll., Erie, Pa. *W. 8th St.*
BENNETT, Prof. A. A., Ph.D. (Princeton) Brown Univ., Providence, R. I.
BENNETT, Prof. J. L., A.M. (Kansas) Physics, Ottawa Univ., Ottawa, Kans.
BENNETT, Asst. Prof. THEODORE, Ph.D. (Illinois) Univ. of Wisconsin, Madison, Wis.
BERGER, EDLA G., A.M. (Minnesota) Mathematician, Actuarial Dept., Equitable Life Assurance Soc., New York, N. Y. *1411-161st St., Beechhurst, L. I., N. Y.*
BERGSTRESSER, C. A., A.M. (Lafayette), M.S. (Pennsylvania) Head of Dept., Jamaica High School, New York, N. Y. *780 St. Marks Ave., Brooklyn, N. Y.*
BERKELEY, L. M., A.M. (Virginia) 36 W. 91st St., New York, N. Y.
BERNSTEIN, Prof. B. A., Ph.D. (California) Univ. of California, Berkeley, Calif. *2785 Shasta Rd.*
BERRY, Prof. E. M., Ph.D. (Iowa) Lynchburg Coll., Lynchburg, Va. *504 Westwood, Ave.*
BERRY, Asso. Prof. GRACE E., A.M. (Mount Holyoke) Pomona Coll., Claremont, Calif. *353 W. 11th St.*
BERRY, Prof. W. J., A.M. (Harvard) Poly. Inst. of Brooklyn, Brooklyn, N. Y. *224 St. John's Pl.*
BERRY, W. J., M.S. (Colorado) Metropolitan Life Ins. Co., New York, N. Y. *609 W. 137 St.*
BERT, Prof. O. F. H., A.M. (Geneva) Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BERTRAND, Sister M. (WALTON), Ph.D. (Fordham) Teacher, Marywood Coll., Scranton, Pa.
BETTINGER, Asst. Prof. A. K., A.M. (Wisconsin) Creighton Univ., Omaha, Nebr.
BETZ, WILLIAM, A.M. (Rochester) Vice-Prin., East High School, Rochester, N. Y. *652 Melville St.*
BEVERIDGE, Prof. H. R., Ph.D. (Illinois) Monmouth Coll., Monmouth, Ill. *316 S. 9th St.*
BIBB, Asst. Prof. S. F., M.S. (Chicago) Armour Inst. of Tech., Chicago, Ill.
BICKERSTAFF, Asst. Prof. T. A., A.M. (Mississippi) Univ. of Mississippi, University, Miss. *Box 23.*
BIGBEE, J. A., B.S. (Drury) High School, Little Rock, Ark. *3110 Battery St.*
BILL, Prof. E. G., Ph.D. (Yale) Dean of Freshmen, Director of Admissions, Dartmouth Coll., Hanover, N. H.
BINGLEY, Prof. G. A., A.M. (Princeton) St. John's Coll., Annapolis, Md. *Brice House.*
BIRCHBY, W. N., A.M. (Colo. Coll.) Instr., California Inst. of Tech., Pasadena, Calif.
BIRCHENOUGH, Prof. HARRY, A.M. (Columbia) State Coll. for Teachers, Albany, N. Y. *687 Hudson Ave.*

BIRKHOFF, Prof. G. D., Ph.D. (Chicago) Harvard Univ., Cambridge, Mass. *984 Memorial Dr.*
BLACK, Asst. Prof. FLORENCE L., Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *1300 Louisiana St.*
BLACK, L. G., A.M. (Colorado) Instr., Purdue Univ., W. Lafayette, Ind. *202 North St.*
BLACK, Prof. L. T., A.M. (Michigan) Ashland Coll., Ashland, Ohio.
BLAIR, Prof. HAROLD, A.M. (Michigan) Western State Teachers Coll., Kalamazoo, Mich. *1220 Academy St.*
BLAIR, Asst. Prof. LEORA, A.M. (Chicago) State Normal Coll., Natchitoches, La.
BLAIR, Asst. Prof. R. V., A.M. (Peabody) Vanderbilt Univ., Nashville, Tenn.
BLAIR, VEVIA, A.M. (Columbia) Teacher, Horace Mann School, New York, N. Y. *509 W. 121 St.*
BLAKE, ARCHIE, M.S. (Chicago) Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C. *212 A Street S.E.*
BLAU, L. W., Ph.D. (Texas) Director of Geophysical Research, Humble Oil and Refining Co., Houston, Tex. *2027 Colquitt Ave.*
BLICHFELDT, Prof. H. F., Ph.D. (Leipzig) Stanford Univ., Stanford University, Calif. *Box 875.*
BLISS, Prof. G. A., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill.
BLUE, Prof. A. H., Ph.D. (Iowa) Western Union Coll., LeMars, Iowa.
BLUMBERG, A. A., A.B. (Texas) Instr., A. and M. Coll., College Station, Tex. *Box 149, Faculty Exchange.*
BLUMBERG, Prof. HENRY, Ph.D. (Göttingen) Ohio State Univ., Columbus, Ohio. *76 E. Blake Ave.*
BLUMENTHAL, L. M., Ph.D. (Johns Hopkins) Instr., Rice Inst., Houston, Tex.
BOBERTZ, W. E., M.S.E. (Michigan) Research Engineer, Union Switch and Signal Co., Swissvale, Pa. *3148 Woodward Ave., Detroit, Mich.*
BOEDER, PAUL, A.M. (Pennsylvania) Univ. of Göttingen, Göttingen, Germany. *Weenderlandstr. 46.*
BOGARD, Prof. A., A.M. (Chicago) Coll. of St. Teresa, Winona, Minn. *1075 W. 7th St.*
BOHNERT, J. I., Jr. Student, Carnegie Inst. of Tech., Pittsburgh, Pa. *109 Grandview Ave., Mt. Washington Sta.*
BOLKS, STANLEY, M.S. (Iowa State Coll.) Instr., Purdue Univ., W. LaFayette, Ind. *153 S. Sheetz St.*
BOLTON, R. W., E.E. (Illinois) 2000 Tenth St., Glendale, Calif.
BOND, Prof. J. D., Ph.D. (Michigan) Univ. of Tennessee, Knoxville, Tenn.
BOND, Pres. O. J., LL.D. (South Carolina) The Citadel, The Milit. Coll. of South Carolina, Charleston, S. C.
BOND, Prof. W. M., A.M. (Columbia) Waynesburg Coll., Waynesburg, Pa. *Fort Jackson Hotel.*
BOND, W. V. Student, Univ. of Göttingen, Göttingen, Germany. *Weenderlandstr. 46.*
BOOTHROYD, Prof. S. L., M.S. (Colo. State Coll.) Astr. and Geodesy, Cornell Univ., Ithaca, N. Y.
BOREL, ÉMILE, Docteur es Sc. (Paris) Prof. à la Faculté des Sciences de Paris, Paris, France. *32 rue du Bac, Paris VII.*
BORGER, Prof. R. L., Ph.D. (Chicago) Ohio Univ., Athens, Ohio. *Fenzel Apts.*
BORGMAN, W. M., Jr. B.S. (Michigan) Instr., Coll. of the City of Detroit, Detroit, Mich.
BORTOLOTTI, Prof. ETTORE, Dottore in mat. (Bologna) Univ. of Bologna, Bologna, Italy. *100 Via Maggiore.*
BOWDEN, Prof. JOSEPH, Ph.D. (Yale) Adelphi Coll., Garden City, N. Y. *33 Exeter St., Brooklyn, N. Y.*
BOWER, JULIA W., A.M. (Syracuse) Grad. Student, Univ. of Chicago, Chicago, Ill.
BOWER, O. K., Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill. *714 Arlington Ct., Champaign, Ill.*
BOWERSOX, Asso. Prof. E. R., E.E. (Iowa) Physics, Butler Univ., Indianapolis, Ind.
BOWLES, Asst. Prof. C. F., Ph.D. (Chicago) State School of Mines, Rapid City, S. Dak. *Box 922.*
BOYCE, M. G., Ph.D. (Chicago) Instr., Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.
BOYD, ELIZABETH N., A.M. (Columbia) Instr., Math. and Chem., Greenbrier Coll., Lewisburg, W. Va.
BOYD, Dean P. P., Ph.D. (Cornell) Head of Dept., Univ. of Kentucky, Lexington, Ky. *119 Waller Ave.*
BRADLEY, A. D., A.M. (Teachers Coll., Columbia) Instr., Hunter Coll., New York, N.Y. *8 S. Broadway, Dobbs Ferry, N. Y.*
BRADLEY, Asso. Prof. H. C., B.S. (Mass. Inst. of Tech.) Mass. Inst. of Tech., Cambridge, Mass.

BRADSHAW, Prof. J. W., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Rd.*
BRAMBLE, Prof. C. C., Ph.D. (Johns Hopkins) Math. and Mech., Postgraduate School, U. S. Naval Acad., Annapolis, Md. *3 Fifth St.*
BRAND, Prof. LOUIS, Ph.D. (Harvard) Univ. of Cincinnati, Cincinnati, Ohio. *2605 University Ct.*
BRANDEBERRY, Prof. J. B., A.M. (Ohio State) Univ. of the City of Toledo, Toledo, Ohio.
BRANDNER, F. A., A.M. (Chicago) Instr., Iowa State Coll., Ames, Iowa. *Dept. of Math.*
BRATTON, Prof. W. A., Sc.D. (Williams) Dean and Prof., Whitman Coll., Walla Walla, Wash. *570 Boyer Ave.*
BRAUN, ANTHONY, A.M. (Catholic Univ.) Franciscan Monastery, Washington, D. C.
BRAY, Asst. Prof. H. E., Ph.D. (Rice) Rice Institute, Houston, Tex.
BRECKENRIDGE, W. E., A.M. (Yale) Stuyvesant High School, New York, N. Y. *345 E. 15th St.*
BREIT, Prof. GREGORY, Ph.D. (Johns Hopkins) Physics, New York Univ., New York, N. Y.
BRENKE, Prof. W. C., Ph.D. (Harvard) Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st St.*
BREWSTER, Asst. Prof. J. A., A.B. (Harvard) Coll. of the City of New York, New York, N. Y. *Hudson View Gardens, 183 and Pinehurst Ave.*
BREZLER, W. J., A.M. (Grove City) Wartburg Normal Coll., Waverly, Iowa.
BRINK, Prof. R. W., Ph.D. (Harvard) Univ. of Minnesota, Minneapolis, Minn. *55 Williams Ave. S.E.*
BRISTOL, W. A., A.M. (Pennsylvania) Instr., Temple Univ., Philadelphia, Pa.
BRITTON, JACK, A.B. (Clark) Engg. Math., Univ. of Colorado, Boulder, Colo. *1145 Pennsylvania Ave.*
BRIXEY, Asst. Prof. J. C., A.M. (Oklahoma) Univ. of Oklahoma, Norman, Okla. *417 Elm St.*
BROCKETT, Asso. Prof. GERTRUDE L., A.M. (Brown) Houghton Coll., Houghton, N. Y.
BRODIE, Prof. W. M., M.E. (Va. Poly. Inst.), A.M. (Columbia) Virginia Poly. Inst., Blacksburg, Va.
BROMWELL, ALICE, A.M. (Nebraska) Instr., Monticello Seminary, Godfrey, Ill.
BROOKE, Prof. W. E., A.M. (Nebraska) Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
BROWN, A. B., Ph.D. (Harvard) Instr., Columbia Univ., New York, N. Y. *2940 Broadway, Room 605.*
BROWN, Asst. Prof. B. H., Ph.D. (Harvard) Dartmouth Coll., Hanover, N. H. *1 S. Balch Rd.*
BROWN, D. M., A.B. (Illinois) Asst., Univ. of Illinois, Urbana, Ill. *160 Math. Bldg.*
BROWN, E. C., A.M. (Maine) Worcester Poly. Inst., Worcester, Mass. *25 Schussler Rd.*
BROWN, Prof. E. W., D.Sc. (Cambridge, Eng.) Yale Univ., New Haven, Conn. *116 Everet St.*
BROWN, G. M., M.Sc. (Univ. of Leeds, England) 3215 N. Keating Ave., Chicago, Ill.
BROWN, Prof. H. S., M.S. (Lafayette) Hamilton Coll., Clinton, N. Y.
BROWN, Prof. LILLIAN O., A.M. (Columbia) Hood Coll., Frederick, Md.
BROWN, Prof. MYRTLE C., A.M. (Texas) Teachers Coll., Denton, Tex. *1415 W. Oak St.*
BROWN, O. E., Ph.D. (Chicago) Instr., Case School of Appl. Sci., Cleveland, Ohio. *1754 Noble Rd., E. Cleveland, Ohio.*
BROWN, Asso. Prof. T. H., Ph.D. (Yale) Business Statistics, Grad. School of Bus. Admin., Harvard Univ., Cambridge, Mass. *25 Meadow Way.*
BROWNE, Asso. Prof. E. T., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N. C. *730 E. Franklin St.*
BRUCE, C. W., A.M. (Virginia) Grad. Student, Univ. of Virginia, University, Va.
BRUCE, Prof. R. E., Ph.D. (Boston Univ.) Boston Univ., Boston, Mass.
BRYAN, Asso. Prof. N. R., Ph.D. (Columbia) Univ. of Maine, Orono, Me. *4 University Pl.*
BRYANT, E. S., Ed.M. (Harvard) Acting Prin., High School, Everett, Mass. *53 Lexington St.*
BUCHANAN, Prof. DANIEL, Ph.D. (Chicago) Dean of Faculty of Arts and Sciences, Univ. of British Columbia, Vancouver, B.C., Can.
BUCHANAN, Prof. H. E., Ph.D. (Chicago) Tulane Univ., New Orleans, La.
BUCHANAN, Asso. Prof. SCOTT, Ph.D. (Harvard) Philosophy, Univ. of Virginia, University, Va.
BUELL, C. E., A.B. (Oberlin) Grad. Student, Ohio State Univ., Columbus, Ohio. *47 E. Woodruff Ave.*
BULLARD, Prof. J. A., Ph.D. (Clark) Math. and Mech., Univ. of Vermont, Burlington, Vt. *110 Summit St.*
BULLARD, N. H., B.S. (Georgia) Head of Dept., High School, Ft. Pierce, Fla. *115 N. 12th St.*
BULLITT, W. M., B.S. (Princeton), LL.B. (Louisville) 1710–1711 Inter Southern Bldg., Louisville, Ky.

BUMER, Asst. Prof. C. T., Ph.D. (Ohio State) Kenyon Coll., Gambier, Ohio.
BUNYAN, Asst. Prof. L. H., Ph.D. (Wisconsin) Rutgers Univ., New Brunswick, N. J. *Dept. of Math.*
BURDICK, R. D., A.M. (Columbia) Instr., Coll. of the City of New York, New York, N. Y. *6083 Broadway.*
BURGESS, H. T., Ph.D. (Yale) Morningside Drive, Morningside, Milford, Conn.
BURGESS, R. W., Ph.D. (Cornell) Chief Statistician, Western Electric Co., New York, N. Y. *195 Broadway.*
BURINGTON, R. S., Ph.D. (Ohio State) Instr., Case School of Appl. Sc., Cleveland, Ohio *3417 Beechwood Ave., Cleveland Heights, Ohio.*
BURKE, J. G., A.M. (Mt. St. Mary's) Vice Pres., Mt. St. Mary's Coll., Emmitsburg, Md.
BURKETT, Asst. Prof. F. J. H., Ph.D. (New York Univ.) Union Coll., Schenectady, N. Y.
BURLEY, J. F., Civil Engineer, Washington, D. C. *3010 Wisconsin Ave.*
BURNAM, Prof. J. E., A.M. (Texas) Simmons Univ., Abilene, Tex.
BURNEY, M. SUE, A.M. (Chicago) 20 Rock Ridge, Greenwich, Conn.
BURROWS, W. R., B.S. (Chicago) Draftsman, Standard Oil Co. (Ind.) Whiting, Ind. *8222 Drexel Ave., Chicago, Ill.*
BURWELL, W. R., Ph.D. (Oxford, England) Cuyahoga Bldg., Cleveland, Ohio.
BUSHEY, Asst. Prof. J. H., Ph.D. (Michigan) Math. and Ins., Hunter Coll., New York, N. Y. *Park Ave. and 68th St.*
BUSHYAGER, G. R., A.M. (Pennsylvania State) Instr., Univ. of Detroit, Detroit, Mich.
BUSSEY, Asst. Dean and Prof. W. H., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *106 Folwell Hall.*
BUTLER, J. F., A.M. (St. Louis Univ.) St. Ignatius Coll., Chicago, Ill. *1076 W. Roosevelt Rd.*
BUTLER, L. G., A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash. *802 Linden Ave.*
BUTTERFIELD, Prof. A. D., A.M. (Columbia) Coll. of Eng., Univ. of Vermont, Burlington, Vt. *25 Colchester Ave.*
BUTTERFIELD, D. D., A.M. (Princeton) Instr., Phillips Exeter Acad., Exeter, N. H. *Webster Hall.*
BUXTON, Prof. W. H., A. M. (Oregon) Whitworth Coll., Spokane, Wash.
BYRD, SAM, A.M. (Arkansas) Teacher, Central High School, Tulsa, Okla. *1344 S. Birmingham Ave.*
BYRNE, Asso. Prof. W. E., Ph.D. (Rensselaer) Virginia Milit. Inst., Lexington, Va.
BYRNES, T. J., B.S. (St. Bonaventure) Teacher, Rochester Shop School, Rochester, N.Y. *73 Martin St.*

CAIRNS, S. S., Ph.D. (Harvard) Instr., Lehigh Univ., Bethlehem, Pa. *525 W. Union Blvd.*
CAIRNS, Prof. W. D., Ph.D. (Göttingen) Oberlin Coll., Oberlin, Ohio. *33 Peters Hall.*
CALKINS, Prof. HELEN, A.M. (Columbia) Pennsylvania Coll. for Women, Pittsburgh, Pa.
CALLAWAY, Prof. THEODOSIA T., B.S. (Columbia) Stephens Coll., Columbia, Mo. *417 Hitt St.*
CAMERON, R. H., A.B. (Cornell) Instr., Cornell Univ., Ithaca, N. Y. *134 College Ave.*
CAMP, Prof. B. H., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn. *110 Mt. Vernon St.*
CAMP, Prof. C. C., Ph.D. (Cornell) Univ. of Nebraska, Lincoln, Nebr. *Dept. of Math.*
CAMP, H. L., A.M. (Oklahoma) Instr., A. and M. Coll. of Texas, College Station, Tex. *P. O. Box 166, Faculty Exchange.*
CAMPBELL, Prof. A. D., Ph.D. (Cornell) Syracuse Univ., Syracuse, N. Y. *208 Westminster Ave.*
CAMPBELL, D. F., Ph.D. (Harvard) Actuary, 160 N. LaSalle St., Chicago, Ill.
CAMPBELL, G. A., Ph.D. (Harvard) Research Engineer, Amer. Tel. and Tel. Co., New York, N. Y. *Room 1654, 195 Broadway.*
CAMPBELL, G. C., M.S. (Iowa) Bell Telephone Labs., New York, N. Y. *463 West St.*
CAMPBELL, H. A., A.M. (Creighton) Teacher, Tech. High School, Omaha, Nebr. *32nd and Cuming.*
CAMPBELL, JESSIE R., A.B. (Syracuse) Instr., Hollywood Jr. Coll., Los Angeles, Calif. *5400 Brynhurst Ave.*
CAMPBELL, Prof. J. W., Ph.D. (Chicago) Univ. of Alberta, Edmonton South, Alta., Can.
CAMPBELL, Prof. W. B., A.M. (Cornell) Head of Dept., Judson Coll., Rangoon, Burma.
CANDY, Prof. A. L. Univ. of Nebraska, Lincoln, Nebr., *Station A.*
CANNING, JOSEPH, A.B. (Intermountain) Machinist, N. P. Ry. Co., Helena, Mont. *1813 Boulder Ave.*
CAPARO, Prof. J. A., Ph.D. (Notre Dame) Notre Dame Univ., Notre Dame, Ind. *1024 Leeper Ave., South Bend, Ind.*
CAPRON, Prof. PAUL, A.M. (Harvard) U. S. Naval Acad., Annapolis, Md. *203 Duke of Gloucester St.*

CAREY, Asso. Prof. E. F. A., M.S. (California) Univ. of Montana, Missoula, Mont.
CARIS, Pres. A. G., A.M. (Defiance) Defiance Coll., Defiance, Ohio.
CARIS, Asst. Prof. P. A., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *717 Runnymede Ave., Drexel Hill, Pa.*
CARLEN, MILDRED E., M.S. (Brown) Registrar, Grad. School, Brown Univ., Providence, R. I. *23 Exeter St.*
CARLSON, J. A., B.S. (Washington) Asso., Univ. of Washington, Seattle, Wash. *Dept. of Math.*
CARLSON, Asst. Prof. S. ELIZABETH, Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *3020 14th Ave. S.*
CARMAN, M. D., Ph.D. (Illinois) Head of Dept., State Teachers Coll., Murray, Ky.
CARMICHAEL, Prof. F. L., A.M. (Princeton) Math. and Statistics, Univ. of Denver, Denver, Colo. *2230 Colorado Blvd.*
CARMICHAEL, Prof. R. D., Ph.D. (Princeton) Univ. of Illinois, Urbana, Ill. *207 W. Washington Blvd.*
CARPENTER, Prof. D. R., A.M. (Princeton) Roanoke Coll., Salem, Va.
CARR, Prof. E. L., A.M. (Chicago) Head of Dept., Union Univ., Jackson, Tenn. *109 Camden St.*
CARR, Prof. F. E., Ph.D. (Chicago) Math. and Astr., Oberlin Coll., Oberlin, Ohio. *On leave 1931–32, Mount Wilson Observatory.*
CARROLL, Asso. Prof. EVELYN T., A.M. (Wells) Wells Coll., Aurora, N. Y.
CARROLL, Asst. Prof. I. S., A.M. (Columbia) Syracuse Univ., Syracuse, N. Y. *511 Comstock Ave.*
CARRUTH, Prof. W. M., A.B. (Cornell) Hamilton Coll., Clinton, N. Y.
CARSCALLEN, Asso. Prof. G. E., A.M. (Illinois) Wabash Coll., Crawfordsville, Ind. *112 N. Barr St.*
CARTER, C. C. Attorney at Law, Bluffs, Ill. *R.D. No. 2.* Honorary Life Member.
CARUS, E. H., Ph.D. (Chicago) LaSalle, Ill.
CARUTHERS, R. W. E., B.S. (Miss. A. and M. Coll.) Engineer, International General Electric Co., Buenos Aires, Argentina. *Victoria 618.*
CARVER, Prof. W. B., Ph.D. (Johns Hopkins) Cornell Univ., Ithaca, N. Y. *White Hall.*
CARY, R. L., A.M. (Haverford) Prinz-Louis-Ferdinandstr. 5, Berlin N W 7, Germany.
CEDERBERG, Prof. W. E., Ph.D. (Wisconsin) Augustana Coll., Rock Island, Ill. *2542 22½ Ave.*
CELL, Asst. Prof. J. W., A.M. (Illinois) School of Eng., Southern Methodist Univ., Dallas, Tex.
CHACE, ARNOLD B., Sc.D. Chancellor, Brown Univ., Providence, R. I. *99 Power St.*
CHAMBERS, Prof. G. G., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *251 S. 38th St.*
CHANG, Prof. H. C., Ph.D. (Michigan) Central Univ., Nanking, China.
CHARLES, Prof. R. L., A.M. (Lehigh) Physics, Franklin and Marshall Coll., Lancaster, Pa. *510 Race Ave.*
CHAROSH, MANNIS, B.S. (C.C.N.Y.) Teacher, New Utrecht High School, Brooklyn, N. Y.
CHASE, L. R. Instr., Rogers High School, Newport, R. I. *Boulevard Ter.*
CHELLEVOLD, Asst. Prof. J. O., A.M. (Northwestern) St. Paul Luther Coll., St. Paul, Minn.
CHENEY, Prof. W. F., Jr., Ph.D. (Mass. Inst. of Tech.) Connecticut Agric. Coll., Storrs, Conn.
CHITTENDEN, Prof. E. W., Ph.D. (Chicago) Univ. of Iowa, Iowa City, Iowa. *1101 Kirkwood Ave.*
CHRISTIAN, R. S., A.B. (Westminster) Instr., Westminster Coll., Fulton, Mo.
CHRISTMAN, LAURA E., A.M. (Wisconsin) Teacher, Senn High School, Chicago, Ill. *1217 Elmdale Ave.*
CHURCH, Asso. Prof. E. F., C.E. (Syracuse) Photogrammetry, Univ. of Syracuse, Syracuse, N. Y. *Parish, N. Y.*
CHURCH, W. R., A.M. (Pennsylvania) Grad. Student, Yale Univ., New Haven, Conn. *463 Skiff St., Hamden, Conn.*
CLACK, Prof. R. W., A.M. (Grinnell) Math. and Astr., Alma Coll., Alma, Mich. *209 W. Downie St.*
CLAIRE, C. N., B.S. (Alfred) Junior Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
CLARK, Asso. Prof. A. G., A.M. (Colorado) Colorado Agric. Coll., Fort Collins, Colo.
CLARK, R. F., A.B. (Williams) Chm., Math. Dept., DeWitt Clinton High School, New York, N. Y. *194 Christie Hts. St., Leonia, N. J.*
CLARKE, Prof. E. H., Ph.D. (Chicago) Math. and Astr., Hiram Coll., Hiram, Ohio. *Box 283.*
CLARKE, J. A., A.M. (Princeton) West Phila. High School for Boys, Philadelphia, Pa. *Devon, Pa.*

CLARKE, Sister M. BORGIA, A.M. (Catholic Univ.) Prof., Webster Coll., Webster Groves, Mo.
CLAWSON, Prof. J. W., M.A. (New Brunswick) Ursinus Coll., Collegeville, Pa.
CLAYTON, J. L., A.M. (Michigan) Instr., U. S. Naval Acad., Annapolis, Md. *20 Jefferson St.*
CLELAND, Prof. W. E., Ph.D. (Princeton) Geneva Coll., Beaver Falls, Pa. *North Hall.*
CLEMENTS, Prof. G. R., Ph.D. (Harvard) U. S. Naval Acad., Annapolis, Md. *7 Thompson St.*
COBB, Prof. H. E., A.M. (Wesleyan) Lewis Inst., Chicago, Ill.
COBLE, Prof. A. B., Ph.D. (Johns Hopkins) Univ. of Illinois, Urbana, Ill. *702 W. Washington Blvd.*
COE, Asst. Prof. C. J., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *2022 Hill St.*
COFFIN, Prof. L. M., A.M. (Michigan) Coe Coll., Cedar Rapids, Iowa. *1027 Second Ave.*
COHEN, Prof. ABRAHAM, Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore, Md.
COHEN, Asst. Prof. L. W., Ph.D. (Michigan) Univ. of Kentucky, Lexington, Ky. *363 Aylesford Pl.*
COHEN, Asst. Prof. TERESA, Ph.D. (Johns Hopkins) Pennsylvania State Coll., State College, Pa. *512 W. Beaver Ave.*
COLAW, J. M., A.M. (Dickinson) Attorney at Law, Monterey, Va.
COLE, Prof. LENA R., A.M. (Missouri) Central Normal Coll., Danville, Ind.
COLEMAN, Prof. J. B., Ph.D. (California) Univ. of South Carolina, Columbia, S. C.
COLEMAN, Prof. R. H., A.B. (Charleston) Coll. of Charleston, Charleston, S. C.
COLLIER, MYRTIE, Ph. D. (Strasbourg) 320½ N. Alexandria Ave., Los Angeles, Calif.
COLLINS, O. C., M.A. (Oxford, England) Instr., Univ. of Nebraska, Lincoln, Nebr. *1920 S. 26th St.*
COLLITON, J. W., C.E., E.M. (Lafayette) Head of Dept., High School, Trenton, N. J. *223 Highland Ave.*
COLPITTS, Asso. Prof. JULIA T., Ph.D. (Cornell) Iowa State Coll., Ames, Iowa. *29 Cranford Apts.*
COLWELL, Prof. R. C., Ph.D. (Princeton) Physics, West Virginia Univ., Morgantown, W. Va. *332 Demain Ave.*
COLYER, Prof. E. E., A.M. (Kansas) State Teachers Coll., Hays, Kans. *408 W. 15th St.*
COMSTOCK, Prof. C. E., A.M. (Knox) Bradley Poly. Inst., Peoria, Ill. *203 Fredonia Ave.*
CONDIT, Prof. I. S., A.M. (Parsons) State Teachers Coll., Cedar Falls, Iowa. *115 E. 11th St.*
CONGDON, Asso. Prof. A. R., A.M. (Nebraska) Pedagogy of Math., Univ. of Nebraska, Lincoln, Nebr. *Station A.*
CONKLING, R. P., A.B. (Cornell) Head of Dept., Newark Tech. School and Central High School, Newark, N. J. *31 N. 10th St.*
CONKWRIGHT, N. B., Ph.D. (Illinois) Asso., Univ. of Iowa, Iowa City, Iowa. *120 Physics Bldg.*
CONLEE, R. H., A.B. (New Mexico) Supt. of Public Schools, Mountainair, N. Mex.
CONSTABLE, MARY LOUISE, A.M. (Pennsylvania) Teacher, Philadelphia High School for Girls, Philadelphia, Pa. *The Whittier, 140 N. 15th St.*
CONSTANTINE, JUNE F., B.S. (Minnesota) Statistician, Child Welfare Research Sta., Univ. of Iowa, Iowa City, Iowa.
CONWELL, G. M., Ph.D. (Princeton) Master in Math., St. Paul's School, Concord, N. H.
CONWELL, Prof. H. H., M.S. (Kansas) Beloit Coll., Beloit, Wis.
COOK, Asso. Prof. A. J., Ph.D. (Chicago) Univ. of Alberta, Edmonton, Alta., Can. *Dept. of Math.*
COOLEY, HOLLIS, A.M. (Middlebury) Instr., New York Univ., New York, N. Y. *32 Waverley Pl.*
COOLIDGE, Prof. J. L., Ph.D. (Bonn) Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*
COOPER, Prof. A. E., Ph.D. (Chicago) Applied Math., Univ. of Texas, Austin, Tex. *Rice Ct., Rio Grande St.*
COOPER, ELIZABETH M., Ph.D. (Illinois) Prin., Buckingham School, Cambridge, Mass. *10 Buckingham St.*
COPE, Prof. T. F., Ph.D. (Chicago) Marietta Coll.. Marietta, Ohio.
COPELAND, Asso. Prof. LENNIE P., Ph.D. (Pennsylvania) Wellesley Coll., Wellesley, Mass. *14 Waban St.*
COPP, P. T., A.M. (Ohio State) Instr., Valparaiso Univ., Valparaiso, Ind. *812 Union St.*
CORAL, MAX, Ph.D. (Chicago) National Research Fellow, Harvard Univ., Cambridge, Mass. *23 Ware St.*
CORBIN, Prof. C. E., A.M. (Northwestern) Coll. of the Pacific, Stockton, Calif. *117 W. Euclid Ave.*
COREY, S. A. 1079 23rd St., Des Moines, Iowa.
CORLISS, J. J., Ph.D. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *5 East Hall.*

CORONA, Sister MARIA, Ph.D. (Fordham) Teacher, Mount St. Joseph Coll., Mount St. Joseph, Ohio.
CORRAL Y ALEMAN, JOSÉ ISAAC, Ing. de Mines. Director de Montes y Mines, Republic of Cuba, Havana, Cuba. *Calzada esquina 13, Vedado.*
COSBY, Prof. BYRON, A.M. (Missouri) Math. and Bus. Admin., State Teachers Coll., Kirksville, Mo. *707 E. Normal Ave.*
COURT, Asso. Prof. N. A., D.Sc. (Ghent, Belgium) Univ. of Oklahoma, Norman, Okla.
COWLES, Prof. W. H. H., A.M. (Columbia) Head of Dept., Math. and English, Pratt Inst., Brooklyn, N. Y. *215 Ryerson St.*
COWLEY, Prof. ELIZABETH B., Ph.D. (Columbia) Vassar Coll., Poughkeepsie, N. Y. *On leave of absence, 913 Arch St., N.S., Pittsburgh, Pa.*
COX, Dean C. S., A.M. (Vanderbilt) Dean and Prof., Southern Coll., Lakeland, Fla.
COX, Asso. Prof. E. F., Ph.D. (Cornell) Howard Univ., Washington, D. C.
CRAIG, Asst. Prof. C. C., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *3020 Angell Hall.*
CRAIG, Adj. Prof. H. V., Ph.D. (Wisconsin) Applied Math., Univ. of Texas, Austin, Tex. *3208 Grandview Ave.*
CRAMLET, Asst. Prof. C. M., Ph.D. (Washington) Univ. of Washington, Seattle, Wash. *Phil. Hall.*
CRANE, Asso. Prof. RUFUS, A.M. (Ohio State) Math. and Eng., Ohio Wesleyan Univ., Delaware, Ohio. *39 Montrose Ave.*
CRATHORNE, Asso. Prof. A. R., Ph.D. (Göttingen) Univ. of Illinois, Urbana, Ill. *802 Pennsylvania Ave.*
CRAWLEY, Prof. E. S., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
CRENSHAW, Prof. B. H., M.E. (Alabama Poly. Inst.) Alabama Poly. Inst., Auburn, Ala.
CROCKETT, Prof. C. W., C.E. (Rensselaer) Math. and Astr., Rensselaer Poly. Inst., Troy, N. Y. *221 Stow Ave.*
CROMWELL, J. W., Jr., A.M. (Dartmouth) Certified Publ. Accountant; Controller, Howard Univ., Washington, D. C. *1815-13th St. N.W.*
CROOM, Pres. A. S. Oklahoma Christian Coll., Cordell, Okla.
CROSS, Asst. Prof. SAVANNAH L., A.M. (Michigan) State Teachers Coll., Nacogdoches, Tex. *On leave 1931–32, Stanford Univ., Stanford University, Calif.*
CROWE, Asso. Prof. S. E., A.M. (Michigan) Michigan State Coll., East Lansing, Mich.
CRUDELI, Prof. UMBERTO, Dr. in mat. (Rome) Director, Math. Inst., Univ. of Cagliari, Sardinia, Italy.
CRUM, Prof. W. L., Ph.D. (Yale) Economic Statistics, Harvard Univ., Cambridge, Mass.
CULMER, ORPHA A., A.M. (Michigan) State Normal School, Florence, Ala. *217 Oneal Ave.*
CUMMING, Asso. Prof. FORREST, A.M. (Georgia) Univ. of Georgia, Athens, Ga. *Box 786.*
CUMMINGS, Prof. LOUISE D., Ph.D. Vassar Coll., Poughkeepsie, N. Y.
CUNHA, Prof. PEDRO JOSÉ DA. Math. Analysis, Faculty of Sc., Lisbon Univ., and Rector of the Univ., Lisbon, Portugal. *Rua de S. Bento, No. 706.*
CURRIER, Asso. Prof. C. H., A.M. (Brown) Brown Univ., Providence, R. I.
CURRY, Asst. Prof. H. B., Ph.D. (Göttingen) Pennsylvania State Coll., State College, Pa. *805 W. Beaver Ave. On leave 1931–32, Univ. of Chicago.*
CURTIS, Prof. H. B., Ph.D. (Cornell) Lake Forest Coll., Lake Forest, Ill. *College Hall.*
CURTISS, Prof. D. R., Ph.D. (Harvard) Northwestern Univ., Evanston, Ill. *2023 Sherman Ave.*
CUTLER, E. H., Ph.D. (Harvard) Instr., Lehigh Univ., Bethlehem, Pa. *Dept. of Math.*
CUTTING, L. H., A.M. (Missouri) Teacher, Westport High School, Kansas City, Mo. *406 E. 43rd St.*

DADOURIAN, Prof. H. M., Ph.D. (Yale) Trinity Coll., Hartford, Conn. *125 Vernon St.*
DAHLENE, Prof. OSCAR, M.S. (Kansas) Eng. and Math., Univ. of Alabama, University, Ala.
DALAKER, Prof. H. H., Ph.D. (Cornell) Math. and Mech., Coll. of Eng. and Arch., Univ. of Minnesota, Minneapolis, Minn. *523 Walnut St. S.E.*
DALE, Asst. Prof. JULIA, Ph.D. (Cornell) Duke Univ., Durham, N. C.
DALTON, Prof. J. P., D.Sc. (St. Andrews; Cape) Univ. of the Witwatersrand, Johannesburg, S. Africa, *P. O. Box 1176.*
DAME, Prof. MABELLE C., Ph.D. (Chicago) 50 Cedar St., Amesbury, Mass.
DANCER, Asst. Prof. C. W., A.M. (Ohio State) Univ. of the City of Toledo, Toledo, Ohio. *1929 Ottawa Dr.*
DANIELLS, Asst. Prof. MARIAN E., M.S. (Iowa State) Iowa State Coll., Ames, Iowa.
DAPPERT, J. W. Civil Engineer, Taylorville, Ill. *Lock Box 141.*
DARKOW, Asst. Prof. MARGUERITE D., Ph.D. (Chicago) Hunter Coll., New York, N. Y.
DARLING, F. W., A.B. (Cornell) Asso. Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

DARNELL, ALBERTUS, Ph.B. (Michigan) Asst. Dean, Coll. of the City of Detroit, Detroit, Mich. *1216 Edison Ave.*
DARRAGH, MARGARET L., A.B. (Hanover Coll.) Teacher, Roosevelt High School, East Chicago, Ind. *4144 Northcote Ave.*
DAUENHAUER, HELEN A., A.B. (Hunter) Tutor, Hunter Coll., New York, N. Y. *217 E. 176 St.*
DAUGHERTY, R. D., M.S. (Iowa) Instr., State Agric. Coll., Manhattan, Kans.
DAUS, Asso. Prof. P. H., Ph.D. (California) Univ. of California at Los Angeles, Los Angeles, Calif. *405 N. Hilgard Ave.*
DAVIS, C. H. 95 Jefferson St., Hartford, Conn.
DAVIS, Prof. D. R., Ph.D. (Chicago) State Teachers Coll., Montclair, N. J.
DAVIS, Asst. Prof. H. A., Ph.D. (Cornell) West Virginia Univ., Morgantown, W. Va. *307 Duquesne Ave.*
DAVIS, Asso. Prof. H. T., Ph.D. (Wisconsin) Indiana Univ., Bloomington, Ind. *712 E. Second St.*
DAVIS, Asso. Prof. J. E., A.M. (Wisconsin) Drexel Inst., Philadelphia, Pa.
DAVIS, Prof. J. M., A.M. (Kentucky) Univ. of Kentucky, Lexington, Ky. *340 Madison Pl.*
DAVIS, J. W., A.M. (Yale) Teacher, High School, Boston, Mass. *17 Wrentham St., Dorchester, Mass.*
DAVIS, W. M., M.S. (Iowa) Instr., Armour Inst. of Tech., Chicago, Ill. *3300 Federal St.*
DAWSON, Prof. J. D., M.S. (Ohio State) Antioch Coll., Yellow Springs, Ohio.
DAY, EDNA L., A.M. (Northwestern) Britt, Iowa.
DEAN, ALICE C., A.M. (Rice) Fellow, Rice Inst., Houston, Tex.
DEAN, MILDRED W. (Mrs. C. E.), Ph.D. (Johns Hopkins) 3908-213 St., Bayside, L. I., N. Y.
DEARMAN, Prof. D. S., A.M. (Vanderbilt) State Teachers Coll., Hattiesburg, Miss.
DECHERD, Adj. Prof. MARY E., A.M. (Texas) Pure Math., Univ. of Texas, Austin, Tex. *2313 Nueces St.*
DECKER, Prof. F. F., Ph.D. (Syracuse) Syracuse Univ., Syracuse, N. Y. *312 Marshall St.*
DECLEENE, Rev. L. A.V., Ph.D. (Catholic Univ.) Prof., St. Norbert Coll., West DePere, Wis.
DECOU, Prof. E. E., M.S. (Chicago) Univ. of Oregon, Eugene, Ore. *929 Hilyard St.*
DEDERICK, L. S., Ph.D. (Harvard) Mathematician, Aberdeen Proving Ground, Md.
DE LA GARZA, E. Box 304, Brownsville, Tex.
DELONG, Prof. I. M., A.M. (Simpson) Emeritus, Univ. of Colorado, Boulder, Colo. *517 Pine St.*
DELURY, Prof. A. T., M.A. (Toronto) Univ. of Toronto, Toronto, Ont., Can.
DEMING, Prof. R. M., B.S. (Iowa State) Upper Iowa Univ., Fayette, Iowa.
DENNISON, C. H. Grad. Mass. Inst. of Tech. Chemist, Amer. Rubber Co., Cambridge, Mass. *183 Norfolk St., Wollaston, Mass.*
DENNY, Prof. A. K., M.S. (Lincoln) Prof. and Registrar, Lincoln Coll., Lincoln, Ill. *221 Oglesby Ave.*
DENTON, Asst. Prof. W. W., Ph.D. (Illinois) Univ. of Michigan, Ann Arbor, Mich. *1014 Cornwell Pl.*
DEUTSCH, RALPH, A.M. (Columbia) Teacher, New Utrecht High School, Brooklyn, N. Y. *654 Warwick St.*
DICE, ELIZABETH S., A.M. (Texas) Teacher, High School, Dallas, Tex.
DICKINSON, Prof. C. N. Hollins College, Hollins, Va.
DICKSON, Prof. L. E., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill., *5535 University Ave.*
DICKSTEIN, Prof. S., Ph.D. Math. and Hist., Univ. of Warsaw, Warsaw, Poland. *Marszalkowska St. 117.*
DILLINGHAM, Prof. ALEXANDER. U. S. Naval Acad., Annapolis, Md.
DIMICK, Prof. C. E., A.M. (Pennsylvania) U. S. Coast Guard Acad., New London, Conn.
DINES, Dean L. L., Ph.D. (Chicago) Univ. of Saskatchewan, Saskatoon, Sask., Can.
DINWIDDIE, Pres. A. B., Ph.D. (Virginia) Pres., and Prof. of Math., Tulane Univ., New Orleans, La. *Station 20.*
DOAK, Prof. ELEANOR C., Ph.B. (Chicago) Mount Holyoke Coll., South Hadley, Mass.
DOAN, Asst. Prof. C. S., A.M. Purdue Univ., W. LaFayette, Ind. *415 Russell St.*
DOBBIN, Sister MARIOLA, A.M. (Wisconsin) Rosary Coll., River Forest, Ill.
DO BELL, Asst. Prof. H. A., Ph.D. (Cornell) State Coll. for Teachers, Albany, N. Y.
DODD, Prof. E. L., Ph.D. (Yale) Univ. of Texas, Austin, Tex. *3012 West Ave.*
DOERMANN, Asst. Prof. F. W., Ph.D. (Vienna) Math. Physics, New York Univ., New York, N. Y. *University Hts.*
DONAGHHO, Prof. J. S., A.M. (Marietta) Univ. of Hawaii, Honolulu, T. H. *961 Alewa Dr.*
DONAHUE, Asso. Prof. J. E., A.M. (Harvard) Univ. of Vermont, Burlington, Vt. *Essex Junction, Vt.*
DORROH, J. L., Ph.D. (Texas) National Research Fellow, Princeton Univ., Princeton, N. J. *I-1 Prospect Apt.*

Dorwart, H. L., Ph.D. (Yale) Instr., Williams Coll., Williamstown, Mass. *Faculty Club.*
Dostal, Asst. Prof. B. F., A.M. (Indiana) Univ. of Florida, Gainesville, Fla. *Engineering Bldg.*
Dotterer, Prof. J. E., A.M. (Illinois) Head of Dept., Manchester Coll., North Manchester, Ind.
Dougherty, Lucy T., A.M. (Kansas) Instr., Jr. Coll., Kansas City, Kans. *Gould Hotel.*
Doughty, Annie W., A.M. (Radcliffe) Teacher, Dana Hall School, Wellesley, Mass.
Douglas, J. L., A.M. (Davidson) Davidson Coll., Davidson, N. C.
Douglass, Prof. R. D., Ph.D. (Mass. Inst. of Tech.) Mass. Inst. of Tech., Cambridge, Mass. *76 Drew Rd., Belmont, Mass.*
Downing, Prof. H. H., Ph.D. (Chicago) Univ. of Kentucky, Lexington, Ky. *138 State St.*
Dresden, Prof. Arnold, Ph.D. (Chicago) Swarthmore Coll., Swarthmore, Pa. *606 Elm Ave.*
Dressel, F. G., M.S. (Michigan) Instr., Duke Univ., Durham, N. C. *Box 4393.*
Drew, J. W., A.M. (Cornell) Head of Dept., Storer Coll., Harpers Ferry, W. Va. *Box 202.*
Driver, D. D., A.M. (Nebraska) Instr., Hesston Coll., Hesston, Kans.
Drummond, Asst. Prof. G. B., M.S. (U. S. Milit. Acad.) Oklahoma A. and M. Coll., Stillwater, Okla. *Box 161.*
Dube, Prof. L. H., Ph.D. (Gregorian Univ., Rome) Higher Math., Ottawa Univ., Ottawa, Ont., Can.
Duerksen, J. A., A.B. (Bethel) Asst. Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C. *3134 Monroe St. N.E.*
Duncan, Bertha K., Ph.D. (Texas) Instr., Phil. and Psych., Texas State Coll. for Women, Denton, Tex. *Box 604, C.I.A. Sta.*
Dunford, Nelson, Ph.B. (Chicago) Grad. Student, Univ. of Chicago, Chicago, Ill. *60 Snell Hall.*
d'Unger, V. C., A.M. (Little Rock) Lincoln Natl. Life Ins. Co., Fort Wayne, Ind.
Dunkel, Prof. Otto, Ph.D. (Harvard) Washington Univ., St. Louis, Mo.
Dunlap, L. T., B.S. (Marquette) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
Du Pasquier, Prof. L. G., Ph.D. (Zurich) Head of Dept., Univ. of Neuchâtel, Neuchâtel, Switzerland. *Beaux-Arts 4.*
Durell, Fletcher, Ph.D. (Princeton) Emeritus, formerly Head of Dept., Lawrenceville School, Lawrenceville, N. J. *Belleplain, N. J.*
Duren, Asst. Prof. W. L., Jr., Ph.D. (Chicago) Tulane Univ., New Orleans, La.
Durfee, Prof. W. H., Ph.D. (Cornell) Hobart Coll., Geneva, N. Y. *403 Pulteney St.*
Durfee, Prof. W. P., Ph.D. (Johns Hopkins) Hobart Coll., Geneva, N. Y. *639 Main St.*
Dustheimer, Prof. O. L., Ph.D. (Michigan) Astr. and Math., Baldwin-Wallace Coll., Berea, Ohio. *272 Beech St.*
Duval, Asso. Prof. E. P. R., A.M. (Harvard) Univ. of Oklahoma, Norman, Okla. *Faculty Exchange.*
Dwyer, Asso. Prof. P. S., A.M. (Penna. State) Antioch Coll., Yellow Springs, Ohio. *Box 94.*
Dye, L. A., Ph.D. (Cornell) Instr., Cornell Univ., Ithaca, N. Y. *White Hall 8.*

Eagles, Prof. T. R., A.M. (North Carolina) Howard Coll., Birmingham, Ala. *8016 Second Ave. S.*
Earhart, Franc C., A.M. (Wisconsin) Lenox Coll., Hopkinton, Iowa.
Earl, Asso. Prof. J. M., Ph.D. (Minnesota) Univ. of Omaha, Omaha, Nebr. *3160 Dodge St.*
Earle, Prof. M. D., A.M. (Furman) Furman Univ., Greenville, S. C.
Echols, Col. C. P. Prof., U. S. Milit. Acad., West Point, N. Y.
Echols, Prof. W. H., C.E. (Virginia) Univ. of Virginia, University, Va.
Edington, Prof. W. E., Ph.D. (Illinois) Head of Dept., DePauw Univ., Greencastle, Ind.
Edmondson, Aubrey, B.S. in C.E. (Miss. A. and M. Coll.) Instr., Miss. A. and M. Coll., Starkville, Miss. *Box 2.*
Edmondson, Prof. T. W., Ph.D. (Clark) New York Univ., New York, N. Y. *University Hts.*
Edmonson, Asst. Prof. Nat, Jr., Ph.D. (Rice) Texas Tech. Coll., Lubbock, Tex. *Box 124.*
Edmonston, J. H., A.B. (George Washington) 1441 Fairmont St. N.W., Washington, D. C.
Edwards, Asso. Prof. P. D., Ph.D. (Indiana) Ball State Teachers Coll., Muncie, Ind. *713 W. North St.*
Eells, Prof. W. C., Ph.D. (Stanford) Educ., Stanford University, Calif. *735 Dolores St.*
Eide, Margaret Chapman (Mrs. R. B.), A.M. (Wisconsin) State Teachers Coll., River Falls, Wis. *308 S. Second St.*
Eiesland, Prof. J. A., Ph.D. (Johns Hopkins) West Virginia Univ., Morgantown, W. Va.
Eisele, Carolyn, A.M. (Hunter) Instr., Hunter Coll., New York, N. Y. *650 E. 231 St.*
Eisenhart, Dean L. P., Ph.D. (Johns Hopkins) Princeton Univ., Princeton, N. J. *73 Nassau St.*

ELDER, J. D., Ph.D. (Calif. Inst. of Tech.) Instr., Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
ELLIOTT, Prof. W. W., Ph.D. (Cornell) Duke Univ., Durham, N. C. *Box 533.*
ELSTON, J. S., A.B. (Cornell) Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
EMCH, Prof. ARNOLD, Ph.D. (Kansas) Univ. of Illinois, Urbana, Ill. *1002 S. Orchard St.*
EMMONS, Prof. C. W., A.M. (Illinois) Simpson Coll., Indianola, Iowa. *1009 N. B St.*
EMMONS, Prof. L. C., A.M. (Harvard) Mathl. Statistics, Michigan State Coll., East Lansing, Mich.
ENGLISH, HARRY, A.B. (Johns Hopkins), LL.M. (Columbia) Head of Dept., Washington High Schools, Washington, D. C.
ENRIQUES, Prof. FEDERIGO. Univ. of Rome, Rome, Italy. *Via Sardegna 50.*
ENTZ, G. G., B.S. (Columbia) 1352 Poinsettia Pl., Hollywood, Calif.
ERICKSON, Asst. Prof. E. E., M.S. (Iowa) Miami Univ., Oxford, Ohio.
ERIKSON, Asst. Prof. C. M., Ph.D. (Michigan) Michigan State Normal Coll., Ypsilanti, Mich. *504 N. Huron St.*
ERNSBERGER, IVA B., A.M. (Nebraska) Teacher, Fullerton High School and Jr. Coll., Fullerton, Calif.
ERWIN, GRACE, A.B. (Nebraska) Grad. Student, Univ. of Minnesota, Minneapolis, Minn. *316-16th Ave. S.E.*
ESCOTT, E. B., M.S. (Chicago) Actuary and Accountant, 1019 S. East Ave., Oak Park, Ill.
ESHLEMAN, J. D., Ph.D. (Chicago) Paradise, Pa.
ESTY, Dean T. C., A.M. (Amherst) Amherst Coll., Amherst, Mass.
ETTINGER, W. J., B.S. in M.E. (Lewis Inst.) Research Engineer, Edison Electric Appliance Co., Chicago, Ill. *5600 W. Taylor St.*
ETTLINGER, Prof. H. J., Ph.D. (Harvard) Pure Math., Univ. of Texas, Austin, Tex. *3110 Harris Park Ave.*
EVANS, Prof. G. C., Ph.D. (Harvard) Pure Math., Rice Inst., Houston, Tex.
EVANS, G. W., A.B. (Harvard) Charlestown High School, 107 Ocean St., Lynn, Mass. *Retired.*
EVANS, Prof. H. B., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
EVANS, Asst. Prof. H. P., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *North Hall.*
EVANS, P. H. Vice-Pres., and Actuary, Northwestern Mut. Life Ins. Co., Milwaukee, Wis. *720 E. Wisconsin Ave.*
EVERETT, Prof. H. S., Ph.D. (Chicago) Extension Prof., Univ. of Chicago, Chicago, Ill. *5545 Woodlawn Ave.*
EVERETT, Prof. J. P., Ph.D. (Columbia) Chm. Dept. of Math., State Teachers Coll., Kalamazoo, Mich. *907 W. South St.*
EVERETT, Asso. Prof. J. R., A.M. (Wisconsin) Colorado School of Mines, Golden, Colo. *1700 Washington St.*
EVERS, Prof. CORNELIUS, M.S. (Michigan State) Central Univ. of Iowa, Pella, Iowa.
EWIN, MARY, A.M. (George Washington) Teacher, Eastern High School, Washington, D. C. *3629 Tenth St. N. W.*
EWING, W. M., A.M. (Rice) Physics, Lehigh Univ., Bethlehem, Pa. *Dept. of Physics.*

FAHNESTOCK, SARAH. Head of Dept., Marymount Coll., Salina, Kans.
FAIRCHILD, Prof. J. T., A.M., C.E. (Ohio Northern), Ph.M. (Carnegie Inst.) Ohio Northern Univ., Ada, Ohio.
FARNUM, FAY, Ph.D. (Cornell) Instr., Washington Square Coll., New York Univ., New York, N. Y. *Washington Sq.*
FAULKNER, Prof. DONALD, A.B. (Stetson) J. B. Stetson Univ., DeLand, Fla.
FEDERICO, P. J., A.M. (George Washington) Asso. Examiner, U. S. Patent Office, Washington, D. C. *1451 Park Rd. N.W.*
FEEMSTER, Prof. H. C., A.M. (Nebraska) York Coll., York, Nebr.
FEENBERG, EUGENE. Asst., Physics, Univ. of Texas, Austin, Tex. *Dept. of Physics.*
FEHN, Asso. Prof. A. R., Ph.B. (Baldwin-Wallace) Centre Coll., Danville, Ky. *421 W. Lexington Ave.*
FEHR, Prof. HENRI, D. es. S. (Geneva) Univ. of Geneva, Geneva, Switzerland. *110 Florissant St.*
FELD, J. M., A.B., Chem.E. (Columbia) Instr., Columbia Coll., New York, N. Y. *Furnald Hall, Columbia Univ.*
FELDER, VIRGINIA I., M.S. (Tulane) Head of Dept., Copiah-Lincoln Jr. Coll., Wesson, Miss.
FELTGES, EDNA M., A.M. (Wisconsin) 2213 Asbury Ave., Evanston, Ill.
FENNER, BEATRICE A., A.M. (Stanford) Margaret Baylor Inn, Santa Barbara, Calif.
FERGUSON, Prof. C. E., A.M. (Missouri) Stephen F. Austin State Teachers Coll., Nacogdoches, Tex.

FERRY, Pres. F. C., Ph.D. (Clark) Hamilton Coll., Clinton, N. Y.
FIELD, FLORENCE E., A.M. (Michigan) Teacher, High School and Jr. Coll., Jackson, Mich. *511 W. Michigan Ave.*
FIELD, Prof. FLOYD, A.M. (Harvard) Dean of Men, Georgia School of Tech., Atlanta, Ga. *Route 1, Decatur, Ga.*
FIELD, Prof. PETER, Ph.D. (Cornell) Univ. of Michigan, Ann Arbor, Mich. *904 Olivia Ave.*
FIELD, S. E., A.M. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *Edgewood Hills, R.F.D. 3.*
FIELDS, Prof. J. C., Ph.D. (Johns Hopkins) Univ. of Toronto, Toronto, Ont., Can.
FINDLAY, Prof. WILLIAM, Ph.D. (Chicago) McMaster Univ., Hamilton, Ont., Can. *131 Chedoke Ave.*
FINKEL, Prof. B. F., Ph.D. (Pennsylvania) Math. and Physics, Drury Coll., Springfield, Mo. Honorary Life Member. *1227 Clay St.*
FINLAY, A. E., A.M. (Peabody) Instr., A. and M. Coll., of Texas, College Station, Tex. *Faculty Exchange.*
FINLEY, Prof. G. W., M.S. (Kansas State) State Teachers Coll., Greeley, Colo. *1933 Ninth Ave.*
FISANICK, GEORGE, A.B. (Penna. State) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *Box 584, Barnesboro, Pa.*
FISCHER, C. H., M.S. (Iowa) Grad. Asst., Univ. of Iowa, Iowa City Iowa. *Dept. of Math.*
FISHER, Prof. IRVING, Ph.D. (Yale) Pol. Econ., Yale Univ., New Haven, Conn. *460 Prospect St.*
FISK, Dean N. C., Sc.D. (Michigan) Itasca Jr. Coll., Coleraine, Minn.
FISKE, Prof. T. S., Ph.D. (Columbia) Columbia Univ., New York, N. Y.
FITCH, ANNIE L. M. (Mrs. Edward), Ph.D. (Cornell) Clinton, N. Y.
FITE, Prof. W. B., Ph.D. (Cornell) Columbia Univ., New York, N. Y. *Hamilton Hall.*
FITTERER, Prof. J. C., C.E. (Colorado) Head of Dept., Colorado School of Mines, Golden, Colo.
FLAGG, Asst. Prof. ELINOR B., M.S. (Illinois) Illinois State Normal Univ., Normal, Ill.
FLANDERS, Asst. Prof. D. A., Ph.D. (Pennsylvania) New York Univ., New York, N. Y. *40 Morningside Ave.*
FLANDERS, Asst. Prof. R. L., B.S. (Norwich) Civ. Eng., Oklahoma A. and M. Coll., Stillwater, Okla. *908 W. Fourth St.*
FLEET, Prof. R. R., Ph.D. (Heidelberg) Math. and Astr., Central Coll., Fayette, Mo.
FLEISHER, Asst. Prof. EDWARD, M.S. (New York Univ.) Brooklyn Coll., Brooklyn, N. Y. *1068 Park Pl.*
FLEMING, Asst. Prof. ANNIE W., A.M. (California) Iowa State Coll., Ames, Iowa. *719 Douglas Ave.*
FLOOD, M. M., A.M. (Nebraska) Asst., Princeton Univ., Princeton, N. J.
FLYNN, B. D., A.M. Secy. and Actuary, Travelers Ins. Co., Hartford, Conn. *700 Main St.*
FLYNN, J. D., A.M. (Tufts) Statistician, Goodwin-Beach & Co., Investment Brokers, Hartford, Conn. *93 N. Beacon St.*
FOARD, Prof. C. W., Ph.D. (Iowa) Math. and Physics, Youngstown Coll., Youngstown, Ohio.
FOBERG, J. A., B.S. (Illinois) State Teachers Coll., California, Pa. *Dept. of Math.*
FOCKE, Dean T. M., Ph.D. (Göttingen) Kerr Prof. of Math., Case School of Appl. Sc., Cleveland, Ohio.
FOLK, PAULINE F., A.M. (Colorado) 519 N. Tejon St., Colorado Springs, Colo.
FOLLEY, Asst. Prof. K. W., Ph.D. (Toronto) Coll. of the City of Detroit, Detroit, Mich.
FORAKER, Prof. F. A., M.S. (Ohio Northern) Univ. of Pittsburgh, Pittsburgh, Pa. *1313 Macon Ave., Swissvale, Pa.*
FORD, Asst. Prof. L. R., Ph.D. (Harvard) Rice Inst., Houston, Tex.
FORD, Prof. W. B., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *904 Forest Ave.*
FORSYTH, Asst. Prof. C. H., Ph.D. (Michigan) Dartmouth Coll., Hanover, N. H.
FORT, Prof. TOMLINSON, Ph.D. (Harvard) Analysis, Lehigh Univ., Bethlehem, Pa.
FOSTER, R. M., B.S. (Harvard) Dept. of Devel. and Research, Amer. Tel. and Tel. Co., New York, N. Y. *195 Broadway.*
FOX, A. H., A.M. (Harvard) Instr., Yale Univ., New Haven, Conn. *115 Pendleton St.*
FRANK, D. H., B.S. (C.C.N.Y.) Teacher, George Washington High School, New York, N. Y. *3135 Decatur Ave., Bronx, N. Y.*
FRANKEL, E. T., B.S. (C.C.N.Y.) Statistician, Natl. Indus. Conference Bd., Inc., New York, N. Y. *295 Convent Ave., Apt. 63.*
FRANKENBUSH, BERTHA E., A.M. (Tulane) Teacher, High School, New Orleans, La. *4502 Prytania St., Apt. B.*
FRANKLIN, Asso. Prof. PHILIP, Ph.D. (Princeton) Mass. Inst. of Tech., Cambridge, Mass.
FREAS, ELIZABETH, A.B. (Lake Erie) Overlook Sanitarium, New Wilmington, Pa.

FRÉCHET, Prof. MAURICE. Prof. à la Faculté des Sciences, Institut Henri Poincaré, Paris, France. *1 Rue Pierre Curie, Paris (5).*
FRECHEVILLE, GEORGE, M.A. (Oxford; Cambridge) Agric. Econ. Research Inst., Univ. of Oxford, Oxford, England. *Parks Road.*
FREMD, Dean LYDIA K., A.M. (Kentucky) Lees Coll., Jackson, Ky. *Eminence, Ky.*
FRINK, Asso. Prof. ORRIN, Jr., Ph.D. (Columbia) Pennsylvania State Coll., State College, Pa.
FRUMVELLER, Prof. A. F. Univ. of Detroit, Detroit, Mich. *651 E. Jefferson Ave.*
FRY, T. C., Ph.D. (Wisconsin) Bell Telephone Labs., New York, N. Y. *463 West St.*
FUBINI, Dr. GUIDO. Prof. of Analysis, Univ. and École Polytech., Turin, Italy. *Via Pietro Micca 12, Turin (108).*
FULLER, GORDON, A.M. (Michigan) Instr., Univ. of New Mexico, Albuquerque, N. Mex. *900 E. Silver Ave.*
FULLER, K. G., A.M. (Nebraska) Instr., Long Island Univ., Brooklyn, N. Y.
FULMER, Asso. Prof. H. K., A.M. (Columbia) Georgia School of Tech., Atlanta, Ga.
FUNK, Asst. Prof. J. C., A.M. (Columbia) Jr. Coll., Santa Maria, Calif. *910 S. McClelland St.*
FUNKHOUSER, H. G., A.M. (Columbia) Instr., Columbia Univ., New York, N. Y. *2940 Broadway.*

GABA, Prof. M. G., Ph.D. (Chicago) Univ. of Nebraska, Lincoln, Nebr.
GAFAFER, W. M., Sc.D. (Johns Hopkins) Research Asso., Johns Hopkins Univ., Baltimore, Md. *185 Glenwood Ave., Leonia, N. J.*
GAGE, Asst. Prof. W. H., M.A. (Univ. of B. C.) Victoria Coll., Victoria, B. C., Can. *Dept. of Math.*
GAINES, Prof. R. E., A.M. (Furman) Univ. of Richmond, Richmond, Va.
GALE, Prof. A. S., Ph.D. (Yale) Univ. of Rochester, Rochester, N. Y. *11 Thayer St.*
GARABEDIAN, Asso. Prof. C. A., Ph.D. (Harvard) St. Stephen's Coll., Columbia Univ., Annandale-on-Hudson, N. Y.
GARABEDIAN, H. A., B.S. (Tufts) Mathematician, John Hancock Mut. Life Ins. Co., Boston, Mass. *197 Clarendon St.*
GARABEDIAN, H. L., Ph.D. (Princeton) Instr., Northwestern Univ., Evanston, Ill. *2125 Ridge Ave.*
GARDINER, J. A., A.M. (Pennsylvania) Teacher, Howard High School, Wilmington, Del. *1305 Tatnall St.*
GARNETT, W. W., A.M. (Kentucky) Actuarial Dept., Central Life Assur. Co., Des Moines, Iowa. *613 Central Y.M.C.A.*
GARRETSON, Asso. Prof. W. V. N., Ph.D. (Michigan) Oklahoma A. and M. Coll., Stillwater, Okla. *623 Hester St.*
GARRETT, Prof. W. H., A.M. (Illinois Coll.) Vice-Pres., and Prof. of Math. and Astr., Baker Univ., Baldwin, Kans.
GARVER, Asso. Prof. RAYMOND, Ph.D. (Chicago) Univ. of California at Los Angeles, Los Angeles, Calif.
GAULT, Asso. Prof. A. E., M.S. (Chicago) Bradley Poly. Inst., Peoria, Ill.
GAVER, H. H., A.M. (Virginia) Headmaster, Black-Foxe Milit. Inst., Los Angeles, Calif. *637 N. Wilcox Ave.*
GAY, Asst. Prof. H. J., A.M. (Clark) Worcester Poly. Inst., Worcester, Mass. *7 Belvidere Ave.*
GAYLORD, H. D., A.M. (Harvard) Asst. Headmaster, Browne and Nichols School, Cambridge, Mass. *Hotel Commander.*
GAYLORD, Asst. Prof. LESLIE J., M.S. (Chicago) Agnes Scott Coll., Decatur, Ga.
GECKELER, Prof. O. T., A.B. (Indiana) Head of Dept., Carnegie Inst. of Tech., Pittsburgh, Pa.
GEHMAN, Prof. H. M., Ph.D. (Pennsylvania) Univ. of Buffalo, Buffalo, N. Y. *163 Winspear Ave.*
GENTRY, F. C., A.M. (Oklahoma) Instr., Tulane Univ., New Orleans, La.
GENTZLER, W. E., A.M. (Columbia) Asst., Columbia Univ., New York, N. Y. *2940 Broadway.*
GEORGES, J. S., Ph.D. (Chicago) Crane Jr. Coll., Chicago, Ill. *2245 Jackson Blvd.*
GERST, Rev. F. J., Ph.D. (Johns Hopkins) Dir. Dept. of Math., Loyola Univ., Chicago, Ill.
GETCHELL, B. C., A.M. (Harvard) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *432 Thompson St.*
GHORMLEY, Asst. Prof. L. O., A.M. (Chicago) Univ. of Tennessee, Knoxville, Tenn.
GIBBENS, Asst. Prof. GLADYS E. C., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *122 Folwell Hall.*
GIBSON, Prof. J. L., Ph.D. (Vienna) Univ. of Utah, Salt Lake City, Utah.

GILL, Asst. Prof. B. P., Ph.D. (Columbia) Coll. of the City of New York, New York, N. Y. *302 Convent Ave.*
GILLESPIE, Prof. D. C., Ph.D. (Göttingen) Cornell Univ., Ithaca, N. Y. *Cayuga Hts.*
GILLESPIE, Prof. WILLIAM, Ph.D. (Chicago) Princeton Univ., Princeton, N. J. *Graduate College.*
GILLEY, C. A., A.B. (Texas) Sul Ross State Teachers Coll., Alpine, Tex.
GILMAN, Asst. Prof. R. E., Ph.D. (Princeton) Brown Univ., Providence, R. I. *44 E. Manning St.*
GINGRICH, Prof. C. H., Ph.D. (Chicago) Math. and Astr., Carleton Coll., Northfield, Minn.
GINNINGS, R. M., M.S. (Chicago) Head of Dept., Western Illinois State Teachers Coll., Macomb, Ill. *314 N. Ward St.*
GITHENS, C. E., Ph.D. (Franklin) Instr., Kiskiminetas Springs School, Saltsburg, Pa.
GLAZIER, Asst. Prof. HARRIET E., A.M. (Chicago) Univ. of California at Los Angeles, Los Angeles, Calif. *1307 Lucile Ave.*
GLEASON, J. M., A.M. (California) Instr., State Teachers Coll., San Diego, Calif.
GLOVER, Prof. B. C., A.M. (Chicago) Otterbein Coll., Westerville, Ohio. *220 Hiawatha Ave.*
GLOVER, J. W., Ph.D. (Harvard) Pres., Teachers Ins. and Annuity Assn., New York, N. Y. *522 Fifth Ave., Rm. 912.*
GOLD, Asst. Prof. J. S., B.S. (Bucknell) Bucknell Univ., Lewisburg, Pa. *306 S. Third St.*
GOLDBERG, MICHAEL, A.M. (George Washington) Engineer, Bureau of Ordnance, Navy Dept., Washington, D. C. *2816 Connecticut Ave.*
GORDON, W. O., A.B. (Bowdoin) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
GORRELL, Prof. G. W., A.M. (Ohio State) Univ. of Denver, Denver, Colo. *2236 S. Milwaukee St.*
GOSNELL, L. P., A.M. (George Washington) Statistician, George Washington Univ., Washington, D. C. *1205 15th St. N.W.*
GOULD, ALICE B., A.B. (Bryn Mawr) Care W. W. Vaughan, 53 State St., Boston, Mass.
GOUWENS, Asso. Prof. CORNELIUS, Ph.D. (Chicago) Iowa State Coll., Ames, Iowa. *109 N. Hyland Ave.*
GRABER, Prof. M. E., Ph.D. (Iowa) Physics, Morningside Coll., Sioux City, Iowa.
GRAESSER, Asst. Prof. R. F., Ph.D. (Illinois) Univ. of Arizona, Tucson, Ariz. *Dept. of Math.*
GRAHAM, MABEL S. (Mrs. F. W.), A.M. (New Mexico) Albuquerque, N. Mex. *Box 528.*
GRAHAM, MARIA D., A.M. (Columbia) E. Carolina Teachers Coll., Greenville, N. C.
GRAHAM, Prof. P. H., A.M. (Virginia) Washington Square Coll., New York Univ., New York, N. Y.
GRANT, ALICE A., A.M. (Brown) Winthrop Coll., Rock Hill, S. C. *Dept. of Math.*
GRANT, Prof. E. D., Ph.D. (Chicago) Math. and Registrar, Earlham Coll., Richmond, Ind. *330 College Ave.*
GRANT, J. D., M.S. (Michigan), A.M. (Illinois) Asst., Univ. of Illinois, Urbana, Ill. *649 Congress St., Indianapolis, Ind.*
GRAUSTEIN, Asso. Prof. W. C., Ph.D. (Bonn) Harvard Univ., Cambridge, Mass. *32 Shepard St.*
GRAVATT, Prof. T. E., M.S. (Rutgers) Pennsylvania State Coll., State College, Pa.
GRAVES, Asso. Prof. G. H., Ph.D. (Columbia) Purdue Univ., W. LaFayette, Ind. *829 Main St.*
GRAVES, Asso. Prof. L. M., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill. *1201 E. 60th St.*
GREEN, Prof. R. L., A.M. (Indiana) Emeritus, Stanford Univ., Stanford University, Calif. *541 Salvatierra St.*
GREENLEAF, Asso. Prof. H. E. H., A.M. (Boston) DePauw Univ., Greencastle, Ind. *1024 S. College Ave. On leave 1931–32, Indiana Univ.*
GRIFFIN, Prof. F. L., Ph.D. (Chicago) Reed Coll., Portland, Ore.
GRIFFIN, HARRIET M., A.M. (Columbia) Instr., Hunter Coll., New York, N. Y. *918 76th St., Brooklyn, N. Y.*
GRIFFITHS, Asst. Prof. LOIS W., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
GRIMES, Prof. N. C., A.M. (Wisconsin) Grove City Coll., Grove City, Pa. *415 Woodland Ave.*
GROVE, Prof. C. C., Ph.D. (Johns Hopkins) School of Commerce, Coll. of the City of New York, New York, N. Y. *143 Milburn Ave., Baldwin, L. I., N. Y.*
GROVE, Asso. Prof. V. G., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *212 W. Grand River Ave.*
GRUMMANN, Asst. Prof. H. R., A.M. (Minnesota) Washington Univ., St. Louis, Mo. *Faculty Box 103.*
GUMMER, Prof. C. F., Ph.D. (Chicago) Queen's Univ., Kingston, Ont., Can. *143 Collingwood St.*

Gummere, H. V., A.M. (Harvard) Lecturer in Astr., Haverford Coll., Haverford, Pa.
Gundersen, Prof. Carl, Ph.D. (Columbia) Oklahoma A. and M. Coll., Stillwater, Okla. *615 College Ave.*
Gunn, Grace T., A.M. (Northwestern) Instr., Univ. of Omaha, Omaha, Nebr. *24th and Pratt St.*
Gunstad, Borghild, B.S. (Minnesota) Asst. in Botany, Univ. of Minnesota, Minneapolis, Minn. *3245 Nicollet Ave.*
Guttman, Solomon. 1801 Sixth Ave. N., Minneapolis, Minn.
Gwinn, Asst. Prof. I. J., M.S. (Iowa) Physics and Math., Morningside Coll., Sioux City, Iowa.
Gwinner, Prof. Harry, M.E. (Maryland) Eng. Math., Univ. of Maryland, College Park, Md. *Box 242.*

Haas, Arthur, A.B. (C.C.N.Y.), LL.B. (New York Univ.) Chm. Dept., Thomas Jefferson High School, Brooklyn, N. Y. *Dumont and Pennsylvania Aves.*
Hacker, S. G., A.B. (Colorado) Asst. in Astr., Princeton Univ., Princeton, N. J.
Hackney, Lilian, A.B. (West Virginia) Marshall Coll., Huntington, W. Va.
Hadamard, Prof. Jacques, LL.D. (Yale) École Polytech. et Coll. de France, Paris, France. *25 rue de Jean Dolent, Paris (14e).*
Hadley, Prof. Laurence, Ph.D. (Michigan) Purdue Univ., W. LaFayette, Ind. *121 Lutz Ave.*
Hadley, S. M., Ph.D. (Wisconsin) Whittier Coll., Whittier, Calif.
Hadlock, E. H. Instr., Cornell Univ., Ithaca, N. Y. *614 E. Buffalo St.*
Haertter, L. D., A.M. (Columbia) John Burroughs School, Clayton, Mo.
Haggard, H. W., B.S. (Denison) Instr., Armour Inst. of Tech., Chicago, Ill. *940 E. 56th St., Hyde Park Sta.*
Haldeman, C. B. Ross, Butler Co., Ohio.
Hall, F. C., A.M. (Columbia) Grad. Student, Columbia Univ., New York, N. Y. *435 W. 119 St., Apt. 10 C.*
Hall, Prof. H. L., A.B. (Indiana) Northwestern State Teachers Coll., Alva, Okla. *923 Seventh St.*
Hall, Ruth H., A.B. (Brown) Dana Hall, Wellesley, Mass.
Hall, Prof. W. S., Sc.D. (Gettysburg) Lafayette Coll., Easton, Pa. *College Campus.*
Halperin, Prof. Hillel, E.E. (Liége), A.M. (Columbia) A. and M. Coll. of Texas, College Station, Tex.
Hamilton, J. A., B.A. (Queen's Univ.) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
Hamilton, W. M., A.M. (Michigan) Asst., U. S. Nautical Almanac Office, U. S. Naval Observ., Washington, D. C.
Hammond, Prof. E. S., Ph.D. (Princeton) Bowdoin Coll., Brunswick, Me. *84 Federal St.*
Hampton, Laurence, A.M. (Nebraska) Instr., Alabama Poly. Inst., Auburn, Ala. *Box 72.*
Hanawalt, Prof. F. W., A.M. (DePauw) Math. and Astr., Coll. of Puget Sound, Tacoma, Wash. *826 N. Steele St.*
Hancock, Clara L., A.M. (Iowa) Instr., Jr. Coll., Virginia, Minn. *331½ Fifth St. S.*
Hancock, Prof. Harris, Ph.D. (Berlin) Univ. of Cincinnati, Cincinnati, Ohio.
Handy, Mrs. Louise L., A.B. (Michigan) Long Beach, Calif.
Hanna, Prof. U. S., Ph.D. (Pennsylvania) Indiana Univ., Bloomington, Ind. *828 Atwater Ave.*
Hanson, H. O., A.B. (Columbia) Mut. Life Ins. Co., Corona, L. I., N. Y. *106–30 Ditmars Blvd.*
Hanthorn, Emma E., A.B. (Nebraska) Teacher, State Teachers Coll., Kearney, Nebr.
Happell, G. E., A.M. (Nebraska) Instr., Purdue Univ., W. Lafayette, Ind. *707 Hayes St.*
Harding, Prof. A. M., Ph.D. (Chicago) Dir., Genl. Extension Service, Univ. of Arkansas, Fayetteville, Ark. *537 Leverett St.*
Harding, Howard, B.M.E. (Michigan) Rochester Gas and Elec. Corp., Rochester, N. Y. *29 Kingston St.*
Harding, J. C. D., A.B. (Pennsylvania) Instr., Univ. of Delaware, Newark, Del. *Mansion Apts., Lansdowne, Pa.*
Hardy, G. H. Trinity Coll., Cambridge, England.
Hardy, Prof. J. G., Ph.D. (Johns Hopkins) Williams Coll., Williamstown, Mass.
Harkin, Asso. Prof. D. C., Ph.D. (Chicago) Alabama Poly. Inst., Auburn, Ala.
Harmount, G. P., A.M. (Ohio State) Head of Dept., East High School, Columbus, Ohio. *2290 Indianola Ave.*
Harp, E. L., Jr. Asst., Univ. of New Mexico, Albuquerque, N. Mex.
Harper, H. D., Sc.B. (New York Univ.) Instr., Murray Hill Voc. School, New York, N. Y. *3457 72nd St., Jackson Hts., N. Y.*

HARRELL, Prof. J. W., A.M. (Brown) Head of Dept., Baylor Univ., Waco, Tex.
HARRINGTON, Asst. Prof. C. E., M.E. (Cornell, M.S. (Buffalo) Asst. to the Dean, Univ. of Buffalo, Buffalo, N. Y. *52 Winter St.*
HARRIS, Mrs. D. P. 102 Middle St., Louisburg, N. C.
HARRIS, ELIZABETH L., M.S. (Washington Univ.) Computer, Southwestern Bell Tel. Co., St. Louis, Mo. *336 W. Madison Ave., Kirkwood, Mo.*
HARRIS, Asso. Prof. ISABEL, A.M. (Columbia) Westhampton Coll., Univ. of Richmond, Richmond, Va.
HARRIS, MARGARET E., A.M. (Teachers Coll., Columbia) Chair of Math., Grenada Coll., Grenada, Miss.
HARRY, S. C., A.B. (Johns Hopkins) Head of Dept., City Coll., Baltimore, Md. *1528 Linden Ave.*
HARSHBARGER, FRANCES, Ph.D. (Illinois) Head of Dept., Constantinople Women's Coll., Istanbul, Turkey. *Galata Post Office, Box 39.*
HARSHBARGER, Prof. W. A., Sc.D. (Washburn) Washburn Coll., Topeka, Kans. *1401 College Ave.*
HART, BERTHA I., A.M. (Cornell) 316 Fayette St., Cumberland, Md.
HART, Prof. J. N., C.E. (Maine), M.S. (Chicago) Univ. of Maine, Orono, Me. *123 Main St.*
HART, Prof. W. L., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn.
HART, Asso. Prof. W. W., A.B. (Chicago) Univ. of Wisconsin, Madison, Wis. *R.F.D. 7.*
HARTER, Prof. G. A., M.A. (St. John's) Univ. of Delaware, Newark, Del.
HARTIG, Asst. Prof. H. E., Ph.D. (Minnesota) Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
HARTNELL, GEORGE. Observer in charge, Cheltenham Magn. Observ., Cheltenham, Md.
HARTUNG, M. L., Ph.D. (Wisconsin) Instr., Teaching of Math., Wisconsin High School, Madison, Wis.
HARTWICK, Prof. F. C., M.Sc. (Fribourg, Switzerland) Prin., Chaminade High School, Dayton, Ohio. *108 Franklin St.*
HARWOOD, Asst. Prof. MAY N., A.M. (Syracuse) Syracuse Univ., Syracuse, N. Y. *200½ Waverly Ave.*
HASKELL, Prof. M. W., Ph.D. (Göttingen) Univ. of California, Berkeley, Calif. *P. O. Box 3.*
HASKELL, R. N., Ph.D. (Rice) 478 Tell St., Vevay, Ind.
HASSLER, Prof. J. O., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla. *425 Lahoma Ave.*
HATCH, Asso. Prof. D. A., Eng. of Mines (Lafayette) Lafayette Coll., Easton, Pa. *705 High St.*
HATCHER, Asso. Prof. T. W., M.E. (Va. Poly. Inst.), M.S. (Iowa State Coll.) Virginia Poly. Inst., Blacksburg, Va. *Box 476.*
HATFIELD, Prof. CHARLES, A.M. (Tennessee) Georgetown Coll., Georgetown, Ky.
HATHAWAY, Prof. A. S., B.S. (Cornell) Retired, Rose Poly. Inst. *Boerne, Tex.*
HAVILAND, E. K., Ph.D. (Harvard) Grad. Student., Johns Hopkins Univ., Baltimore, Md. *Port Deposit, Md.*
HAWKES, Dean H. E., Ph.D. (Yale) Columbia Univ., New York, N. Y. *Hamilton Hall.*
HAYASHI, Prof. TSURUICHI, M.Sc. (Tokyo Imper. Univ.), D.Sc. (Government) Emeritus, Coll. of Science, Tohoku Imperial Univ., Sendai, Japan. *Kamoncho 10.*
HAZARD, Asso. Prof. C. T., A.M. (Indiana) Purdue Univ., W. LaFayette, Ind., *344 N. Western Ave.*
HAZELTINE, Prof. B. A., B.S. (Tufts) Dean of Men, Middlebury Coll., Middlebury, Vt. *38 South St.*
HAZLETT, Asso. Prof. OLIVE C., Ph.D. (Chicago) Univ. of Illinois, Urbana, Ill. *Box 574, Sta. A., Champaign, Ill.*
H'DOUBLER, F. T., Ph.D. (Wisconsin), M.D. (Harvard) Surgeon, Springfield, Mo. *804 Medical Arts Bldg.*
HEATH, Prof. D. F., A.M. (Illinois) Franklin Coll., Franklin, Ind. *700 E. Jefferson St.*
HEDRICK, Prof. E. R., Ph.D. (Göttingen) Univ. of California at Los Angeles, Los Angeles, Calif.
HENFER, R. A., Ph.D. (Chicago) Instr., Georgia School of Tech., Atlanta, Ga. *946 Juniper St.*
HEINS, A. E. Student, Mass. Inst. of Tech., Cambridge, Mass. *127 Homes Ave., Dorchester, Mass.*
HENDERSON, Prof. ARCHIBALD, Ph.D. (North Carolina; Chicago) Head of Dept. Univ. of North Carolina, Chapel Hill, N. C. *721 E. Franklin St.*
HENDERSON, ROBERT, D.Sc. (Toronto) Vice-Pres. and Actuary, Equitable Life Assur. Soc., New York, N. Y. *393 Seventh Ave.*
HENDRICKS, A. F., B.S. (Valparaiso) Pres. Emeritus, Mayfield Coll., Marble Hill, Mo. *Wingate, N. C.*
HENDRICKS, S. F., A.B. (Berea) Grad. Asst., Univ. of Kentucky, Lexington, Ky. *182 E. High St.*

HENNEL, Asso. Prof. CORA B., Ph.D. (Indiana) Indiana Univ., Bloomington, Ind. *822 Atwater Ave.*
HENROTEAU, F. C., Dr. of Physics and Math. (Brussels) Head of Astrophys. Dept., Dominion Observ., Ottawa, Ont., Can.
HENRY, ETTA M., B.S. (Michigan) 60 E. Valley Stream Blvd., Valley Stream, N. Y.
HEREN, Asst. Prof. MABEL M., M.S. (Northwestern) Knox Coll., Galesburg, Ill.
HERR, Asso. Prof. GERTRUDE A., M.S. (Iowa State Coll.) Iowa State Coll., Ames, Iowa.
HERRON, Prof. C. L., A.M. (Hillsdale) Hillsdale Coll., Hillsdale, Mich. *118 Hillsdale St.*
HESS, Asso. Prof. G. W., Ph.D. (Michigan) Howard Univ., Birmingham, Ala. *8009 Berney Ave.*
HESSELTINE, EVELYN, A.M. (Nebraska) Teacher, Normal School, Spearfish, S. D.
HICKEY, Prof. DEBORAH M., Ph.D. (Rice) Delta State Teachers Coll., Cleveland, Miss.
HICKS, Prof. H. C., Ph.D. Carnegie Inst. of Tech., Pittsburgh, Pa. *906 Kennebec St.*
HICKSON, Asst. Prof. A. O., Ph.D. (Chicago) Duke Univ., Durham, N. C.
HIGGINS, VIRGINIA I., A.B. (Marquette) Teacher, French Inst. of Notre Dame de Sion, Kansas City, Mo.
HIGH, M. DETURK, A.B. (Franklin and Marshall) Head of Dept. and Dean of Men, State Teachers Coll., Lock Haven, Pa.
HIGHTOWER, Prof. RUBY U., Ph.D. (Missouri) Shorter Coll., Rome, Ga.
HILDEBRANDT, MARTHA, M.S. (Chicago) Teacher, Proviso High School, Maywood, Ill. *808 S. Second Ave.*
HILDEBRANDT, Prof. T. H., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Rd.*
HILDNER, R. C., B.S. (Wooster) Grad. Asst., Ohio State Univ., Columbus, Ohio. *Dept. of Math.*
HILL, Prof. A. L., A.B. (Doane) Math. and Physics, State Teachers Coll., Peru, Nebr.
HILL, J. D., A.B. (U.C.L.A.) Univ. of California at Los Angeles, Los Angeles, Calif. *1116 S. Windsor Blvd.*
HILL, Asso. Prof. L. S., Ph.D. (Yale) Hunter Coll., New York, N. Y. *1180 Anderson Ave.*
HILL, Asst. Prof. M. A., Jr., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N. C. *Box 215.*
HILL, Adj. Prof. P. R., M.S. (Georgia) Univ. of Georgia, Athens, Ga. *190 Morton Ave.*
HILL, Prof. R. E., A.M. (Louisville) Head of Dept., Univ. of Louisville, Louisville, Ky. *113 Coral Ave.*
HILL, Prof. W. H., A.M. (Colorado) State Teachers Coll., Pittsburg, Kans. *1612 S. Walnut St.*
HILLARD, Asso. Prof. C. R., A.M. (Columbia) Wheaton Coll., Wheaton, Ill.
HILLE, Asso. Prof. EINAR, Ph.D. (Stockholm) Princeton Univ., Princeton, N. J. *174 Prospect Ave.*
HINSCH, Asso. Prof. V. B., E.M. (Missouri School of Mines) School of Mines and Metallurgy, Rolla, Mo.
HIRSCH, BLANCHE, B.S. (Columbia) Prin., Alcuin Prep. School, New York, N. Y. *48 W. 86th St.*
HIRSCHLER, Prof. E. J., M.S. (Chicago) Bluffton Coll., Bluffton, Ohio.
HITCHCOCK, Prof. R. R., A.M. (Northwestern) Univ. of North Dakota, University, N. D.
HOARE, Prof. A. J., A.M. (Michigan) Head of Dept., Univ. of Wichita, Wichita, Kans. *1717 Holyoke Ave.*
HOBENSACK, CLARICE S., A.M. (Ohio State) Ludlow and Telford St., Cincinnati, Ohio.
HODGE, Asst. Prof. F. H., A.M. (Boston) Purdue Univ., W. LaFayette, Ind. *820 N. Main St.*
HOLDER, Prof. F. J., Ph.D. (Yale) Math., and Dean of School of Commerce, Mercer Univ., Macon, Ga. *912 College St.*
HOLGATE, Prof. T. F., Ph.D. (Clark) Northwestern Univ., Evanston, Ill. *617 Library Pl.*
HOLLCROFT, Prof. T. R., Ph.D. (Cornell) Wells Coll., Aurora, N. Y.
HOLLIS, ELINOR V., M.S. (Chicago) The Madeira School, Greenway, Va. *Fairfax Co.*
HOLLOWAY, W. R. Prin., High School, Basco, Ill.
HOLMES, Dean C. F., M.S. (Howard) Texas Coll., Tyler, Tex.
HOLMES, Asst. Prof. C. T., A.M. (Harvard) Bowdoin Coll., Brunswick, Me. *6 Longfellow Ave.*
HOLMES, Asst. Prof. F. F., A.B. (Amherst) Norwich Univ., Northfield, Vt. *12 Byam St.*
HOLT, E. W., A.B. (Colgate) 7 Jefferson St., Attleboro, Mass.
HOME, MAURICE S., M.S. (McGill) Lecturer, Physics and Appl. Math., Univ. of Bishops Coll., Lennoxville, P. Q., Can.
HOOK, C. W., A.M. (North Carolina) Instr., Georgia School of Tech., Atlanta, Ga.
HOOPES, M. F., A.B. (Oberlin) Teacher, Southern State Teachers Coll., Springfield, S. D.
HOOVER, Asso. Prof. B. P., Ph.D. (Illinois) Carnegie Inst. of Tech., Pittsburgh, Pa. *2040 Fairlawn Ave., Wilkinsburg Manor.*

HOPKINS, FANNIE, A.B. (Franklin) 234 E. Park Ave., Waukesha, Wis.
HOPKINS, Asso. Prof. L. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.
HOPPER, GRACE M., A.M. (Yale) Asst., Vassar Coll., Poughkeepsie, N. Y. *40 Morningside Ave., New York, N. Y.*
HORN, MARVEL C., A.M. (Ohio State) Teacher, High School, Columbus, Ohio. *69 Sherman Ave.*
HORSBURGH, E. M., D.Sc. (Edinburgh) Lecturer in Technical Math. and Eng. Dyn., Univ. of Edinburgh, Edinburgh, Scotland. *11 Granville Ter.*
HORSFALL, I. O., A.M. (Chicago) Instr., Cornell Univ., Ithaca, N. Y. *119 Blair St.*
HORTON, Adj. Prof. GOLDIE P., Ph.D. (Texas) Pure Math., Univ. of Texas, Austin, Tex. *2402 Windsor Rd.*
HOSFORD, Prof. H. M., Ph.D. (Illinois) Univ. of Arkansas, Fayetteville, Ark.
HOSKINS, Prof. L. M., C.E. (Wisconsin) Emeritus, Stanford Univ. *1240 Waverly St., Palo Alto, Calif.*
HOUSEHOLDER, Asst. Prof. A. S., A.M. (Cornell) Washburn Coll., Topeka, Kans.
HOUSEHOLDER, Asso. Prof. F. C., A.B. (Kansas) North Dakota Agric. Coll., Fargo, N. D. *1114–13th St. N.*
HOWARD, Prof. C. M., E.Mines (Ala. Poly. Inst.) Texas Woman's Coll., Fort Worth, Tex. *3318 Ave. I.*
HOWARD, Prof. W. E., Ph.D. (Indiana) Head of Dept., Univ. of Tulsa, Tulsa, Okla.
HOWE, ANNA M., Ph.D. Box 245, Jordan, N. Y.
HOWE, G. K., B.S. (Worcester Poly. Inst.) Treasurer, Tougaloo Coll., Tougaloo, Miss.
HOWIE, Prof. J. M., A.M. (Columbia) Nebraska Wesleyan Univ., Lincoln, Nebr. *5218 Walker Ave.*
HOWLAND, Prof. L. A., Ph.D. (Munich) Fisk Prof. of Math., Wesleyan Univ., Middletown, Conn. *34 Home Ave.*
HOYT, R. S., M.S. (Princeton) Telephone Research Engr., Amer. Tel. and Tel. Co., New York, N. Y. *195 Broadway.*
HUBBS, Asst. Prof. H. N., A.M. (Rochester) Hobart Coll., Geneva, N. Y. *251 Washington St.*
HUBER, Prof. C. M., Ph.D. (Illinois) Head of Dept. and Dir. of Extension, Atlantic Univ. Virginia Beach, Va.
HUBERT, Asso. Prof. W. G., Sc.D. (New York Univ.) Coll. of the City of New York. *125 Lee Ave., Yonkers, N. Y.*
HUFFER, Asso. Prof. R. C., A.M. (Illinois) Beloit Coll., Beloit, Wis. *729 Hobart Pl.*
HUGHES, Asst. Prof. H. K., Ph.D. (Michigan) Purdue Univ., W. LaFayette, Ind. *617 Dodge St.*
HUGHES, Asst. Prof. JEWELL C., Ph.D. (Chicago) Hunter Coll., New York, N. Y. *605 W. 115 St.*
HUGHES, W. M., A.M. (Baylor) 517 Texas St., Denton, Tex.
HUME, Pres. ALFRED, D.Sc. (Vanderbilt) Branham and Hughes Milit. Acad., Spring Hill, Tenn.
HUNT, Asst. Prof. G. H., C.E. (Cornell) Univ. of California at Los Angeles, Los Angeles, Calif. *926 Hyperion Ave.*
HUNT, Prof. MILDRED, Ph.D. (Chicago) Illinois Wesleyan Univ., Bloomington, Ill. *1211 Fell Ave.*
HUNTINGTON, A. H., A.M. (Columbia) Asst. Prin., Beaumont High School, St. Louis, Mo. *736 Fairview Ave., Webster Groves, Mo.*
HUNTINGTON, Prof. E. V., Ph.D. (Strassburg) Mech., Harvard Univ., Cambridge, Mass.
HURRY, Prof. J. A., A.M. (California) Head of Dept., Physics, Jr. Coll., San Antonio, Tex.
HURST, Asso. Prof. J. W., Ph.D. (Illinois) State Coll., Bozeman, Mont. *621 S. 7th St.*
HURWITZ, SOLOMON, A.M. (Columbia) Instr., Townsend Harris Hall High School, New York, N. Y. *1601 University Ave., Bronx.*
HURWITZ, Prof. W. A., Ph.D. (Göttingen) Cornell Univ., Ithaca, N. Y. *White Hall 8.*
HUSSEY, L. W., A.M. (Harvard) 55 Monroe Pl., Bloomfield, N. J.
HUTCHERSON, Prof. W. R., Ph.D. (Cornell) Head of Dept., Berea Coll., Berea, Ky.
HUTCHINS, MABEL, A.B. (Blue Mountain) Head of Dept., Blue Mountain Coll., Blue Mountain, Miss.
HUTCHINSON, Prof. C. A., A.M. (Wittenberg) Eng. Math., Univ. of Colorado, Boulder, Colo. *837 Fifteenth St.*
HUTCHINSON, JEAN A., A.B. (Hunter) Substitute, Hunter Coll., New York, N. Y. *410 Park Ave.*
HYDE, Asso. Prof. EMMA, A.M. (Chicago) State Agric. Coll., Manhattan, Kans.

ICAMEN, B. R., A.B. (Occidental), B.S.C.E. (South Carolina) Acting Head of Dept., Silliman Inst., Maasin, Leyte, P. I.

ILOFF, Prof. P. M., A.M. (Michigan) State Teachers Coll., Chico, Calif. *1329 Esplanade.*
INGALLS, E. E., M.S. (Iowa) Albion Coll., Albion, Mich. *416 E. Cass St.*
INGOLD, Prof. BYRON, A.M. (Central Wesleyan) Culver-Stockton Coll., Canton, Mo.
INGOLD, Prof. LOUIS, Ph.D. (Chicago) Univ. of Missouri, Columbia, Mo. *206 Thilly Ave.*
INGRAHAM, Prof. M. H., Ph.D. (Chicago) Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
IRWIN, Asso. Prof. FRANK, Ph.D. (Harvard) Univ. of California, Berkeley, Calif. *2921 Regent St.*
ISAACS, Prof. C. A., A.M. (Columbia) Head of Dept., State Coll., Pullman, Wash.
IVANOFF, V. F., A.B. (California) Grad. Student, Univ. of California, Berkeley, Calif. *1646–26th Ave., San Francisco, Calif.*

JABLONOWER, JOSEPH, Pd.M. (New York Univ.) Teacher, Fieldston School, New York, N. Y. *238th St. and Riverdale Ave.*
JACKSON, Prof. DUNHAM, Ph.D. (Göttingen) Univ. of Minnesota, Minneapolis, Minn. *119 Folwell Hall.*
JACKSON, Prof. J. B., A.M. (Columbia) Univ. of South Carolina, Columbia, S. C. *227 S. Waccamaw Ave.*
JACKSON, Prof. ROSA L., Ph.D. (Chicago) Alabama Coll., Montevallo, Ala.
JAEGER, Prof. C. G., Ph.D. (Missouri) Pomona Coll., Claremont, Calif. *1224 Harvard Ave.*
JAMES, Asso. Prof. GLENN, Ph.D. (Columbia) Univ. of California at Los Angeles, Los Angeles, Calif. *405 Hilgard Ave., Brentwood Hts. Sta.*
JAMISON, Prof. G. H., A.M. (Chicago) State Teachers Coll., Kirksville, Mo. *Box 116.*
JANES, Asst. Prof. W. C., A.M. (Nebraska) State Agric. Coll., Manhattan, Kans.
JARRETT, ETHEL L., A.B. (Cornell) Chicago Latin School for Girls, Chicago, Ill. *59–69 Scott St.*
JEFFERS, H. M., Ph.D. (California) Asst. Astronomer, Lick Observ., Mt. Hamilton, Calif.
JEFFERY, R. L., Ph.D. (Cornell) Acadia Univ., Wolfville, N. S.
JEKEL, A. H., A.B. (Michigan) Box 296, Boulder, Colo.
JENKINS, W. A., A.M. (Michigan) 308 S. Seminole Circle, Fort Wayne, Ind.
JENSEN, Asst. Prof. C. M., Ph.D. (Minnesota) Macalester Coll., St. Paul, Minn. *3315 17th Ave. S., Minneapolis, Minn.*
JERBERT, Asst. Prof. A. R., Ph.D. (Washington) Univ. of Washington, Seattle, Wash. *4824–38 N.E.*
JEROME, Prof. JESSIE M., A.M. (Hiram) Hiram Coll., Hiram, Ohio.
JIMENEZ, Rev. FELIPE, B.S. (Notre Dame) Colegio de San Juan de Letran, Manila, P.I. *200 Beaterio St.*
JOFFE, S.A., M.S. (New York Univ.) Asst. Actuary, Mut. Life Ins. Co., New York, N. Y. *32 Nassau St.*
JOHN, F. W., M.E. (Cornell) Instr., Washington Square Coll., New York Univ., New York, N. Y. *199 N. Broadway, Yonkers, N. Y.*
JOHNSON, ABIGAIL E., A.M. (Columbia) Head of Dept., High School, Morristown, N. J. *2 Rosedale Ave., Morris Plains, N. J.*
JOHNSON, Prof. B. F., A.M. State Teachers Coll., Cape Girardeau, Mo. *530 N. Pacific St.*
JOHNSON, C. L., B.S. (Ore. Agric.) Head of Dept., Oregon Agric. Coll., Corvallis, Ore. *1001 Jefferson St.*
JOHNSON, E. H., A.M. (Lehigh) Instr., Univ. of Detroit, Detroit, Mich. *17130 Birchcrest Dr.*
JOHNSON, Prof. E. N., A.M. (Drake), M.S. (Kansas) Butler Coll., Indianapolis, Ind. *304 Downey Ave.*
JOHNSON, H. F., A.M. (Cincinnati) State Teachers Coll., Bowling Green, Ky.
JOHNSON, Asst. Prof. MARIE M., Ph.D. (Chicago) Oberlin Coll., Oberlin, Ohio. *116 Elm St.*
JOHNSON, Asso. Prof. R. A., Ph.D. (Harvard) Brooklyn Coll., Brooklyn, N. Y. *66 Court St.*
JOHNSON, Asso. Prof. R. P., A.M. (Virginia; Harvard) Carnegie Inst. of Tech., Pittsburgh, Pa. *Dept. of Math.*
JOHNSON, W. W. Instr., Applied Math. and Mech., Cleveland Y.M.C.A. School of Tech., Cleveland, Ohio. *1394 E. 109th St.*
JOHNSTON, Asso. Prof. F. E., Ph.D. (Illinois) George Washington Univ., Washington, D.C.
JOHNSTON, Asst. Prof. K. P., B.A., B.Sc. (Queen's) Queen's Univ., Kingston, Ont., Can. *Dept. of Math.*
JOHNSTON, Asso. Prof. L. S., A.M. (Missouri) Coll. of Eng., Univ. of Detroit, Detroit, Mich.
JOHNSTON, NELLIE M., A.B (Kearney State Teachers Coll.) Warren Twp. High School, Gurnee, Ill.
JONAH, Asst. Prof. F. C., Ph.D. (Brown) Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.

JONES, AGNES A. Larimer, Pa.
JONES, Asst. Prof. B. W., Ph.D. (Chicago) Cornell Univ., Ithaca, N. Y. *Dept. of Math.*
JONES, Prof. E. H., A.M. (Harvard) Southern Methodist Univ., Dallas, Tex.
JONES, MARGARET E., A.M. (Ohio State) Instr., Ohio State Univ., Columbus, Ohio. *164–13th Ave.*
JONES, OLIVE M., A.M. (Columbia) Head of Dept., Queen's-Chicora Coll., Charlotte, N. C.
JONES, P. C., B.S. (Mass. Inst. of Tech.) Bureau of Publication, Bell Telephone Labs., New York, N. Y. *463 West St.*
JONES, Asst. Prof. R. W., M.S. (Delaware), A.M. (Pennsylvania) Univ. of Delaware, Newark, Del. *204 W. 17th St., Wilmington, Del.*
JONES, S. I., B.S. (N. Tex. Normal Coll.) Asst. Secy. and Treas., Life and Casualty Ins. Co., Nashville, Tenn.
JORDAN, Asst. Prof. E. E., M.A. (Dalhousie) Univ. of British Columbia, Vancouver, B.C., Can. *4195 W. 15th Ave.*
JORDAN, Asso. Prof. H. E., Ph.D. (Chicago) Univ. of Kansas, Lawrence, Kans. *1600 Kentucky St.*
JORDAN, Asso. Prof. M. F., A.M. (Maine) Astr., Univ. of Maine, Orono, Me.
JORDAN, Prof. WILLIAM, A.M. (DePauw) Head of Dept., Oakland City Coll., Oakland City, Ind.
JOSE, P. D., A.M. (Syracuse) Instr., Syracuse Univ., Syracuse, N. Y. *618 Allen St.*
JUDD, L. W. Jr. Engineer, Panama Canal, Canal Zone. *Box Q, Balboa Hts.*
JURDAK, Prof. M. H., M.A. (Amer. Univ. of Beirut) American Univ. of Beirut, Beirut, Syria.
JUSTICE, Asst. Prof. H. K., C.E. (Cincinnati) Univ. of Cincinnati, Cincinnati, Ohio. *4767 High Ridge Ave., Price Hill.*
JUSTIN, E. M., M.S. (Case School) Case School of Appl. Science, Cleveland, Ohio.

KALTENBORN, H. S., M.S. (Michigan) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *617 Hale St., Wilkinsburg Sta.*
KANIES, ELIZABETH S. (Mrs. W. F.), 7907 Sunset Drive, Elmwood Park, Chicago, Ill.
KARAPETOFF, Prof. VLADIMIR, M.M.E. (Leningrad) Elec. Eng., Cornell Univ., Ithaca, N. Y. *520 E. Buffalo St.*
KARNOW, HERMAN, A.M. (Colorado) 230 Liberty Ave., Brooklyn, N. Y.
KARPINSKI, Prof. L. C., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Rd.*
KARR, LOIS, A.M. (Wisconsin) Instr., Lindenwood Coll., St. Charles, Mo.
KASNER, Prof. EDWARD, Ph.D. (Columbia) Columbia Univ., New York, N. Y. *22 W. 119th St.*
KAZARINOFF, K., Agrégé (Moscow) Instr., Univ. of Michigan, Ann Arbor, Mich. *1515 Cambridge Rd.*
KEARNEY, DORA E., A.M. (Minnesota) State Teachers Coll., Cedar Falls, Iowa. *1310 W. 22nd St.*
KEELER, C. A., A.M. (California) Grad. Student, Univ. of California, Berkeley, Calif. *2426 Fulton St.*
KEITH, Asst. Prof. MARY N., A.B. (Wellesley) Univ. of Redlands, Redlands, Calif. *930 Campus Ave.*
KELLAM, C. E., A.M. (Chicago) Head of Dept., East Chicago High Schools, East Chicago, Ind. *4239 Baring Ave.*
KELLOGG, Prof. O. D., Ph.D. (Göttingen) Harvard Univ., Cambridge, Mass. *18 Craigie St.*
KELLS, Asst. Prof. L. M., Ph.D. (Columbia) U. S. Naval Acad., Annapolis, Md. *23 Thompson St.*
KELLY, K. D., M.S. (Chicago) Georgetown, Ind.
KEMPNER, Prof. A. J., Ph.D. (Göttingen) Head of Dept., Univ. of Colorado, Boulder, Colo.
KENDALL, Asso. Prof. CLARIBEL, Ph.D. (Chicago) Univ. of Colorado, Boulder, Colo. *1305 Euclid Ave.*
KENNEDY, Adj. Prof. E. C., A.M. (Texas) Coll. of Mines, El Paso, Tex.
KENNEDY, KATHRYN M., A.M. (Columbia) Critic, Coll. Training High School, Terre Haute, Ind.
KENNELLY, Prof. A. E., Sc.D. (Pittsburgh; Toulouse) Emeritus, Elec. Eng., Harvard Univ., Cambridge, Mass.
KENNISON, L. S., A.M. (Brown) Research Asst., Princeton Univ., Princeton, N. J. *30 Murray Pl.*
KERSTEN, Asst. Prof. H. J., A.M. (Chicago) Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math., Coll. of Eng. and Commerce.*
KEULEGAN, G. H., Ph.D. (Johns Hopkins) Asso. Physicist, Bureau of Standards, Washington, D. C. *1614 17th St. N.W.*

KEVLES, DAVID, A.B. (Pennsylvania) Teacher, Harding Jr. High School, Philadelphia, Pa. *862 N. Marshall St.*

KEYES, Mrs. JULIA ATKINSON, A.M. (California) Asst. Prof., Univ. of Arizona, Tucson, Ariz. *Box 46, University Sta.*

KICHLINE, W. L., M.S. (Lehigh) Univ. of New Hampshire, Durham, N. H.

KIEFER, Prof. E. C., M.S. (Michigan) James Millikin Univ., Decatur, Ill. *1229 W. Macon St.*

KIEFFER, NORA A., A.M. (Columbia) State Teachers Coll., Shippensburg, Pa.

KILLEBREW, Prof. C. D., M.S. (Alabama Poly. Inst.) Alabama Poly. Inst., Auburn, Ala.

KILLEN, Asso. Prof. C. G., M.S. (Louisiana) Louisiana State Normal Coll., Natchitoches, La.

KIMBALL, Asso. Prof. B. F., Ph.D. (Cornell) Univ. of New Hampshire, Durham, N. H., *Box 63.*

KIMBALL, R. S., A.M. (Brown) Supt. of Schools, North Brookfield, Mass. *43 Gilbert St.*

KIMBALL, T. C., A.M. (Princeton) Teacher, Lawrenceville School, Lawrenceville, N. J. *Box 207.*

KINDLE, Prof. J. H., A.M. (Ohio State) Eng. Coll., Univ. of Cincinnati, Cincinnati, Ohio.

KING, EULA WEEKS (Mrs. H. L.), Ph.D. (Missouri) 3233 Copelin Ave., St. Louis, Mo.

KINGERY, Prof. D. N., A.M. Macalester Coll., St. Paul, Minn. *135 Amherst St.*

KINGSTON, Prof. H. R., Ph.D. (Chicago) Math. and Astr., Univ. of Western Ontario, London, Ont., Can.

KINNEY, J. M., Ph.D. (Chicago) Head of Dept., Crane Jr. Coll., Chicago, Ill. *8058 Bennett Ave.*

KIRCHNER, Prof. W. H., B.S. (Worcester Poly. Inst.) Drawing and Descr. Geom., Univ. of Minnesota, Minneapolis, Minn. *208 Main Eng. Bldg.*

KIRKHAM, W. J., A.M. (Indiana) Instr., Oregon State Coll., Corvallis, Ore. *Dept. of Math.*

KLAUBER, L. M., A.B. (Stanford) San Diego Cons. Gas and Elec. Co., San Diego, Calif. *233 W. Juniper St.*

KLINE, Prof. J. R., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *529 Riverview Rd., Swarthmore, Pa.*

KLINGER, E. L., A.M. (Illinois) Instr., Purdue Univ., W. LaFayette, Ind. *126 North St.*

KNAPP, Prof. G. A., A.M. (Hamilton) Maryville College, Maryville, Tenn.

KNEBELMAN, Asst. Prof., Ph. D. (Princeton) Princeton Univ., Princeton, N. J. *9 Aiken Ave.*

KNEDLER, P. A., A.M. (Pennsylvania) Instr., Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

KNEPPER, Prof. M. MYRTLE, A.M. (Missouri) State Teachers Coll., Cape Girardeau, Mo. *909 College Hill.*

KNIGHT, Asst. Prof. ELIZABETH E., A.M. (Illinois) State Teachers Coll., Milwaukee, Wis. 2731 N. Downer Ave.

KNIGHT, Asst. Prof. L. C., Ph.B. (Wooster) Coll. of Wooster, Wooster, Ohio.

KNISELY, ALEXANDER. Columbia City, Ind.

KNOX, Prof. J. J., S.M. (Chicago) Dakota Wesleyan Univ., Mitchell, S. D. *1219 W. University Blvd.*

KOCH, E. H. Jr., B.S. in E. E. (Pennsylvania) Chm., Applied Math., Tech. High School, Brooklyn, N.Y. *208 N. Maple Ave., East Orange, N. J.*

KOKOMOOR, Prof. F. W., Ph.D. (Michigan) Univ. of Florida, Gainesville, Fla.

KONANTZ, Prof. EMMA L., A.M. (Ohio Wesleyan) Yeu Ching Univ., Peking, China.

KOVARIK, Prof. A. F., Ph.D. (Minnesota), ScD. (Manchester) Physics, Yale Univ., New Haven, Conn. *Sloane Lab.*

KRATHWOHL, Prof. W. C., Ph.D. (Chicago) Armour Inst. of Tech., Chicago, Ill.

KRETH, DANIEL, C. E. Surveyor and Engr., Wellman, Iowa.

KRYDER, LYLAH, A.M. (Columbia) Instr., Brooklyn Coll., Brooklyn, N. Y. *66 Court St.*

KRYLOFF, Prof. NICOLAS, Doctor of Math. *honoris causa* (Kieff) Member of the Acad. of Sci. of Ukraine and of U.S.S.R. Univ. of Kieff, Kieff, Ukraine, U.S.S.R. *Box 135.*

KUHN, Prof. H. W., Ph.D. (Cornell) Ohio State Univ., Columbus, Ohio. *1179 Fairview Ave.*

KULLBACK, SOLOMON, A.M. (Columbia) 1900 F St. N.W., Washington, D. C.

KUNKEL, P. V., A.M. (Pennsylvania) Head of Dept., State Teachers Coll., Kutztown, Pa. *Box 104, Trexlertown, Pa.*

KUNTE, HELEN L., A.M. (Columbia) Instr., Hunter Coll., New York, N. Y. *Park Ave. and 68th St.*

KURZIN, W. H., M.S. (Chicago) Teacher, Crane Jr. Coll., Chicago, Ill. *1631 W. Roosevelt Rd.*

KUSNER, J. H., A.M. (Columbia) Asst., Univ. of Pennsylvania, Philadelphia, Pa. *266 S. 38th St.*

LABOCCETTA, Ing. LETTERIO, C. E. (Naples) Via S. Basilio 50, Rome (105), Italy.

LACY, Prof. CHARLOTTE M., A.M. (Chicago) West Virginia State Coll., Institute, W. Va. *Box 155.*
LADNER, Asst. Prof. A.C., A.M. (Brown) Math. and Eng., Denison Univ., Granville, Ohio. *Box 253.*
LAMBERT, W. D., A.M. (Harvard) Mathematician, U.S. Coast and Geodetic Survey, Washington, D. C.
LAMPLAND, C. O., A.M. (Indiana) Astronomer, Lowell Observ., Flagstaff, Ariz.
LANDIS, Prof. W. W., A.M. (Dickinson) Dickinson Coll., Carlisle, Pa.
LANDRY, Prof. A. E., Ph.D. (Johns Hopkins) Catholic Univ. of America, Washington, D. C. *3624 13th St., Brookland, D. C.*
LANE, Prof. E. P., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill.
LANGE, LUISE, Ph.D. (Göttingen) 5728 Blackstone Ave., Chicago, Ill.
LANGER, Prof. R. E., Ph.D. (Harvard) Univ. of Wisconsin, Madison, Wis. *521 S. Randall Ave.*
LANGMAN, HARRY, Ph.D. (Columbia) Prop., Solar Chem. Mfg. Co., Brooklyn, N. Y. *6919 Burchell St., Arverne, L. I., N. Y.*
LANHAM, D. V. Student, New River State Coll., Montgomery, W.Va. *1658 Franklin Ave., Charleston, W. Va.*
LA PAZ, Asst. Prof. LINCOLN, Ph.D. (Chicago) Ohio State Univ., Columbus, Ohio. *Dept. of Math.*
LAREW, Prof. GILLIE A., Ph.D. (Chicago) Randolph–Macon Woman's Coll., Lynchburg, Va.
LARKIN, E. J., B.S. (Cooper Union) Instr., Cooper Union, New York, N. Y. *Westbury, L. I., N. Y. Box 595.*
LARSON. Asst. Prof. OLGA, A.M. (Missouri) State Coll. for Women, Tallahassee, Fla.
LASLEY, Prof. J. W., Ph.D. (Chicago) Pure Math., Univ. of North Carolina, Chapel Hill, N. C.
LATIMER, Prof. C. G., Ph.D. (Chicago) Univ. of Kentucky, Lexington, Ky. *Dept of Math.*
LATSHAW, ELMER. 1839 N. 60th St., Philadelphia, Pa.
LATSHAW, V. V., Ph.D. (Indiana) Instr., Lehigh Univ., Bethlehem, Pa.
LAURENTINE MARIE, Sister, A.B. (Trinity Coll., Washington) Instr., Emmanuel Coll., Boston, Mass. *32 The Riverway.*
LAVES, Asso. Prof. KURT, Ph.D. (Berlin) Univ. of Chicago, Chicago, Ill. *5611 Kenwood Ave.*
LEAVENS, D. H., A.M. (Yale) Research Staff, Harvard Business School. *Soldiers Field, Boston, Mass.*
LEFEVER, RALPH, A.M. (Nebraska) Strang, Nebr.
LEFSCHETZ, Prof. SOLOMON, Ph.D. (Clark) Princeton Univ., Princeton, N. J.
LEHMAN, Prof. D. A., A.M. (Western Reserve) Goshen Coll., Goshen, Ind.
LEHMANN, C. H., A.M. (Columbia) Instr., Cooper Union Inst. of Tech., New York, N. Y. *14425-33rd Ave., Flushing, L. I., N. Y.*
LEHMER, D. H., Ph.D. (Brown) National Research Fellow, Stanford Univ., Stanford University, Calif. *Dept. of Math.*
LEHMER, Prof. D. N., Ph.D. (Chicago) Univ. of California, Berkeley, Calif. *2736 Regent St.*
LEHR, MARGUERITE, Ph.D. (Bryn Mawr) Asso., Bryn Mawr Coll., Bryn Mawr, Pa. *Low Bldgs.*
LEIB, Prof. D. D., Ph.D. (Johns Hopkins) Connecticut Coll., New London, Conn. *358 Mohegan Ave.*
LEIPER, Prof. C. L., Grad. (U. S. Naval Acad.) U. S. Naval Acad., Annapolis, Md.
LEITH, Asso. Prof. J. D., A.M. (Columbia) Univ. of North Dakota, Grand Forks, N. D.
LEMME, M. M., A.M. (Indiana) Instr., Univ. of the City of Toledo, Toledo, Ohio.
LENNAHAN, C. M. (Creighton Univ.) Observer, U. S. Weather Bureau, Washington, D. C. *1414-2s1t St. N.W.*
LENNES, Prof. N. J., Ph.D. (Chicago) Univ. of Montana, Missoula, Mont.
LEONARD, Prof. H. B., Ph.D. (Colorado) Univ. of Arizona, Tucson, Ariz. *Univ. Sta., Box 24.*
LEONARD, KATHARINE, A.M. (Vermont) Head of Dept., State Teachers Coll., Moorhead, Minn. *Box 48.*
LEPESHKIN, S. A., A.M. (Columbia) Instr., Worcester Poly. Inst., Worcester, Mass.
LEPOWSKY, Mrs. FRANCES R., A.B. (Hunter) Grad. Student, Coll. of the City of New York; Substitute, James Madison High School, New York, N.Y. *229 E. 18th St., Flatbush, Brooklyn, N. Y.*
LERCH, F. S., A.B. (Lehigh) Instr., Union Coll., Schenectady, N. Y.
LESTER, CAROLINE A., A.M. (Cornell) Instr., State Coll. for Teachers, Albany, N. Y.
LESTER, Prof. O. C., Ph.D. (Yale) Dean of Grad. School and Prof. of Physics, Univ. of Colorado, Boulder, Colo. *1061 Eleventh St.*
LESTOURGEON, Asso. Prof. F. ELIZABETH, Ph.D. (Chicago) Univ. of Kentucky, Lexington, Ky. *630 Maxwelton Ct.*

LEVINE, JACK, A.B. (Univ. of Calif. at L.A.) Part-time Instr., Princeton Univ., Princeton, N. J. *32 Bank St.*
LEVY, Asst. Prof. HARRY, Ph.D. (Princeton) Univ. of Illinois, Urbana, Ill. *356 Math. Bldg.*
LEVY, Asst. Prof. SOPHIA H., Ph.D. (California) Univ. of California, Berkeley, Calif. *453 Wheeler Hall.*
LEWANDOWSKI, STEPHEN, B.S. (Marquette) Instr., Marquette Univ., Milwaukee, Wis. *1020 E. Wright St.*
LEWIS, Prof. ANNA D., Ph.D. (Carleton) Lake Erie Coll., Painesville, Ohio. *38 Paige Lane.*
LEWIS, Asst. Prof. A. J., A.M. (Denver) Univ. of Denver, Denver, Colo. *2112 S. Columbine.*
LEWIS, Asso. Prof. C. F., M.S. (K.S.A.C.) State Agric. Coll., Manhattan, Kans.
LEWIS, Asso. Prof. F. A., A.M. (Alabama) Univ. of Alabama, University, Ala.
LEWIS, Prof. FLORENCE P., Ph.D. (Johns Hopkins) Goucher Coll., Baltimore, Md. *2435 N. Charles St.*
LIBMAN, E. E., Ph.D. (Illinois) Genl. Elec. Co., Marine and Air Craft Eng. Dept., Schenectady, N. Y. *River Rd. 1, Bldg. 23, Rm. 334.*
LIEBER, LILLIAN ROSANOFF (Mrs. H. G.), Ph.D. (Clark) Teacher, Commercial High School, Brooklyn, N. Y. *258 Clinton Ave.*
LIGHT, Prof. G. H., Ph.D. (Yale) Univ. of Colorado, Boulder, Colo. *801 Base Line.*
LINARES, ENRIQUE, Jr., B.S. in E. E. (Univ. of Santa Clara) Actg. Chief Engr., Central Board of Roads, Panama City, R.P. *P.O. Box 540.*
LINDEMANN, Prof. C. A., A.M. (Bucknell) Pure Math., and Secy. of Faculty, Bucknell Univ., Lewisburg, Pa. *30 Brown St.*
LINDQUIST, Prof. THEODORE, Ph.D. (Chicago) State Normal Coll., Ypsilanti, Mich. *103 Elm St.*
LINDSEY, Prof. LOUIS, Ph.D. (Syracuse) Applied Math., Syracuse Univ., Syracuse, N. Y. *201 Scottholm Blvd.*
LINEHAN, Prof. P. H., Ph.D. (Columbia) Coll. of the City of New York, New York, N. Y. *346 Convent Ave.*
LINFIELD, Asso. Prof. B. Z., Ph.D. (Harvard) Univ. of Virginia, University, Va. *42 University Pl.*
LING, Dean G. H., Ph.D. (Columbia) Univ. of Saskatchewan, Saskatoon, Sask., Can.
LINKER, Prof. J. B., Ph.D. (Johns Hopkins) Univ. of North Carolina, Chapel Hill, N. C.
LINTON, M. ALBERT, A.M. (Haverford) Pres., The Provident Mut. Life Ins. Co., Philadelphia, Pa. *1632 Chestnut St.*
LISH, HERBERT. Draftsman, Moore Steam Turbine Corp., Wellsville, N. Y. *46 Grover St.*
LITTAUER, S. B., Ch.E. (Rensselaer), A.M. (Columbia) National Research Fellow, Harvard Univ., Cambridge, Mass. *9 Ware St.*
LITTERICK, W. S., M.S. (Brown) Instr., Peddie School, Hightstown, N. J.
LITTLE, NEIL, A.M. (Michigan) Instr., Purdue Univ., W. LaFayette, Ind. *425 Littleton St.*
LIVINGSTON, Asso. Prof. G. R., A.M. (California) State Teachers Coll., San Diego, Calif.
LOCKE, L. L., A.M. (Grove City Coll.) Teacher, Brooklyn Tr. School for Teachers, Brooklyn, N. Y. *950 St. John's Pl.*
LOCKWOOD, E. C., A.M. (Brown) Instr., American Coll., Madura, S. India. *Tallakulam P. O.*
LODWICK, DECA, Ph.B. (Iowa) Head of Dept., High School, Long Beach, Calif. *4112 Colorado St.*
LOEWEN, Prof. O. B., A.M. (Kansas) Ottawa Univ., Ottawa, Kans.
LOGSDON, Mrs. MAYME I., Ph.D. (Chicago) Asso. Prof., Univ. of Chicago, Chicago, Ill.
LONG, Asst. Prof. FLORENCE, M.S. (Illinois) Earlham Coll., Earlham, Ind.
LONG, J. K., A.B. (Washington and Jefferson) Instr., Purdue Univ., W. LaFayette, Ind. *138 Chauncey Ave.*
LONG, Asst. Prof. T. R., A.M. (Rochester) Univ. of Rochester, Rochester, N. Y. *75 Cypress St.*
LONG, Prof. W. F., A.B. (Franklin and Marshall) Math. and Astr., Franklin and Marshall Coll., Lancaster, Pa.
LONGENECKER, J. V., M.S. (Iowa) Actuary, Farmers and Bankers Life Ins. Co., Wichita, Kans. *701 Beacon Bldg.*
LONGLEY, R. W., A.M. (Harvard) Grad. Student, Harvard School of Educ., Cambridge, Mass. *82 Kirkland St.*
LONGLEY, Prof. W. R., Ph.D. (Chicago) Yale Univ., New Haven, Conn.
LOVE, Prof. C. E., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1915 Scottwood Ave.*
LOVETT, Pres. E. O., Ph.D. (Virginia; Leipzig) Rice Inst., Houston, Tex.
LOVITT, Prof. W. V., Ph.D. (Chicago) Colorado Coll., Colorado Springs, Colo.
LOWENSTEIN, L. L., A.B. (Cornell) Instr., Cornell Univ., Ithaca, N. Y. *White Hall.*
LOWNEY, R. E., A.M. (Mich. State Coll.) Instr., Univ. of Wisconsin, Madison, Wis. *North Hall.*

LUBBEN, Adj. Prof. R. G., Ph.D. (Texas) Univ. of Texas, Austin, Tex. *708 W. 22½ St.*
LUBIN, Asst. Prof. C. I., Ph.D. (Harvard) Coll. of Eng., Univ. of Cincinnati, Cincinnati, Ohio. *3612 Washington Ave.*
LUBY, W. A., A.M. (Kansas) Jr. Coll., Kansas City, Mo. *5411 Rockhill Rd.*
LUCK, Prof. J. J., Ph.D. (Virginia) Univ. of Virginia, University, Va. *Colonnade Club.*
LUFKIN, H. M., Ph.D. (Pennsylvania) Instr., Univ. of Pennsylvania, Philadelphia, Pa. *601 Woodcrest Ave., Ardmore, Pa.*
LUNN, Prof. A. C., Ph.D. (Chicago) Applied Math., Univ. of Chicago, Chicago, Ill.
LUTZ, Asst. Prof. JUNA M., A.M. (Chicago) Butler Univ., Indianapolis, Ind. *727 Fairfield Ave., Apt. 6.*
LYLE, G. A., M.S. (Lehigh) Instr., U. S. Naval Acad., Annapolis, Md. *112 High St.*
LYMAN, Prof. E. A., A.B. (Michigan) State Normal Coll., Ypsilanti, Mich. *308 Ellis St.*
LYON, ELLEN S. (Mrs. A. R.), A.M. (Louisville) Teacher, Jr. Coll., Campbellsville, Ky. *404 Columbia Ave.*
LYONS, Asso. Prof. W. H., A.M. (Denver) State Agric. Coll., Manhattan, Kans. *1507 Fifth Ave. N., Fort Dodge, Iowa.*
LYTLE, Asso. Prof. E. B., Ph.D. (Yale) Teaching of Math., Univ. of Illinois, Urbana, Ill. *903 S. Busey Ave.*

MACCREADIE, Asst. Prof. W. T., Ph.D. (Cornell) Bucknell Univ., Lewisburg, Pa. *1425 W. Market St.*
MACCULLOUGH, Prof. R. H., M.S. (Lafayette) Defiance Coll., Defiance, Ohio.
MACDONALD, I. J., M.S. (St. Bonaventure's) Oratory Prep. School, Summit, N. J.
MACDONALD, Prof. S. L., M.S. (Columbia) State Agric. Coll., Fort Collins, Colo.
MACDONALD, Prof. W. E., A.M. (Harvard) Lingnan Univ., Canton, China.
MACDOUGAL, H. B., M.S. (Iowa) South Dakota State Coll., Brookings, S. D. *Dept. of Math.*
MACDUFFEE, Asso. Prof. C. C., Ph.D. (Chicago) Ohio State Univ., Columbus, Ohio. *365 Oakland Park Ave.*
MACGREGOR, F. S., A.B. (Harvard) Mgr. Textbook Dept., Harper and Bros., New York, N. Y. *49 E. 33rd St.*
MACKAY, ROY, A.B. (New Mexico) Grad. Student, Univ. of New Mexico; Teacher, High School, Albuquerque, N. M. *917 E. Grand Ave.*
MACKIE, Asso. Prof. E. L., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N. C. *702 Gimghoul Rd.*
MACLEAN, Lieut.-Col. N. B., Ph.D. (Chicago) Prof., Applied Math., McGill Univ., Montreal, P.Q., Can.
MACMILLAN, Prof. W. D., Ph.D. (Chicago) Astr., Univ. of Chicago, Chicago, Ill.
MAC NEISH, Asso. Prof. H. F., Ph.D. (Chicago) Brooklyn Coll., Brooklyn, N.Y. *Scarsdale, N. Y.*
MACNUTT, BARRY, M.S. (Lehigh) 643 Washington St., Gloucester, Mass.
MCALISTER, Prof. E. H., A.M. (Oregon) Mech. and Astr., Univ. of Oregon, Eugene, Ore. *334 Pearl St.*
MCCAIN, Prof. GERTRUDE I., Ph.D. (Indiana) Marymount Coll., Salina, Kans.
MCCARTHY, E. D., A.B. (Cornell) Instr., Eng. Coll., Univ. of Detroit, Detroit, Mich.
MCCARTY, A. L., A.M. (California) Head of Dept., Lowell High School, Berkeley, Calif. *2137 Parker St.*
MCCLELLAN, ADA A., B.S. (Chicago) 313 N. New Hampshire Ave., Los Angeles, Calif.
MCCLENON, Prof. R. B., Ph.D. (Yale) Grinnell Coll., Grinnell, Iowa. *1105 Park St.*
MCCORMICK, Prof. CLARENCE, A.M. (Clark) State Teachers Coll., Weatherford, Okla.
MCCORMICK, W. W., M.S. (Michigan) Instr., Math. and Physics, Geneva Coll., Beaver Falls, Pa. *On leave 1931–32, Univ. of Michigan, Ann Arbor, Mich.*
MCCOY, Prof. DOROTHY, Ph.D. (Iowa) Belhaven Coll., Jackson, Miss.
MCCOY, M. HELEN, A.M. (Illinois) Address unknown.
MCCOY, Asst. Prof. N. H., Ph.D. (Iowa) Smith Coll., Northampton, Mass. *29 Kensington Ave.*
MCCUE, Prof. M. J., C. E., M.S. (Notre Dame) Coll. of Eng., Univ. of Notre Dame, Notre Dame, Ind. *130 Fremont St., Woodstock, Ill.*
MCDILL, Prof. R. M., A.M. (Indiana) Hastings Coll., Hastings, Nebr. *129 E. 7th St.*
MCDONALD, EMMA WHITON (Mrs. J. H.), A.M. (California) Hotel Claremont, Berkeley, Calif.
MCDONALD, JANET, A.M. (Tulane) Teacher, Mississippi Synodical Coll., Holly Springs, Miss. *Box 244.*
MCDONOUGH, D. L., A.M. (Pennsylvania) Teacher, Overbrook High School; Drexel Inst., Philadelphia, Pa. *18 Sunshine Rd., Upper Darby, Pa.*
MCDOUGLE, EDITH A., A.B. (Delaware) Instr., Women's Coll., Univ. of Delaware, Newark, Del.

McEwen, Prof. G. F., Ph.D. (Stanford) Oceanographer, Scripps Inst., Univ. of California. *Box 68, La Jolla, Calif.*
McEwen, W. H., Ph.D. (Minnesota) Mount Allison Univ., Sackville, N. B., Can. *Box 475.*
McFarlan, Asst. Prof. L. H., Ph.D. (Missouri) Univ. of Washington, Seattle, Wash.
McFarland, Asst. Prof. Dora, A.M. (Oklahoma) Univ. of Oklahoma, Norman, Okla.
McFarland, Elsie J., Ph.D. (California) 122 E. Jefferson Ave., Kirkwood, Mo.
McGavock, Asso. Prof. Martha P., A.M. (Chicago) Rockford Coll., Rockford, Ill.
McGaw, Prof. F. M., A.M. (Phila. City Coll.) Math. and Eng. Drawing, Cornell Coll., Mt. Vernon, Iowa.
McGiffert, Prof. James, Ph.D. (Columbia) Pure Math., Grad. Dept., Rensselaer Poly. Inst., Troy, N. Y. *169 Eighth St.*
McGinley, Margaret F., A.M. (Colorado) Teacher, Denver Schools, Denver, Colo. *1601 E. Mississippi Ave.*
McKelvey, Asso. Prof. J. V., Ph.D. (Cornell) Iowa State Coll., Ames, Iowa. *2117 Graeber St.*
McKelvey, Martha M. (Mrs. J. V.), M. S. (Iowa) Instr., Iowa State Coll., Ames, Iowa. *2117 Graeber St.*
McLaughlin, Prof. J. A., M.S. (Colgate) Head of Dept., St. Bonaventure's Coll., Allegany, N. Y.
McMaster, A. S., M.S. in C.E. (Colorado) Instr., Eng. Math., Univ. of Colorado, Boulder, Colo. *Coll. of Engineering.*
McMillan, Mary B., A.M. (Wisconsin) 1341 Third St. S., Wisconsin Rapids, Wis.
McNatt, J. Q. Div. Engr., Colorado Fuel and Iron Co., Canon City, Colo. *910 Glenwood Ave.*
McNelly, Sister Mary of the Presentation, A.M. (Catholic Univ.) Prof., Our Lady of the Lake Coll., San Antonio, Tex.
McShane, E. J., Ph.D. (Chicago) National Research Fellow, Univ. of Chicago, Chicago, Ill. *Eckhart Hall.*
McSweeny, Prof. A. A., A.M. (Montana) Head of Dept., John Tarleton Agric. Coll., Stephenville, Tex.
Maddox, Prof. A. C., A.M. (Columbia) State Normal Coll., Natchitoches, La. *405 New Second St.*
Magee, S. R. Kinderhook, Columbia Co., N. Y.
Maiden, W. M., A.M. (Virginia) Washington Square Coll., New York Univ., New York, N. Y. *Washington Sq. E.*
Maizlish, Asso. Prof. Israel, Ph.D. (Minnesota) Physics, Centenary Coll., Shreveport, La.
Majella, Brother, A.M. (Catholic Univ.) St. John's Prep. School, Danvers, Mass.
Male, Asst. Prof. C. T., M.C.E. (Union) Union Coll., Schenectady, N. Y. *Niskayuna, Schenec. Co., N. Y.*
Mallory, Asso. Prof. V. S., A.M. (Columbia) State Teachers Coll., Montclair, N. J.
Manchester, Prof. R. E., A.M. (Michigan) Head of Dept., and Dean of Men, Kent State Coll., Kent, Ohio.
Maney, Prof. C. A., M.S. (Chicago) Transylvania Coll., Lexington, Ky. *670 N. Broadway.*
Mangold, Prof. M. Cecilia, Ph.D. (Catholic Univ.) Trinity Coll., Washington, D. C.
Manning, Asst. Prof. F. L., M.S. (Rutgers) Ursinus Coll., Collegeville, Pa.
Manning, Dr. H. M. U. S. Public Health Service, c/o American Consulate, Rotterdam, Holland.
Manning, Asso. Prof. H. P., Ph.D. (Johns Hopkins) Emeritus, Brown Univ., Providence, R. I. *106 Carrington Ave., E. Side Sta.*
Manson, Prof. E. S., M.S. (Mass. Inst. of Tech.) Astr., Ohio State Univ., Columbus, Ohio.
Many, Asst. Prof. Anna E., A.M. (Tulane) Counselor to Women, Sophie Newcomb Coll., New Orleans, La.
March, Asst. Prof. G. E., B.S. (S. Dak. School of Mines) South Dakota State School of Mines, Rapid City, S. D. *1216 Fulton St.*
Markowitz, Max. Student, Brooklyn Coll., Brooklyn, N. Y. *2037-71st. St.*
Marm, Prof. Anna, A.M. (Kansas) Bethany Coll., Lindsborg, Kans. *741 N. 2nd St.*
Marriott, Prof. R. W., Ph.D. (Pennsylvania) Swarthmore Coll., Swarthmore, Pa. *213 Lafayette Ave.*
Marshall, Prof. William, Ph.D. (Zürich) Head of Dept., Purdue Univ., LaFayette, Ind. *1017 State St.*
Martin, Prof. A. W., Ph.D. (Chicago) Coll. of Puget Sound, Tacoma, Wash. *3209 N. 15th St.*
Martin, Prof. Emilie N., Ph.D. (Bryn Mawr) Mount Holyoke Coll., South Hadley, Mass.
Martin, Mary C., B.S. (Knox) Teacher, High School, Freeport, Ill. *831 W. Stephenson St.*
Martyn, W. J., M.A. (New Zealand) Mathematical Master, Otago Boys High School, Dunedin, New Zealand.

MARY JOAN, SISTER, A.B. (St. Catherine's Coll.) Teacher, St. Catherine's High School, Racine, Wis. *1209 Park Ave.*
MASON, Prof. T. E., Ph.D. (Indiana) Purdue Univ., LaFayette, Ind. *103 Waldron St.*
MASON, Asst. Prof. W. E., M.E. (Michigan) Applied Math., Univ. of California at Los Angeles, Los Angeles, Calif.
MATHESON, Prof. JOHN, M.A. (Queen's) Queen's Univ., Kingston, Ont., Can.
MATHEWS, JANE H., B.S. (Columbia) Teacher, Peabody High School, Pittsburgh, Pa. *Cathedral Mansions, Ellsworth Ave. E. E.*
MATHEWSON, Asst. Prof. L. C., Ph.D. (Illinois) Dartmouth Coll., Hanover, N. H.
MATHIAS, FLORENTINA, A.M. (Ohio State) Teacher, High School, Chillicothe, Ohio. *38 Ewing St.*
MATHIAS, H. R., A.M. (Indiana) Instr., Bowling Green State Coll., Bowling Green, Ohio. *303 N. Prospect St.*
MATHIS, BUENA C., A.B. (Kentucky) Grad. Asst., Univ. of Kentucky, Lexington, Ky. *R.F.D. 9.*
MATTESON, Asso. Prof. JANE L. Michigan State Normal Coll., Ypsilanti, Mich. *318 W. Cross St.*
MAURUS, Prof. E. J., M.S. (Notre Dame) Univ. of Notre Dame, Notre Dame, Ind. *L.B. 75.*
MAY, LIDA B., A.M. (Texas) Teacher, High School, Borger, Tex. *Box 1403.*
MAYER, JOANNA I., Ph.D. (Marquette) 234 S. Vermont St., Sedalia, Mo.
MEADE, MARY E., A.M. (Virginia) Instr. and Acting Dean of Women, Martin Coll., Pulaski, Tenn.
MEACHAM, Prof. E. D., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla. *712 Lindsay Ave.*
MEARS, Asst. Prof. FLORENCE M., Ph.D. (Cornell) George Washington Univ., Washington, D. C.
MEBANE, Asst. Prof. W. N., Jr., A.M. (Cornell) Davidson Coll., Davidson, N. C. *Box 274.*
MEDER, Asst. Prof. A. E., A.M. (Columbia) New Jersey Coll. for Women, New Brunswick, N. J.
MEEK, KATE M. High School, Pasadena, Calif. *919 E. California St.*
MELVILLE, Asso. Prof. C. E., A.B. (Northwestern) Math. and Registrar, Clark Univ., Worcester, Mass.
MENDENHALL, Pres. W. O., Ph.D. (Michigan) Friends Univ., Wichita, Kans.
MENKE, Asst. Prof. H. E., A.M. (Ohio State) Otterbein Coll., Westerville, Ohio. *124 W. Home St.*
MENSENKAMP, L. E., A.M. (Illinois) Instr., High School, Freeport, Ill. *808 W. American St.*
MENUET, Asso. Prof. R. L., B.E. (Tulane) Tulane Univ., New Orleans, La.
MERCEDES, Sister M., M.S. (Notre Dame) Instr., Mary Manse Coll., Toledo, Ohio.
MERGENDAHL, Asst. Prof. T. E., M.S. (Tufts) Math. and Registrar, Tufts Coll., Tufts College, Mass. *128 Professors Row.*
MERRILL, Prof. A. S., Ph.D. (Chicago) Univ. of Montana, Missoula, Mont. *541 Beckwith Ave.*
MERRILL, Prof. HELEN A., Ph.D. (Yale) Wellesley Coll., Wellesley, Mass.
MERRILL, J. E., Ph.D. (Princeton) Univ. of Illinois Observ., Urbana, Ill.
MERRIMAN, Asst. Prof. G. M., Ph.D. (Cincinnati) Pure Math., Univ. of Cincinnati, Cincinnati, Ohio.
MERRISS, A. A. 913 East 23rd St. N., Portland, Ore.
MESSICK, Asso. Prof. C. A., Ph.D. (Chicago) Lincoln Memorial Univ., Harrogate, Tenn.
MESSICK, Prof. J. F., Ph.D. (Johns Hopkins) Emory Univ., Emory University, Ga.
MEYER, Prof. H. A., Ph.D. (Iowa) Hanover Coll., Hanover, Ind. *Box 3.*
MEYER, Prof. J. B., M.S. (Purdue) Head of Dept., State Teachers Coll., Valley City, N. D. *814 Fifth Ave.*
MEYER, Rev. J. H. Loretto House, Hot Springs, N. C.
MICHAL, Asso. Prof. A. D., Ph.D. (Rice) California Inst. of Tech., Pasadena, Calif.
MICKELSON, Prof. E. L., Ph.D. (Minnesota) Math. and Physics, State Teachers Coll., Silver City, N. Mex. *503 Kelly St.*
MICKLE, M. KATHERINE, A.B. (Alabama) Roanoke, Ala.
MIDDLEMISS, Asst. Prof. R. R., M.S. (Colorado) Washington Univ., St. Louis, Mo.
MIKAMI, YOSHIO. Research Assoc., Imperial Acad., Tokyo, Japan. *7 Daini Mejiro Bunkamura, 1321 Shimo Ochiai.*
MIKESH, J. S. Lawrenceville School, Lawrenceville, N. J.
MILES, Asso. Prof. E. J., Ph.D. (Chicago) Yale Univ., New Haven, Conn. *87 Marvel Rd.*
MILES, H. J., Ph.D. (California) Instr., Univ. of Illinois, Urbana, Ill. *369 Math. Bldg.*
MILLER, A. L., Ph.D. (Harvard) 25 Clinton Road, Brookline 46, Mass.
MILLER, Prof. E. B., A.M. (Chicago) Illinois Coll., Jacksonville, Ill. *1205 W. College Ave.*
MILLER, F. H., M.S. (Cornell) Instr., Columbia Univ., New York, N. Y. *45 Sickles St.*

MILLER, Prof. G. A., Ph.D. (Cumberland) Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
MILLER, G. T., M.S. (Iowa State Coll.) Instr., Purdue Univ., LaFayette, Ind. *222 Harrison St.*
MILLER, Prof. I. L., A.M. (Indiana) South Dakota State Coll., Brookings, S. D. *811 12th Ave.*
MILLER, Prof. J. A., Ph.D. (Chicago) Astr., Swarthmore Coll., Swarthmore, Pa.
MILLER, Prof. J. S., Sc.D. (Virginia) Emory and Henry Coll., Emory, Va.
MILLER, Prof. NORMAN, Ph.D. (Harvard) Queen's Univ., Kingston, Ont., Can.
MILLER, W. I., A.M. (Bucknell) Fellow, Univ. of Pittsburgh, Pittsburgh, Pa. *722 Broad St., Sewickley, Pa.*
MILLER, Asst. Prof. W. M., Ph.D. (Illinois) Tufts Coll., Medford, Mass. *37 Chetwynd Rd., W. Somerville, Mass.*
MILLIKAN, Asst. Prof. C. B., Ph.D. (Calif. Inst. of Tech.) Grad. School of Aeronautics, California Inst. of Tech, Pasadena, Calif.
MILLINGTON, Asst. Prof. H. G., C.E. (Rensselaer) Coll. of Eng., Univ. of Vermont, Burlington, Vt. *224 Plattsburg Ave.*
MILLS, Prof. C. N., A.M. (Indiana) Illinois State Normal Univ., Normal, Ill. *602 N. School St.*
MILLS, VICENTE. Chief Surveyor, Bureau of Lands, Manila, P.I. *P. O. Box 2442.*
MILNE, Asst. Prof. T. H., M.A. (Toronto) Univ. of Manitoba, Winnipeg, Man., Can.
MILNE, Prof. W. E., Ph.D. (Harvard) Univ. of Oregon, Eugene, Ore. *1862 Kincaid St.*
MIRICK, G. R., A.M. (Michigan) Teacher, Lincoln School., Teachers Coll., New York, N. Y. *425 W. 123 St.*
MISER, NELLIE P. (Mrs. W. L.), A.B. (Huron Coll.) Teacher, Ward-Belmont School, Nashville, Tenn. *1715 Ashwood Ave.*
MISER, Prof. W. L., Ph.D. (Chicago) Vanderbilt Univ., Nashville, Tenn. *1715 Ashwood Ave.*
MISH, A. F., A.B. (West Virginia) Instr., High School, Grafton, W. Va. *Inwood, Berkeley Co., W. Va.*
MITCHELL, Asst. Prof. A. K., Ph.D. (Johns Hopkins) Trinity Coll., Hartford, Conn. *149 N. Whitney St.*
MITCHELL, Prof. B. E., Ph.D. (Columbia) Millsaps Coll., Jackson, Miss.
MITCHELL, Mrs. DECIMA E., M.A. (Alberta) 600 W. 32nd St., Austin, Tex.
MITCHELL, Prof. H. H., Ph.D. (Princeton) Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
MITCHELL, Prof. U. G., Ph.D. (Princeton) Head of Dept., Univ. of Kansas, Lawrence, Kans. *1313 Massachusetts Ave.*
MITRA, Prof. N. B., M.A. (Calcutta Univ.) Ewing Christian Coll., Allahabad City, U.P., India.
MIZE, CLARA P., M.S. (Chicago) Head of Dept., Virginia Intermont Coll., Bristol, Va.
MODE, Prof. E. B., A.M. (Harvard) Boston Univ., Boston, Mass. *688 Boylston St.*
MOENCH, Prof. L. W., A.B. (Macalester) German and Religion, Prin. of Acad., St. Paul Luther Coll., St. Paul, Minn.
MOLINA, E. C. Tel. Engr., Amer. Tel. and Tel. Co. New York, N. Y. *195 Broadway.*
MONASTERIO, Prof. J. O., C.E. (Chapultepec Mil. Acad.) Math. and Physics, Loyola Univ., New Orleans, La.
MONTAGUE, HARRIET F., A.M. (Buffalo) Instr., Univ. of Buffalo, Buffalo, N. Y. *3425 Main St.*
MOODY, ETHEL I., Ph.D. (Cornell) Instr., Sweet Briar Coll., Sweet Briar, Va.
MOODY, Prof. W. A., A.M. (Bowdoin) Emeritus, Bowdoin Coll., Brunswick, Me. *60 Federal St.*
MOORE, Asso. Prof. C. L. E., Ph.D. (Cornell) Mass. Inst. of Tech., Cambridge, Mass.
MOORE, Prof. C. N., Ph.D. (Harvard) Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, Prof. E. H., Ph.D. (Yale) Univ. of Chicago, Chicago, Ill.
MOORE, Asso. Prof. F. C., A.B. (Dartmouth) Mass. Agric. Coll., Amherst, Mass.
MOORE, G. E., M.S. (Illinois) Asst., Univ. of Illinois, Urbana, Ill. *712 W. Arlington Ct., Champaign, Ill.*
MOORE, Asst. Prof. L. T., Ph.D. (Johns Hopkins) Brooklyn Coll., Brooklyn, N. Y. *59 Livingston St.*
MOORE, Prof. R. L., Ph.D. (Chicago) Pure Math., Univ. of Texas, Austin, Tex.
MOORE, Asst. Prof. T. W., Ph.D. (Yale) Indiana Univ., Bloomington, Ind. *Dept. of Math.*
MOORE, Prof. W. A., A.M. (Chicago) Birmingham Southern Coll., Birmingham, Ala.
MOORE, Asso. Prof. W. L., Ph.D. (Illinois) Univ. of Louisville, Louisville, Ky. *4521 S. 2nd St.*
MOOTS, E. E., Ph.D. (Iowa) Chm., Math. and Eng., Cornell Coll., Mt. Vernon, Iowa. *824 Summit Ave.*

MOREHOUSE, T. C. Mgr., College Dept., The Macmillan Co., New York, N.Y. *60 Fifth Ave.*
MORENO, Prof. H. C., Ph.D. (Clark) Applied Math., Stanford Univ., Stanford University, Calif. *684 Mirada Ave.*
MORENUS, Prof. EUGENIE M., Ph.D. (Columbia) Sweet Briar Coll., Sweet Briar, Va.
MORGAN, F. M., Ph.D. (Cornell) Dir., Clark School, Hanover, N. H.
MORGAN, W. D. 740 Curfew Ave., St. Paul, Minn.
MORIARTY, M. M. S., A.B. (Holy Cross) Emeritus, High School, Holyoke, Mass. *Petersham, Mass.*
MORITZ, Prof. R. E., Ph.D. (Strassburg) Univ. of Washington, Seattle, Wash.
MORLEY, Prof. R. K., Ph.D. (Clark) Worcester Poly. Inst., Worcester, Mass. *43 Laconia Rd.*
MORRILL, W. K., Ph.D. (Johns Hopkins) Instr., Johns Hopkins Univ., Baltimore, Md. *5203 Gwynn Oak Ave.*
MORRIS, Prof. C. C., A.M. (Harvard) Ohio State Univ., Columbus, Ohio.
MORRIS, Prof. F. R., Ph.D. (California) State Teachers Coll., Fresno, Calif.
MORRIS, Asst. Prof. MAX, Ph.D. (Chicago) Case School of Appl. Sc., Cleveland, Ohio.
MORRIS, Prof. RICHARD, Ph.D. (Cornell) Rutgers Univ., New Brunswick, N. J. *12 Johnson St.*
MORRIS, Prof. S. S., M.S. (West Virginia) Broaddus Coll., Philippi, W. Va.
MORROW, E. B., A.B. (Princeton) Headmaster, Gilman Country School, Roland Park, Md.
MORSE, Prof. D. S., Ph.D. (Cornell) Union Coll., Schenectady, N. Y. *1372 Nott St.*
MORSE, Prof. MARSTON, Ph.D. (Harvard) Harvard Univ., Cambridge, Mass. *1737 Cambridge St.*
MORSE, Prof. W. P., A.M. (Maine) Ricker Jr. Coll., Houlton, Me.
MORTON, Prof. A. B., A.M. (Brown) Pure Math., Georgia School of Tech., Atlanta, Ga.
MORTON, NELLIE C., A.B. (Brown) Asst. in Astr., Wellesley Coll., Wellesley, Mass.
MOSKOVITZ, David, M.S. (Carnegie Inst. of Tech.) Brown Univ., Providence, R. I.
MOSSMAN, Asst. Prof. THIRZA A., A.M. (Chicago) State Agric. Coll., Manhattan, Kans.
MOULTON, Dean E. J., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *1114 Colfax St.*
MOULTON, F. R., Ph.D. (Chicago) Dir., Utilities Power and Light Corp., Chicago, Ill. *327 S. LaSalle St.*
MOURAD, Lieut. Commander SALIH (Turkish Navy; retired) Prof. of Physics, Govt. Eng. Coll; Appl. Math., Robert Coll., Constantinople, Turkey. *Robert College, Bebek.*
MOYLE, K. E., A.M. (Bucknell) Teacher, Moses Brown School, Providence, R. I.
MUEHLMAN, Rev. PAUL, A.M. (St. Louis Univ.) St. Louis Univ., St. Louis, Mo. *221 Grand Blvd.*
MUIR, Sir THOMAS, D.Sc. (Univ. of Cape Town) Late Supt.-Genl. of Educ., Cape Colony. *Elmcote, Sandown Rd., Rondebosch, S. Africa.*
MULLEMEISTER, Asst. Prof. HERMANCE, Ph.D. (Utrecht, Holland) Univ. of Washington, Seattle, Wash. *Dept. of Math.*
MULLEN, Mrs. SELAH W., B.S. (Michigan) Teacher, North East High School, Detroit, Mich. *286 E. Grand Blvd.*
MULLER, JESSIE J. (Mrs. HERMANN J.), Ph.D. (Illinois) Univ. of Texas, Austin, Tex.
MULLINGS, M. E., Ph.D. (Cincinnati) Instr., Eng. Math., Univ. of Cincinnati, Cincinnati, Ohio. *Dept of Math.*
MULLINS, Prof. G. W., Ph.D. (Columbia) Barnard Coll., Columbia Univ., New York, N. Y. *Barnard Coll., 119th St. and Broadway.*
MUNDHJELD, SIGURD, A.B. (Concordia, Moorhead, Minn.) Instr., Waldorf Coll., Forest City, Iowa.
MUNROE, FLORENCE L., A.B. (Wellesley) Teacher, High School, Northampton, Mass. *5 Franklin St.*
MURFEE, Col. W. L., A.M. (Virginia) Pres., Marion Inst., Marion, Ala.
MURNAGHAN, Prof. F. D., Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore, Md.
MURRAY, Asso. Prof. C. A., A.M. (Texas) State Teachers Coll., Canyon, Tex. *On leave 1931–32, Univ. of Texas, Austin, Tex.*
MURRAY, Prof. D. A., Ph.D. (Johns Hopkins) Applied Math., McGill Univ., Montreal, P. Q., Can. *3653 University St.*
MURRAY, F. H., Ph.D. (Harvard) Address unknown.
MURRAY, G. H., B.S. in Bus. Adm. (Missouri) 2934 Victor St., Kansas City, Mo.
MURTO, Miss ARRIA, M.S. (Chicago) Teacher, High School, Carthage, Mo. *914 Howard Ave.*
MUSSELMAN, Prof. J. R., Ph.D. (Johns Hopkins) Western Reserve Univ., Cleveland, Ohio.
MYERS, Prof. H. S., A.M. (Chicago) Southwestern Coll., Winfield, Kans. *1501 E. 3rd Ave.*

NASH, F. P., A.M. (Columbia) Teacher, Groton School, Groton, Mass.
NASSAU, Prof. J. J., Ph.D. (Syracuse) Astr., Case School of Appl. Sci., Cleveland, Ohio.

NEELLEY, Prof. J. H., Ph.D. (Yale) Carnegie Inst. of Tech., Pittsburgh, Pa. *300 Broadmoor Ave., Sunset Hills, Mt. Lebanon, Pa.*
NEFF, Prof. I. F., M.S. (Drake; Chicago) Drake Univ., Des Moines, Iowa. *2801 Brattleboro Ave.*
NEIKIRK, Asst. Prof. L. I., Ph.D. (Pennsylvania) Univ. of Washington, Seattle, Wash. *4723-21st. Ave. N.E.*
NELSON, Prof. A. L., Ph.D. (Chicago) Coll. of the City of Detroit, Detroit, Mich.
NELSON, Asso. Prof. C. A., Ph.D. (Chicago) New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N. J.
NELSON, W. A., A.M. (Texas) Teacher, Jr. Coll., Tyler, Tex. *Box 676.*
NELSON, Asso. Prof. W. K., M.S.; E.E. (Colorado) Eng. Math., Univ. of Colorado, Boulder, Colo. *925 Grandview Ave.*
NESS, MARIE M., A.M. (Minnesota) Research Assoc., Univ. of Minnesota, Minneapolis, Minn. *2530 Dupont Ave. S.*
NEUBAUER, GRETA, A.M. (Wyoming) Instr., Univ. of Wyoming, Laramie, Wyo. *Ivinson Hall.*
NEWELL, M. J., A.M. (Michigan) Teacher, High School, Evanston, Ill. *2226 Hartzell St.*
NEWLIN, Asst. Prof. R. L., M.S. (Chicago) Ohio Wesleyan Univ., Delaware, Ohio. *163 N. Sandusky St.*
NEWSOM, Prof. C. V., Ph.D. (Michigan) Univ. of New Mexico, Albuquerque, N. Mex.
NEWSON, Prof. MARY W., Ph.D. (Göttingen) Eureka Coll., Eureka, Ill.
NEWTON, F. E., Ph.B. (Yale) Instr., Phillips Acad., Andover, Mass. *9 Salem St.*
NEWTON, Prof. G. A., A.M. (Trinity Univ.) Head of Dept., Trinity Univ., Waxahachie, Tex.
NICHOLS, G. D., A.M. (Nebraska) Univ. of Arkansas, Fayetteville, Ark. *Dept. of Math.*
NICHOLS, Prof. I. C., Ph.D. (Michigan) Applied Math., Louisiana State Univ., Baton Rouge, La.
NICKOL, Prof. J. P., Ph.D. (Fribourg) Physics, St. Bonaventure's Coll., St. Bonaventure, N. Y.
NICOLET, JUSTIN, C.E. 1849 Belle Plaine Ave., Chicago, Ill.
NIXON, J. C., M.S. (Chicago) 430 Diversey Parkway, Chicago, Ill.
NOBLE, Prof. C. A., Ph.D. (Göttingen) Univ. of California, Berkeley, Calif. *2224 Piedmont Ave.*
NORBERT, Brother, M.S. (Notre Dame) Instr., St. Edwards Univ., Austin, Tex.
NORDGAARD, Prof. M. A., Ph.D. (Columbia) Wagner Coll., Staten Island, N. Y. *Grymes Hill.*
NYBERG, J. A., M.S. (Chicago) Instr., Hyde Park High School, Chicago, Ill. *7753 East End Ave., Windsor Park Sta.*
NYSWANDER, Asst. Prof. J. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.

OAKLEY, Asst. Prof. C. O., Ph.D. (Illinois) Brown Univ., Providence, R. I.
O'DONNELL, G. A., A.M. (Woodstock; Georgetown) Grad. Student, St. Louis Univ., St. Louis, Mo. *Grand and Pine Blvds.*
OERGEL, C. T., B.S. in M.E. (Penna. State Coll.) Student Engr., Genl. Elec. Co., Schenectady, N. Y. *1054 Glenwood Blvd.*
OGLESBY, Prof. E. J., A.M. (Virginia) Eng. Math., Univ. of Virginia, University, Va. *P. O. Box 1032.*
OKEAN, HARRY, B.S. (Alfred) Address unknown.
OLDENBURGER, RUFUS, M.S. (Chicago) Instr., Case School of Appl. Sc., Cleveland, Ohio. *Main 67.*
OLDHAM, Asst. Prof. MABEL R., A.M. (Texas) State Coll. for Women, Denton, Tex. *Box 895, C.I.A. Sta.*
OLDS, Asso. Prof. E. G., Ph.D. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *1424 Barnsdale St.*
OLLIVIER, ARTHUR, M.S. (Iowa) Instr., State Agric Coll., Manhattan, Kans. *Dept. of Math.*
OLSON, EMMA J., A.M. (Chicago) 2218 S. St. Aubin St., Morningside, Sioux City, Iowa.
OLSON, Asst. Prof. H. L., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *544 Forest Ave.*
OPP, J. E., A.M. (Nebraska) Supt. City Schools, Burwell, Nebr.
O'QUINN, R. L., A.B. (Louisiana) Instr., Louisiana State Univ., Baton Rouge, La.
ORANGE, WILLIAM, A.M. (California) Chm., Math. Dept., Jr. Coll., Los Angeles, Calif. *855 N. Vermont Ave.*
ORE, Prof. OYSTEIN, Ph.D. (Yale) Yale Univ., New Haven, Conn.
ORR, EUNICE C., A.M. (Indiana) Teacher, High School, Arcadia, Ind. *Scircleville, Ind.*
OSBORN, Prof. JESSE, Ph.D. (Cornell) Harris Teachers Coll., St. Louis, Mo. *3239 Lafayette Ave.*
OSGOOD, Prof. W. F., Ph.D. (Erlangen) Harvard Univ., Cambridge, Mass. *74 Avon Hill St.*

O'Shaughnessy, Prof. Louis, Ph.D. (Pennsylvania) Applied Math., Virginia Poly. Inst., Blacksburg, Va. *Box 177.*
O'Toole, A. L., Ph.D. (Michigan) National Research Fellow, Univ. of Minnesota, Minneapolis, Minn. *1109 Sixth St. S.E.*
Ott, Prof. W. P., Ph.D. (Chicago) Univ. of Alabama, University, Ala.
Overman, J. R., Ph.D. (Michigan) Head of Dept., State Coll., Bowling Green, Ohio.
Owens, Prof. F. W., Ph.D. (Chicago) Head of Dept., Pennsylvania State Coll., State College, Pa. *526 E. Foster Ave.*
Owens, Helen B. (Mrs. F. W.), Ph.D. (Cornell) 526 E. Foster Ave., State College, Pa.
Oxsheer, Asso. Prof. Lela, A.M. (Teachers Coll., Columbia) Stephen F. Austin Teachers Coll., Nacogdoches, Tex. *516 North St.*

Paaswell, George, C.E. (Cornell) Consulting Engr., New York, N. Y. *1950 Andrews Ave., Bronx.*
Packer, Margaret C., A.M. (Brown) Bell Telephone Labs., New York, N. Y. *463 West St. Room 957.*
Palmié, Prof. Anna H., Ph.B. (Cornell) Emeritus, Coll. for Women, Western Reserve Univ., Cleveland, Ohio. *Ormond Beach, Fla.*
Paradiso, L. J. A.M. (Ohio State) Cornell Univ., Ithaca, N. Y., *129 Linden Ave.*
Park, Prof. R. S., Ph.D. (Kentucky) Eastern Teachers Coll., Richmond, Ky. *213 Burnam Ct.*
Parker, Prof. W. P., M.A. (Emory Univ.) Union Christian Coll., Pyengyang, Korea.
Parkinson, Asso. Prof. G. A., Ph.D. (Wisconsin) Univ. of Wisconsin, Extension Div., Milwaukee, Wis. *623 W. State St.*
Parson, S. F. Head of Dept., Northern Illinois State Teachers Coll., De Kalb, Ill. *305 College Ave.*
Partridge, E. A., Ph.D. (Pennsylvania) Science, West Phila. High School for Boys, Philadelphia, Pa. *48th and Walnut Sts.*
Passano, Asso. Prof. L. M., A.B. (Johns Hopkins) Mass. Inst. of Tech., Cambridge, Mass.
Patten, Prof. W. E., C.E. (Cornell) 401 Conklin Ave., Binghamton, N. Y.
Patterson, Asso. Prof. B. C., Ph.D. (Johns Hopkins) Hamilton Coll., Clinton, N. Y. *College Hill.*
Pattillo, Dean N. A., Ph.D. (Johns Hopkins) Randolph-Macon Women's Coll., Lynchburg, Va.
Patton, Bess, M.S. (Chicago) Teacher, Girls' High School, Atlanta, Ga. *1585 N. Decatur Rd. N.E.*
Paula, Sister Mary, M.S. (Notre Dame) Prof., Marygrove Coll., Detroit, Mich.
Paulinus, Brother, A.B. (Mt. St. Joseph) Instr., St. Xavier High School, Louisville, Ky.
Paxton, Asso. Prof. E. K., A.M. (Columbia) Washington and Lee Univ., Lexington, Va.
Paxton, Mary S., A.M. (Indiana) Teacher, South Side High School, Fort Wayne, Ind. *125 E. Branning Ave.*
Payne, C. K., M.B.A. (New York Univ.) Instr., Washington Square Coll., New York, N. Y. *100 Washington Sq. E.*
Peck, W. D., A.B. (Lebanon Valley) Head of Dept., Asheville School for Boys, Asheville, N. C.
Peckham, Asso. Prof. Anna B., A.M. (Denison) Denison Univ., Granville, Ohio.
Pedersen, Asso. Prof. F. M., Sc.D. (New York Univ.) Coll. of the City of New York, New York, N. Y. *520 W. 114th St.*
Pegram, Prof. Annie M., A.M. (Duke) Greensboro Coll., Greensboro, N. C.
Pehrson, Asso. Prof. E. W., A.M. (California) Univ. of Utah, Salt Lake City, Utah.
Pelletier, Prof. Arthur. Higher Alg., École Poly., Montreal, Can. *8456 Drolet St.*
Pence, Sallie, E., A.M. (Kentucky) Instr., Univ. of Kentucky, Lexington, Ky. *635 Maxwelton Ct.*
Penn, S. S. 860 E. 161st. St., New York, N. Y.
Pennell, W. O., B.S. (Mass. Inst. of Tech.) Chief Engr., So. Western Bell Tel. Co., St. Louis, Mo. *330 Oakwood Ave., Webster Groves, Mo.*
Perkins, F. W., Ph.D. (Harvard) Instr., Dartmouth Coll., Hanover, N. H. *201 Hallgarten Hall.*
Perkins, H. A., A.B. (Colby) Hampton Inst., Hampton, Va.
Perkins, Prof. L. R., A.M. (Tufts) Middlebury Coll., Middlebury, Vt. *6 Franklin St.*
Perry, Asst. Prof. Ruth C., A.B. (Wellesley) Dean of Women, Univ. of Chattanooga, Chattanooga, Tenn.
Peters, J. W., Ph.D. (Johns Hopkins) Instr., Univ. of Illinois, Urbana, Ill. *360 Math. Bldg.*
Peterson, J. K., A.M. (Harvard) 1915 Blakemore Ave., Nashville, Tenn.
Peterson, Prof. O. J., Ph.D. (Michigan) State Teachers Coll., Emporia, Kans. *1402 Highland Pl. St.*

PETERSON, T. S., Ph.D. (Ohio State) Instr., Univ. of Michigan, Ann Arbor, Mich. *1513 S. University Ave.*
PETTERSEN, C. A., Ph.B. (Northwestern) Asst. Prin., Schurz High School, Chicago, Ill. *3922 Lowell Ave.*
PETTIT, Prof. H. P., Ph.D. (Illinois) Marquette Univ., Milwaukee, Wis. *1661 Bartlett Ave.*
PHALEN, Dean H. R., Ph.D. (Chicago) St. Stephen's Coll., Annandale-on-Hudson, N. Y.
PHILIPS, Prof. A. W., A.M. (Chicago) State Teachers Coll., Emporia, Kans.
PIAGGIO, Prof. H. T. H., D.Sc. (London) University Coll., Nottingham, England.
PICKETT, JONATHAN. Realitos, Tex.
PIERCE, Prof. JESSE, Ph.D. (Michigan) Heidelberg Univ., Tiffin, Ohio. *138 Greenfield St.*
PIERCE, Prof. T. A., Ph.D. (California) Univ. of Nebraska, Lincoln, Nebr. *1811 Pepper Ave.*
PIERPONT, Prof. JAMES, Ph.D. (Vienna) Yale Univ., New Haven, Conn. *102 Avon St.*
PIERSON, A. D., A.M. (Missouri) Jr. Coll., Kansas City, Mo. *7217 Summit St.*
PINCHERLE, Prof. SALVATORE, Sc.D. (Bologna) Univ. of Bologna, Bologna, Italy. *Viale Panzacchi 3.*
PINE, WINNIFRED F., A.B. (Brown) Teacher, St. Margaret's School, Waterbury, Conn.
PINKERTON, R. M., B.S. (Bradley Poly. Inst.) Grad. Fellow, Univ. of Virginia. *1110 Westland Ave., Charlottesville, Va.*
PIRENIAN, Asst. Prof. Z. M., M.S. (Florida) Univ. of Florida, Gainesville, Fla. *Dept. of Math.*
PLANT, Prof. L. C., M.S. (Chicago) Michigan State Coll., East Lansing, Mich.
PLIMPTON, G. A., LL.D. (Rochester), L.H.D. (New York Univ.) Sr. member, Ginn & Co., New York, N. Y. *70 Fifth Ave.*
PLOENGES, Prof. E. W., A.M. (Michigan) Kansas Wesleyan Univ., Salina, Kans.
PLYMALE, Prof. R. B., A.M. (Mercer; Columbia) Bessie Tift Coll., Forsyth, Ga.
POBANZ, J. F., Ph.D. (California) Jr. Coll., Modesto, Calif. *309 Hackberry Ave.*
POLAN, L. R., A.B. (Milton Coll.) Instr., Alfred Univ., Alfred, N. Y.
POLLARD, Asst. Prof. H. S., M.S. (Iowa) Miami Univ., Oxford Ohio.
POLLOCK, Asst. Prof. SAUL, Ph.D. (California) State Teachers Coll., Terre Haute, Ind. *2520 Deming St.*
POOL, H. A., A.M. (Minnesota). Address unknown.
POOLER, L. G., Ph.D. (Columbia) Physics, Hunter Coll., New York, N.Y. *456 Riverside Dr.*
PORITSKY, HILLEL, Ph.D. (Cornell) Teacher and Mathematician, Genl. Elec. Co., Schenectady, N. Y.
PORTER, Asso. Prof. C. S., A.M. (Clark) Amherst Coll., Amherst, Mass. *Box 735.*
PORTER, T. I., A.B. (Missouri) State Agric. Coll., Manhattan, Kans.
POST, E. L., Ph.D. (Columbia) Teacher, High School, New York, N. Y. *625 W. 152 St.*
POUND, Prof. V. E., Ph.D. (Toronto) Univ. of Buffalo, Buffalo, N. Y. *122 Berkshire Ave.*
POUNDER, Asso. Prof. I. R., Ph.D. (Chicago) Univ. of Toronto, Toronto, Ont., Can.
POWELL, J. E., M.S. (Ohio State) Instr., Michigan State Coll., East Lansing, Mich.
POWELSON, INEZ D., A.M. (California) Teacher, High School, Berkeley, Calif. *2548 Benvenue Ave.*
PRASAD, Prof. GANESH, D.Sc. (Allahabad), M.A. (Cambridge) Calcutta Univ., Calcutta, India. *2 Samavaya Mansions, Corporation St.*
PRETZ, PIUS S., A.M. (St. John's Univ.) St. Benedict's Coll., Atchison, Kans.
PRICE, G. B., A.M. (Harvard) Grad. Student and Instr., Harvard Univ., Cambridge, Mass. *J-43, John Winthrop House.*
PRIDE, Asst. Prof. H. H., Ph.D. (New York Univ.) New York Univ., New York, N. Y.
PRIESTER, Prof. G. C., Ph.D. (Michigan) Mech., Univ. of Minnesota, Minneapolis, Minn. *Coll. of Eng. and Archit.*
PRUSLIN, JOHANNA L., B.S. (New York Univ.) Instr., Hunter Coll., New York, N. Y. *132–70 Sanford Ave., Flushing, L. I., N. Y.*
PUGH, G. T., Ph.D. (Vanderbilt) Head of Dept., Winthrop Coll., Rock Hill, S. C.
PUGSLEY, Asso. Prof. D. W., M.S. (Michigan) Berea Coll., Berea, Ky. *Box 91, College P. O.*
PURCELL, E. J., A.M. (Colorado) Grad. Student, Cornell Univ., Ithaca, N. Y. *104 Overlook Rd.*
PURDIE, Asst. Prof. K. S., B.S. (Va. Milit. Inst.) Virginia Milit. Inst., Lexington, Va.
PUTNAM, Asso. Prof. R. G., Ph.D. (Chicago) New York Univ., New York, N. Y. *115 Riverview Ave., Tarrytown, N. Y.*
PUTNAM, Prof. T. M., Ph.D. (Chicago) Prof. of Math. and Dean of Undergrads., Univ. of California, Berkeley, Calif.
PYKE, A. J., B.A. (Toronto) 1228 East 56th St., Chicago, Ill.

QUARLES, LOUIS, A.B. (Michigan) Lawyer, Milwaukee, Wis. *Box 489, Route 6, Sta. C.*
QUERFELD, D. W., A.B. (James Millikin) Research Assoc., U. S. Bureau of Standards, Washington, D. C. *1301-15th St. N.W.*

QUIGLEY, Prof. MARY J., A.B. (Radcliffe) Teachers Coll. of the City of Boston. *332 Savin Hill Ave., Dorchester, Mass.*
QUILTY, PATRICK, C.E. (Cooper Union) Instr., Cooper Union, New York, N. Y. *601 W. 163 St.*
QUINN, J. J., Ph.D. (Villanova) St. Mary's Coll., St. Mary's, Kans.

RABE, L. L., A.M. (Illinois) Instr., LaSalle-Peru-Oglesby Jr. Coll., LaSalle, Ill. *1047 Joliet St.*
RADO, Prof. TIBOR, Ph.D. (Szeged) Ohio State Univ., Columbus, Ohio. *407 King Ave.*
RAGSDALE, Prof. VIRGINIA, Ph.D. (Bryn Mawr) North Carolina Coll. for Women, Greensboro, N. C.
RAIFORD, T. E., A.M. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *1618 Shadford Rd.*
RAINICH, Asso. Prof. G. Y., Master in Pure Math. (Kazan) Univ. of Michigan, Ann Arbor, Mich. *602 Oswego St.*
RAINVILLE, E. D., A.B. (Colorado) Instr., Eng. Math., Univ. of Colorado, Boulder, Colo. *P. O. Box 173.*
RAMAGE, C. J., Ph.D. (Grove City) Lawyer, Saluda, S. C.
RAMBO, Asso. Prof. SUSAN M., Ph.D. (Michigan) Smith Coll., Northampton, Mass. *12 Barrett Pl.*
RAMLER, Prof. O. J., Ph.D. (Catholic Univ.) Catholic Univ., Washington, D. C. *12 Girard St. N.E.*
RAMOS, DORA, A.M. (Columbia) Prin., High School, San German, P. R.
RAMSDELL, Prof. G. E., A.M. (Harvard) Bates Coll., Lewiston, Me. *40 Mountain Ave.*
RAMSEY, Asso. Prof. MARGARET, A.M. (Oregon) Albany Coll., Albany, Ore.
RANDALL, Prof. A. W., A.M. (Colorado) State Indus. Coll., Prairie View, Tex. *Box 63.*
RANDOLPH, J. F., A.M. (Michigan) Cornell Univ., Ithaca, N. Y. *White Hall.*
RANKIN, A. W. Elec. Lab., Elec. Service Supplies Co., Philadelphia, Pa. *4526 N. Mole St.*
RANKIN, Prof. J. M. Coll. of Idaho, Caldwell, Idaho. *1810 Ash St.*
RANKIN, Prof. W. W., Jr., A.M. (North Carolina) Duke Univ., Durham, N. C.
RANSOM, Prof. W. R., A.M. (Tufts; Harvard) Tufts Coll., Tufts College, Mass. *29 Sawyer Ave.*
RANUM, Prof. ARTHUR, Ph.D. (Chicago) Cornell Univ. Ithaca, N. Y. *3 Central Ave.*
RASCHE, Prof. W. H. Mechanism, Virginia Poly. Inst., Blacksburg, Va.
RASEL, D. M., M.S. (Washington and Jefferson) Instr., Physics, Washington and Jefferson Coll., Washington, Pa.
RASMUSEN, RUTH B., M.S. (Chicago) Instr., South Dakota State Coll., Brookings, S. D.
RASOR, Prof. S. E., M.S. (Chicago) Ohio State Univ., Columbus, Ohio. *1594 Neil Ave.*
RAU, Prof. A. G., Ph.D. (Lehigh) Moravian Coll., Bethlehem, Pa. *38 W. Market St.*
RAUDENBUSH, H. W., Jr., A.M. (Columbia) Instr., Columbia Univ., New York, N. Y. *Hamilton Hall.*
RAWLINS, Asso. Prof. C. H., Jr., Ph.D. (Johns Hopkins) Postgrad. School, U. S. Naval Acad., Annapolis, Md. *13 Franklin St.*
RAYNOR, Asst. Prof. G. E., Ph.D. (Princeton) Lehigh Univ., Bethlehem, Pa. *704 Fifth Ave.*
REA, Asst. Prof. P. L., A.M. (Illinois) Marietta Coll., Marietta, Ohio. *On leave 1931–32, Ohio State Univ., Columbus, Ohio.*
REAGAN, Prof. C. A., A.M. (Kansas) Friends Univ., Wichita, Kans.
REAVES, Prof. CAROLINE M., A.M. (Oklahoma) Coker Coll., Hartsville, S. C.
REAVES, Prof. S. W., Ph.D. (Chicago) Dean of Coll. of Arts and Sc., Univ. of Oklahoma, Norman, Okla. *527 Chautauqua Ave.*
REBARKER, Prof. HERBERT, Ph.D. (Peabody) East Carolina Teachers Coll., Greenville, N. C.
RECHARD, Prof. O. H., Ph.D. (Wisconsin) Univ. of Wyoming, Laramie, Wyo. *601 S. 13th St.*
RECHT, A. W., A.M. (Denver) Instr., Math. and Astr., Univ. of Denver, Denver, Colo. *935 E. Ellsworth Ave.*
REDDEN, Asso. Prof. J. E., M.S. (Iowa State Coll.) John Tarleton Agric. Coll., Stephenville, Tex. *Box 568.*
REDDICK, Prof. H. W., Ph.D. (Columbia) Cooper Union, New York, N. Y.
REDDITT, B. H., A.M. (Johns Hopkins) Columbia, La.
REECE, Prof. R. H., A.M. (Colorado) New Mexico School of Mines, Socorro, N. Mex.
REED, Asso. Prof. F. W., Ph.D. (Virginia) Ohio Univ., Athens, Ohio.
REED, L. J., Ph.D. (Pennsylvania) School of Hygiene, Baltimore, Md. *615 N. Wolfe St.*
REEN, Asst. Prof. C. G., M.S.E. (Michigan) Civil Eng., Gettysburg Coll., Gettysburg, Pa. *25 Chambersburg St.*
REES, Asso. Prof. C. J., A.M. (Chicago) Univ. of Delaware, Newark, Del.
REES, Asso. Prof. E. L., C.E. (Kentucky), A.M. (Chicago) Univ. of Kentucky, Lexington, Ky. *200 University Ave.*

REES, MINA S., Ph.D. (Chicago) Instr., Hunter Coll., New York, N. Y. *2292 Loring Pl.*
REES, P. K., A.M. (Texas) Grad. Student, Rice Inst., Houston, Tex. *1719 Brun St.*
REES, W. A., A.M. (Texas) Instr., Jr. Coll., Houston, Tex. *1719 Brun St.*
REEVE, Prof. W. D., Ph.D. (Minnesota) Teachers Coll., Columbia Univ., New York, N. Y.
REID, Prof. L. W., Ph.D. (Göttingen) Haverford Coll., Haverford, Pa.
REID, W. T., Ph.D. (Texas) Instr., Univ. of Chicago, Chicago, Ill. *Box 37, Eckhart Hall.*
REILLY, Prof. J. F., Ph.D. (Iowa) Univ. of Iowa, Iowa City, Iowa. *Room 212, Physics Bldg.*
REINSCH, Prof. B. P., Ph.D. (Illinois) School of Eng., Southern Methodist Univ., Dallas, Tex.
REISING, J. A. Teacher, Central High School and Indiana Univ. Extension, Ft. Wayne, Ind. *2527 S. Harrison St.*
REMICK, Prof. B. L., Ph.M. (Cornell Coll.) State Agric. Coll., Manhattan, Kans.
RENNER, THERESA M., M.S. (Illinois) Registrar, Blackburn Coll., Carlinville, Ill. *324 N. Oak St.*
REYNOLDS, Prof. C. N., Ph.D. (Harvard) West Virginia Univ., Morgantown, W. Va.
REYNOLDS, Prof. J. B., Ph.D. (Moravian) Mech., Lehigh Univ., Bethlehem, Pa. *732 W. Broad St.*
REYNOLDS, LENA E., A.M. (California) Head of Dept., Jr. Coll., Fullerton, Calif.
REYNOLDS, W. F., A.B. (Johns Hopkins) Sr. Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C. *848 W. 37th St., Baltimore, Md.*
RHODES, C. E., A.B. (Cornell) Instr., Univ. of Cincinnati, Cincinnati, Ohio.
RHODES, Prof. M. C., Ph.D. (Peabody) Head of Dept., Univ. of Mississippi, University, Miss. *Box 422.*
RICE, Prof. HARRIS, A.M. (Harvard) Worcester Poly. Inst., Worcester, Mass.
RICE, J. M., E.E. (Lehigh) Exec. Engr., Curtiss Aeroplane and Motor Co., Buffalo, N. Y. *279 Colvin Ave.*
RICE, Asso. Prof. J. N., Ph.D. (Catholic Univ.) Catholic Univ., Washington, D. C. *3326 Thirteenth St. N.E.*
RICE, Asst. Prof. L. H., A.B. (Syracuse) Mass. Inst. of Tech., Cambridge, Mass.
RICHARDS, W. A., A.M. (Chicago) Teacher, Morton Jr. Coll., Cicero, Ill. *3305 Ridgeland Ave., Berwyn, Ill.*
RICHARDSON, Prof. A. V., M.A. (Cambridge) Bishops Coll., Lennoxville, Quebec, P. Q., Can.
RICHARDSON, Prof. C. H., Ph.D. (Michigan) Bucknell Univ., Lewisburg, Pa.
RICHARDSON, Dean R. G. D., Ph.D. (Yale) Grad. School, Brown Univ., Providence, R. I.
RICHERT, Prof. D. H., A.M. (Colorado) Bethel Coll., Newton, Kans.
RICHESON, Asst. Prof. A. W., Ph. D. (Johns Hopkins) Univ. of Maryland, College Park, Md. *306 E. 32nd St., Baltimore, Md.*
RICHMOND, SUSAN V., A.B. (Randolph-Macon Woman's Coll.) Teacher, Western High School, Washington, D. C. *2126 Connecticut Ave. N. W.*
RICHTMEYER, Asst. Prof. C. C., A.M. (Peabody) State Teachers Coll., Mt. Pleasant, Mich.
RICKARD, Asst. Prof. HORTENSE, A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *79 Beechwold Rd.*
RIDER, Asso. Prof. P. R., Ph.D. (Yale) Washington Univ., St. Louis, Mo.
RIES, H. F., A.B. (Michigan) Asst. Actuary, Guaranty Life Ins. Co., Davenport, Iowa. *2529 Middle Rd.*
RIESBECK, LAURA, B.S. in Educ. (Ohio Univ.) Teacher, High School, Zanesville, Ohio. *1001 Greenwood Ave.*
RIETZ, Prof. H. L., Ph.D. (Cornell) Univ. of Iowa, Iowa City, Iowa.
RIGGS, Prof. N. C., M.S. (Harvard) Mech., Carnegie Inst. of Tech., Pittsburgh, Pa.
RISLEY, W. J., A.M. (Illinois) Plaza Hotel, Denver, Colo.
RISSELMAN, Prof. W. C., Ph.D. (Minnesota) Northern Arizona Teachers Coll., Flagstaff, Ariz.
RITER, H. E., M.A. (Manitoba) Math. Master, Provincial Normal School, Winnipeg, Man., Can. *909 Strathcona St.*
RITT, Prof. J. F., Ph.D. (Columbia) Columbia Univ., New York, N. Y. *Dept. of Math.*
ROBB, D. L., A.M. (Pittsburgh) Grad. Asst., Univ. of Pittsburgh. *403 Federal St., Butler, Pa.*
ROBB, J. M., A.M. (Michigan) Teacher, Math. and Physics, Jr. Coll., Taft, Calif. *727 San Emidio St.*
ROBBINS, ALICE V., Ph.M. (Chicago) Teacher, State Teachers Coll., Mankato, Minn. *418 S. 4th St.*
ROBBINS, Asst. Prof. C. K., A.M. (Harvard) Purdue Univ., W. LaFayette, Ind. *418 Vine St.*
ROBERTS, Prof. B. D., A.M. (Indiana) Parsons Coll., Fairfield, Iowa. *409 W. Carpenter St.*
ROBERTS, H. M., B.M.E. (Minnesota) Efficiency Engr., Sears, Roebuck & Co., Minneapolis, Minn. *3225 2nd Ave S.*

ROBERTS, Asst. Prof. J. H., Ph.D. (Texas) Duke Univ., Durham, N. C. *Box 4708.*
ROBERTS, Prof. MARIA M., B.L. (Iowa State Coll.) Prof. of Math. and Dean, Iowa State Coll., Ames, Iowa.
ROBERTS, WALTER, B.S. (Pennsylvania) Head of Dept., W. Philadelphia High School, Philadelphia, Pa. *3419 N. 22nd St.*
ROBERTSON, FRED, A.M. (Indiana) Instr., Iowa State Coll., Ames, Iowa. *Dept of Math.*
ROBINSON, Asst. Prof. A. J., M.S. (Emory) Alabama Poly. Inst., Auburn, Ala.
ROBINSON, G. EDNA, A.M. (Missouri) 1507 Windsor St., Columbia, Mo.
ROBINSON, Prof. H. A., Ph.D. (Johns Hopkins) Agnes Scott Coll., Decatur, Ga. *123 College Pl.*
ROBINSON, L. V., Ph.D. (Harvard) Astronomer, Harvard Coll. Observ., Cambridge, Mass.
ROBINSON, ROBIN, Ph.D. (Harvard) Instr., Dartmouth Coll., Hanover, N. H. *21 Prospect St.*
ROBINSON, FLORA EATON (Mrs. W. F.) Registrar and Head of Dept., Mars Hill Coll., Mars Hill, N. C.
ROBISON, G. M., Ph.D. (Cornell) Faculty Club, Durham, N. C.
RODGERS, Dean T. G., A.M. (Wisconsin) New Mexico Normal Univ., Las Vegas, N. Mex. *1018 Fourth St.*
ROESER, H. M., M.S. (George Washington) Engr., Physicist, U. S. Bureau of Standards, Chicago, Ill. *5800 W. 69th St., Clearing Sta.*
ROEVER, Prof. W. H., Ph.D. (Harvard) Pure Math., Washington Univ., St. Louis, Mo.
ROGERS, Dean J. C., A.M. (Columbia) Dean and Prof. of Math., Piedmont Coll., Demorest, Ga.
ROGERS, H. P., A.M. (Illinois) Asst., Univ. of Illinois. *616 E. Daniel St., Champaign, Ill.*
ROMAN, Asst. Prof. IRWIN, Ph.D. (Chicago) Math. and Physics, Michigan Coll. of Mining and Tech., Houghton, Mich. *Hubbell Hall.*
ROOD, MARION B., M.S. (Michigan) 1014 Church St., Ann Arbor, Mich.
ROOS, C. F., Ph.D. (Rice) Permanent Secy., Amer. Assn. for the Adv. of Sc., Washington, D. C. *Smithsonian Inst. Bldg.*
ROOT, Prof. R. E., Ph.D. (Chicago) U. S. Naval Acad., Annapolis, Md. *7 Franklin St.*
ROOTS, Prof. Y. K., M.S. (New York Univ.) Math. and Physics, Findlay Coll., Findlay, Ohio.
ROSEBRUGH, Prof. T. R., M.A. (Toronto) Elec. Eng., Univ. of Toronto, Toronto, Ont., Can. *92 Walmer Rd.*
ROSENBACH, Asso. Prof. J. B., M.S. (Illinois) Carnegie Inst. of Tech., Pittsburgh, Pa. *2526 Beechwood Blvd.*
ROSENBAUM, JOSEPH, Ph.D. (Cornell) Instr., The Milford School, Milford, Conn.
ROSENGARTEN, Prof. GEORGE, Ph.D. (Pennsylvania) W. Phila. High School for Boys, Philadelphia, Pa. *1105 Edgewood Rd., Upper Darby, Pa.*
ROSSKOPF, M. F., A.M. (Minnesota) Instr., Brown Univ., Providence, R. I. *63 Charles Field St.*
ROTE, L. J. Practicing Attorney, Denver, Colo. *260 S. Broadway.*
ROTH, Asst. Prof. W. E., Ph.D. (Wisconsin) Univ. of Wisconsin, Extension Div., Milwaukee, Wis. *2103 Wisconsin Ave.*
ROTHERMEL, FLORENCE. Teacher, W. Phila. High School, Philadelphia, Pa. *320 S. 42nd St.*
ROTHROCK, Dean D. A., Ph.D. (Leipzig) Indiana Univ., Bloomington, Ind. *1000 Atwater Ave.*
ROULTON, J. A. Research Engr., Atlantic Refining Co., Philadelphia, Pa. *LeRoy Court Apts., 60th St. and Warrington Ave.*
ROUSE, Asst. Prof. L. J., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich.
ROWE, Prof. C. H., M.A. (Dublin) Univ. of Dublin, Dublin, Ireland. *38 Trinity College.*
ROWE, Pres. J. E., Ph.D. (Johns Hopkins) Clarkson Coll. of Tech., Potsdam, N. Y.
ROWLAND, Prof. S. A. Ohio Wesleyan Univ., Delaware, Ohio. *49 Oak Hill Ave.*
RUDERMAN, H. D. Student, Coll. of the City of New York. *718 Ave. P, Brooklyn, N. Y.*
RUMNEY, ETHEL A., M.S. (Chicago) 309 N. Main St., Sandwich, Ill.
RUMSEY, MARY B., A.M. (Illinois) Asst., Univ. of Illinois, Urbana, Ill. *Univ. Sta.*
RUNDSTROM, Prof. INEZ, Fil. Kand. (Uppsala, Sweden) Gustavus Adolphus Coll., St. Peter, Minn.
RUNGE, Asst. Prof. LULU L., A.M. (Wisconsin) Univ. of Nebraska, Lincoln, Nebr.
RUNNING, Prof. T. R., Ph.D. (Wisconsin) Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, Asso. Prof. C. A., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa.
RUSK, Prof. W. J., M.A. (Toronto) Math. and Astr., Grinnell Coll., Grinnell, Iowa. *1415 Park St.*
RUSSELL, BEULAH, A.M. (Chicago) Williamsburg, Va.

RUSSELL, Prof. W. P., A.M. (Cumberland) Pomona Coll., Claremont, Calif.
RUTLEDGE, Asso. Prof. GEORGE, Ph.D. (Illinois) Mass. Inst. of Tech., Cambridge, Mass. *29 Bellevue Rd., Belmont, Mass.*
RYAN, MARGARET M., A.M. (Syracuse) Grad. Asst., Syracuse Univ., Syracuse, N. Y. *415 Bryant Ave.*

SABIN, MARY S., A.M. (Denver) 730 Emerson St., Denver, Colo.
SAFFORD, Prof. F. H., Ph.D. (Harvard) Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
SAGEN, G. O., A.M. (California) Instr., Kern Co. Jr. Coll., Bakersfield, Calif.
SAIBEL, Asst. Prof. E. A., Ph.D. (Mass. Inst. of Tech.) Carnegie Inst. of Tech., Pittsburgh, Pa.
SALAS-EDWARDS, Prof. RAMON. Dynamics, Univ. of Chile, Santiago, Chile. *Delicias 1876.*
SALKOVER, Asst. Prof. MEYER, Ph.D. (Yale) Univ. of Cincinnati, Cincinnati, Ohio. *3467 Ruther St.*
SALVOSA, Prof. L. R., Ph.D. (Michigan) Forest Eng., Univ. of the Philippines, Manila, P. I.
SANCHEZ-DIAZ, Asst. Prof. RAFAEL, B.S. (Porto Rico) Coll. of A. and M., Univ. of Porto Rico, Mayaguez, P. R. *P. O. Box 231.*
SANDERS, Prof. S. T., M.S. (Chicago) La. State Univ., Baton Rouge, La. *714 Mills Ave.*
SANFORD, Asst. Prof. VERA, Ph.D. (Columbia) School of Education, Western Reserve Univ. Cleveland Ohio.
SAUNDERS, FAITH, A.M. (Missouri) Maysville, DeKalb Co., Mo.
SAUREL, Prof. PAUL, Sc.D. (Bordeaux, France) Coll. of the City of New York, New York, N. Y.
SAUTÉ, Asst. Prof. GEORGE, A.M. (Brown) Cleveland Coll., Cleveland, Ohio. *3528 Ingleside Rd., Shaker Hts.*
SAVARY, C. M., C.E. Chief of Geodesy and Cartography, Dept. of Lands and Forests, Quebec, P. Q., Can. *586 St. Foye Rd.*
SCAMMON, Prof. R. E., Ph.D. (Harvard) Anatomy, Univ. of Minnesota, Minneapolis, Minn. *Inst. of Anatomy, Univ. of Minn.*
SCARBOROUGH, Asst. Prof. J. B., Ph.D. (Johns Hopkins) U. S. Naval Acad., Annapolis, Md. *P. O. Box 332.*
SCARBOROUGH, Prof. J. H., Ph.D. (Vanderbilt) State Teachers Coll., Warrensburg, Mo.
SCHAEFFER, NEDA F. (Mrs. W. E.), A.M. (Boston) 331 E. Main St., Kent, Ohio.
SCHEIER, Prof. MAURICE, A.M. (St. Bonaventure's) Holy Name Coll., Catholic Univ., Washington, D. C. *18th and Sheppard Sts. N.E.*
SCHELKUNOFF, S. A., Ph.D. (Columbia) Bell Telephone Labs. New York, N. Y. *463 West St.*
SCHEY, OLE, B.S. (Minnesota) Asst. in Educ. Psych., Univ. of Minnesota, Minneapolis, Minn. *502-15th Ave. S.E.*
SCHLAUCH, HELEN M., A.M. (Cornell) Instr., Hunter Coll., New York, N. Y. *219 Division Ave., Hasbrouck Hts., N. J.*
SCHLAUCH, Asst. Prof. W. S., A.M. (Columbia) Acctg., School of Commerce, New York Univ., New York, N. Y. *219 Division Ave., Hasbrouck Hts., N. J.*
SCHMEISER, MABEL F., Ph.D. (Ohio State) Head of Dept., State Teachers Coll., Wayne, Nebr. *315 E. Tenth St.*
SCHNELL, L. J., B.S. in E.E. (Colorado) Instr. and Acting Head of Dept., Augsburg Coll., Minneapolis, Minn. *8th St. and 21st Ave. S.*
SCHOONMAKER, Prof. HAZEL E., Ph.D. (Cornell) Hartwick Coll., Oneonta, N. Y.
SCHOONOVER, Asst. Prof. R. H., A.M. (Michigan) Coll. of the City of Detroit, Detroit, Mich. *4841 Cass Ave.*
SCHORLING, Asso. Prof. RALEIGH, Ph.D. (Michigan) School of Educ., Univ. of Michigan, Ann Arbor, Mich.
SCHOUTEN, Prof. Dr. J. A. Tech. Univ., Delft, Holland. *Rotterdamsche Weg 111.*
SCHREIBER, Asso. Prof. E. W., A.M. (Chicago) State Teachers Coll., Macomb, Ill. *456 S. Edwards St.*
SCHROEDER, Asso. Prof. C. E., A.M. (Boston Coll.) Boston Coll., Boston, Mass.
SCHUB, PINCUS, Ph.D. (Dropsie) Teacher, Gratz Coll., New York, N. Y. *207 E. Broadway.*
SCHULTZ, N. L. Student Asst., Univ. of Maine, Orono, Me. *40 Middle St.*
SCHUYLER, ELMER, M.S. (Lafayette) Instr., Bay Ridge High School, Brooklyn, N. Y. *87 71st St.*
SCHWEITZER, A. R., Ph.D. (Chicago) 452 Oakdale Ave., Chicago, Ill.
SCOTT, Prof. T. P., A.M. (Mississippi) State Teachers Coll., Memphis, Tenn.
SEARCY, Asst. Prof. C. L., C.E. (Purdue), A.M. (California) Univ. of Nevada, Reno, Nev.
SEELY, CAROLINE E., Ph.D. (Columbia) 501 W. 116th St., New York, N. Y.
SEIDEL, WLADIMIR, Ph.D. (Munich) National Research Fellow, Harvard Univ., Cambridge, Mass. *30 Mount Auburn St.*

SEIDLIN, Prof. JOSEPH, Ph.D. (Columbia) Alfred Univ., Alfred, N. Y.
SEIVERLING, I. F., A.M. (Columbia) Head of Dept., State Teachers Coll., Millersville, Pa.
SELHEIMER, C. W., Jr., M.S. (Michigan) Chem. Engr., Chemical Div., Proctor and Gamble Co., Ivorydale, Ohio.
SELL, WILLIAM, A.M. (Alabama) Instr., Univ. of Alabama, University, Ala. *Box 931.*
SELLEW, Prof. G. T., Ph.D. (Yale) Knox Coll., Galesburg, Ill.
SENSENIG, WAYNE, Ph.D. (Pennsylvania) 309 Bangor Rd., Bala-Cynwyd, Mont. Co., Pa.
SEWELL, J. S., Grad. (U. S. Milit. Acad.) Pres., Alabama Marble Co., Birmingham, Ala. *2801 Mountain Ave.*
SHANNON, Prof. J. I., A.M. (St. Louis) Dean, School of Philos. and Sc., St. Louis Univ., St. Louis, Mo. *215 N. Grand Ave.*
SHAUB, Asst. Prof. H. C., A.M. (Dartmouth) Washington and Jefferson Coll., Washington, Pa. *440 E. Bean St.*
SHAW, Asst. Prof. A. A., Ph.D. (California) Univ. of Arizona, Tucson, Ariz.
SHAW, R. S., A.M. (Columbia) Physics, Coll. of the City of New York, New York, N. Y.
SHEFFER, Asst. Prof. I. M., Ph.D. (Harvard) Pennsylvania State Coll., State College, Pa. *12 Liberal Arts Bldg.*
SHELDON, Prof. E. W., Ph.D. (Yale) Univ. of Alberta, Edmonton South, Alta., Can.
SHELLENBARGER, R. C., A.M. (Michigan) Head of Dept., Jr. Coll., Bay City, Mich. *248 N. Grant St.*
SHENTON, Prof. W. F., Ph.D. (Johns Hopkins) American Univ., Washington, D. C. *3605 Porter St. N.W.*
SHERER, C. R., A.M. (Nebraska) Head of Dept., Texas Christian Univ., Fort Worth, Tex.
SHERMAN, C. C., M.S. (Iowa) 505 K Ave. East, Oskaloosa, Iowa.
SHERWOOD, Prof. G. E. F., Ph.D. (Chicago) Univ. of California at Los Angeles, Los Angeles, Calif. *926 Hyperion Ave.*
SHEWHART, W. A., Ph.D. (California) Engineer, Bell Telephone Labs., New York, N. Y. *158 Lake Drive, Mountain Lakes, N. J.*
SHIBLI, Asst. Prof. JABIR, A.M. (Wisconsin) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
SHIRK, Prof. J. A. G., M.S. (Kansas) State Teachers Coll., Pittsburg, Kans. *116 E. Lindberg Ave.*
SHIRLEY, PAULINE, A.B. (Baylor) Teacher, Jr. Coll., Wichita Falls, Tex. *2141 Ave. G.*
SHIVELY, Prof. L. S., Ph.D. (Chicago) Ball Teachers Coll., Muncie, Ind. *407 Riverside Ave.*
SHOHAT, Asst. Prof. J. A., Magister of Pure Math. (Petrograd) Univ. of Pennsylvania, Philadelphia, Pa. *R 308, Bennett Hall.*
SHOOK, Asst. Prof. C. A., Ph.D. (Johns Hopkins) Lehigh Univ., Bethlehem, Pa. *1624 Millard St.*
SHORR, ROSE, A.B. (Hunter) Teacher, Manual Training High School, Brooklyn, N. Y. *29 Schenck Ave.*
SHORT, Pres. C. A., M.S. (Delaware) Wesley Collegiate Inst., Dover, Del.
SHORT, Asst. Prof. JESSIE M., A.M. (Carleton) Reed Coll., Portland, Ore.
SHORT, Prof. W. T., A.B. (Okla. Baptist Coll.,) Head of Dept., Oklahoma Baptist Coll., Shawnee, Okla.
SHOVER, C. GRACE, A.M. (Ohio State) Instr., Connecticut Coll., New London, Conn.
SHOWMAN, H. M., A.M. (Harvard) Lecturer and Univ. Recorder, Univ. of California at Los Angeles, Los Angeles, Calif. *912 Hyperion Ave.*
SHRINER, W. O., Ph.D. (Michigan) Head of Dept., State Teachers Coll., Terre Haute, Ind.
SHUMAN, J. W., B.S. (Mass. Inst. of Tech.) Secy-Treas., Power Eng. Co.; Consulting Engr.; Minneapolis, Minn. *716 Metro. Life Bldg.*
SHUMWAY, Asso. Prof. R. R., A.B. (Minnesota) Univ. of Minnesota, Minneapolis, Minn.
SHUSTER, Asst. Prof. C. N., A.M. (Columbia) Head of Dept., State Teachers Coll., Trenton, N. J. *2393 Pennington Rd.*
SICELOFF, Asso. Prof. L. P., Ph.D. (Columbia) Columbia Univ., New York, N. Y.
SILVERMAN, Prof. L. L., Ph.D. (Missouri) Dartmouth Coll., Hanover, N. H.
SIMESTER, Asst. Prof. J. H., M.A. (Toronto) Univ. of Louisville, Louisville, Ky.
SIMMONS, Asso. Prof. H. A., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *2125½ Ridge Ave.*
SIMON, Prof. W. G., Ph.D. (Chicago) Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.
SIMONS, Prof. LAO G., Ph.D. (Columbia) Hunter College, New York, N. Y. *180 W. 88th St.*
SIMPSON, Prof. T. M., Ph.D. (Wisconsin) Univ. of Florida, Gainesville, Fla. *547 S. 9th St.*
SIMPSON, Prof. T. McN., Jr., Ph.D. (Chicago) Randolph-Macon Coll., Ashland, Va.
SINCLAIR, Prof. MARY EMILY, Ph.D. (Chicago) Oberlin Coll., Oberlin, Ohio. *260 Oak St.*
SINGER, Prof. S. A., A.M. (Capital) Capital Univ., Columbus, Ohio. *2322 E. Main St.*

SINKOV, ABRAHAM, A.M. (Columbia) Jr. Mathematician, U. S. Civil Serv., Washington, D. C. *1412 Chapin St. N.W.*
SIROKY, Asso. Prof. EDMOND, M.S. (Washington Univ.) Washington Univ., St. Louis, Mo.
SISAM, Prof. C. H., Ph.D. (Cornell) Colorado Coll., Colorado Springs, Colo. *816 N. Weber St.*
SKARSTEDT, Prof. MARCUS, Ph.D. (California) Whittier Coll., Whittier, Calif.
SKELDING, A. Z., A.B. (C.C.N.Y) Asst. Actuary, Natl. Council on Compensation Ins., New York, N. Y. *116 Stanton Ave., Baldwin, N. Y.*
SKILES, Dean W. V., A.M. (Harvard) Georgia School of Tech., Atlanta, Ga.
SKINNER, Prof. E. B., Ph.D. (Chicago) Univ. of Wisconsin, Madison, Wis. *210 Lathrop St.*
SLAUGHT, Prof. H. E., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill.
SLEIGHT, Prof. E. R., A.M. (Albion) Albion Coll., Albion, Mich.
SLOBIN, Prof. H. L., Ph.D. (Clark) Univ. of New Hampshire, Durham, N. H.
SMAIL, Prof. L. L., Ph.D. (Columbia) Lehigh Univ., Bethlehem, Pa.
SMILEY, Asst. Prof. C. H., Ph.D. (California) Brown Univ., Providence, R. I.
SMINK, R. D., M.S. (Bucknell) Teacher, Sr. High School, Williamsport, Pa. *667 Sixth Ave.*
SMITH, A. W., Ph.D. (Chicago) Colgate Univ., Hamilton, N. Y.
SMITH, Prof. C. D., Ph.D. (Iowa) Mississippi A. and M. Coll., A. and M. College, Miss.
SMITH, Prof. CLARA E., Ph.D. (Yale) Wellesley Coll., Wellesley, Mass. *14 Waban St.*
SMITH, Mrs. CONSTANCE FITCH, A.M. (Arizona) Pima Co. School Supt., Tucson, Ariz. *Box 457, R.F.D. 2.*
SMITH, C. W., A.B. (Minnesota) State Teachers Coll., Superior, Wis.
SMITH, Prof. DAVID EUGENE, Ph.D., LL.D. (Syracuse) Emeritus, Columbia Univ., New York, N. Y. *501 W. 120th St.*
SMITH, Prof. D. M., Ph.D. (Chicago) Georgia School of Tech., Atlanta, Ga.
SMITH, Prof. EDWIN R., Ph.D. (Munich) Head of Dept., Iowa State Coll., Ames, Iowa.
SMITH, Prof. ELMER R., A.M. (Vanderbilt) Secy. to the Faculty, State Coll. for Women, Tallahassee, Fla. *79 College Ave.*
SMITH, Prof. E. S., Ph.D. (Virginia) Univ. of Cincinnati, Cincinnati, Ohio.
SMITH, F. E., Ph.D. (Catholic Univ.) Instr., Brooklyn Coll., Brooklyn, N. Y. *Box 212. Wortendyke, N. J.*
SMITH, Asso. Prof. G. W., Ph.D. (Illinois) Univ. of Kansas, Lawrence, Kans. *1730 Illinois.*
SMITH, Asso. Prof. H. L., Ph.D. (Chicago) Louisiana State Univ., Baton Rouge, La.
SMITH, Asst. Prof. H. W., M.S. (Chicago) Oklahoma A. and M. Coll., Stillwater, Okla.
SMITH, Prof. I. W., A.M. (Illinois) North Dakota Agric. Coll., Fargo, N. D. *1126 13th St. N.*
SMITH, Rev. J. P., A.M. (Woodstock) St. Peters Coll., Jersey City, N. J. *144 Grand St.*
SMITH, Prof. L. W., Ph.D. (Washington and Lee) Washington and Lee Univ., Lexington, Va.
SMITH, Pres. M. G., Ph.D. (Illinois) Spring Arbor Sem. and Jr. Coll., Spring Arbor, Mich.
SMITH, Prof. MARTHA L., M.S. (Chicago) Virginia Union Univ., Richmond, Va.
SMITH, Prof. P. F., Ph.D. (Yale) Yale Univ., Sheffield Sc. Sch., New Haven, Conn. *330 Willow St.*
SMITH, Asst. Prof. R. F., M.S. (New York Univ.) Coll. of the City of New York, New York, N. Y. *463 W. 144th St.*
SMITH, R. R., A.B. (Yale) 12 E. 41st St., New York, N. Y.
SMITH, S. J., A.M. (Pittsburgh) Instr., State Teachers Coll., Lock Haven, Pa.
SMITH, Asst. Prof. W. F., A.M. (Kentucky) New River State Coll., Montgomery, W. Va. *615 First Ave.*
SMITH, Prof. W. M., Ph.D. (Columbia) Lafayette Coll., Easton, Pa.
SMYTH, Sister M. PAULINE, A.M. (Canisius) Aquinas Inst., Rochester, N. Y. *1127 Dewey Ave.*
SMYTH, RUTH B. (Mrs. B. J.), A.M. (Oberlin) Instr., part-time, Oberlin Coll., Oberlin, Ohio. *93 E. Lorain St.*
SNEDECOR, Asso. Prof. G. W., A.M. (Michigan) Physics, Iowa State Coll., Ames, Iowa. *807 Hodge Ave.*
SNYDER, Asst. Prof. A. D., A.M. (Wisconsin) Union Coll., Schenectady, N. Y. *1592 Union St.*
SNYDER, Prof. VIRGIL, Ph.D. (Göttingen) Cornell Univ., Ithaca, N. Y. *214 University Ave.*
SOKOLNIKOFF, Asst. Prof. I. S., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
SOKOLNIKOFF, ELIZABETH STAFFORD (Mrs. I. S.), Ph.D. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis.
SOLUM, Asst. Prof. A. K., A.B. (St. Olaf) St. Olaf Coll., Northfield, Minn. *307 Manitou St.*
SOUSLEY, Prof. C. P., Ph.D. (Johns Hopkins) Rose Poly. Inst., Terre Haute, Ind. *92 Potomac St.*
SPANN, Asst. Prof. J. T., B.S. (Mississippi) Univ. of Maryland, College Park, Md.

SPARKS, Prof. F. W., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex. *Box 94, Tech. Sta.*
SPARROW, Prof. C. M., Ph.D. (Johns Hopkins) Physics, Univ. of Virginia, University, Va.
SPEAR, Prof. JOSEPH, A.B. (Harvard) Chm. of Dept., School of Eng., Northeastern Univ., Boston, Mass. *316 Huntington Ave.*
SPEEKER, Asso. Prof. G. G., A.M. (Indiana) Michigan State Coll., East Lansing, Mich.
SPENCELEY, Asso. Prof. G. W., A.M. (Harvard) Miami Univ., Oxford, Ohio. *100 E. Walnut St.*
SPENCER, Prof. MARY C., M.S. (Cornell). Sophie Newcomb Coll., New Orleans, La.
SPENCER, VIVIAN E., A.M. (Oberlin) Grad. Asst., Univ. of Pittsburgh, Pittsburgh, Pa. *Webster Hall, Oakland Sta.*
SPENCER, W. A., B.S. (Nebraska) Instr., Armour Inst. of Tech., Chicago, Ill.
SPERRY, MAY J., A.M. (Brown) Instr., Syracuse Univ., Syracuse, N. Y. *Dept. of Math.*
SPERRY, Asso. Prof. PAULINE, Ph.D. (Chicago) Univ. of California, Berkeley, Calif. *1194 Cragmont Ave.*
SPIES, Prof. ANTOINETTE, Ph.D. (Fordham) Head of Dept., Coll. of the Sacred Heart, Manhattanville, New York, N. Y.
SPINKS, M. J. Chief Engr., Champion Bridge Co., Wilmington, Ohio. *Box 594.*
SPOONER, Prof. C. C., A.M. (Amherst) State Teachers Coll., Marquette, Mich. *210 E. Prospect St.*
SROMOVSKY, Rev. R. A., A.M. (Wisconsin) Grad. Student, Univ. of Wisconsin, Madison, Wis. *120 W. Johnson St.*
STAFFORD, ANNA A., A.B. (Western Coll.) Grad. Student, Univ. of Chicago, Chicago, Ill. *184 N. Wildwood Rd., Lake Forest, Ill.*
STAGER, H. W., Ph.D. (California) 833 Lincoln Ave., Palo Alto, Calif.
STAGNER, MARGUERITE E., B.S. (Iowa State Coll.) Teacher, High School, Glenham, S. D.
STANWICK, C. A., B.S. in E.E. (Washington) Elec. Engr., 33 Beech St., East Orange, N. J.
STARK, Asst. Prof. MARION E., Ph.D. (Chicago) Wellesley Coll., Wellesley, Mass. *Wilder Hall.*
STARRETT, A. L., A.M. (Harvard) Instr., Hobart Coll., Geneva, N. Y.
STAUFFER, J. R. K., M.S. (Chicago) Grad. Student, Univ. of Chicago, Chicago, Ill. *6020 Harper Ave.*
STEED, Asso. Prof. D. V., Ph.D. (California) Univ. of Southern California, Los Angeles, Calif. *1733 W. 80th St.*
STEIMLEY, L. L., Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill.
STEININGER, EDITH, A.M. (Kansas) Instr., Jr. Coll., Coffeyville, Kans. *912 Willow St.*
STEIRNAGLE, W. M., A.B. (Indiana) United Film Industries, New York, N. Y. *535 Fifth Ave.*
STELSON, Asst. Prof. H. E., Ph.D. (Iowa) Kent State Coll., Kent, Ohio.
STEPHENS, Asst. Prof. EUGENE, M.S. (Washington Univ.) Washington Univ., St. Louis, Mo. *5863 Bartmer Ave.*
STEPHENS, R. C., Ph.D. (Iowa) Instr., Knox Coll., Galesburg, Ill.
STEPHENS, Prof. R. P., Ph.D. (Johns Hopkins) Univ. of Georgia, Athens, Ga.
STETSON, Prof. J. M., Ph.D. (Princeton) Coll. of William and Mary, Williamsburg, Va.
STEVENS, W. R., A.B. (George Washington) Meteorologist, U. S. Weather Bureau, Washington, D. C.
STEVENSON, Asso. Prof. GUY, Ph.D. (Illinois) Univ. of Louisville, Louisville, Ky.
STEWART, I. D., A.B. (Muskingum) Instr., Whitman Coll., Walla Walla, Wash. *424 Boyer Ave.*
STEWART, J. K., M.S. (West Virginia) Grad. Asst., West Virginia Univ., Morgantown, W. Va.
STOKES, ELLEN C., A.M. (Brown) Instr., State Coll. for Teachers, Albany, N. Y. *31 Hampton Ave., Schenectady, N. Y.*
STOKES, E. LOUISE, Ed.M. (Harvard) Teacher, Virginia State Coll., Ettrick, Va.
STONE, JOSEPHINE, A.M. (Peabody) Teacher, Athens Coll., Athens, Ala.
STONE, Prof. ORMOND, A.M. (Old Univ. of Chicago) Retired, Astr., Univ. of Virginia, Charlottesville, Va. *Clifton Station, Fairfax Co., Va.*
STONE, Asso. Prof. R. B., A.M. (Harvard) Purdue Univ., W. LaFayette, Ind. *615 Russell St.*
STONER, P. W., M.S. (California) Teacher, High School, Pasadena, Calif. *2131 Spaulding Pl.*
STOUFFER, Prof. E. B., Ph.D. (Illinois) Univ. of Kansas, Lawrence, Kans.
STOUT, C. E., A.M. (Wisconsin) Head of Dept., Genl. Motors Inst. of Tech., Flint, Mich. *1917 Monteith St.*
STOWELL, Prof. C. J., Ph.D. (Illinois) McKendree Coll., Lebanon, Ill.
STRATTON, Prof. W. T., Ph.D. (Washington) State Agric. Coll., Manhattan, Kans.

STREATOR, G. W., A.M. (Western Reserve) Instr., Bennett Coll. for Women, Greensboro, N. C. *949 E. Washington St.*
STREET, R. E. Student, Rensselaer Poly. Inst., Troy, N. Y. *16 Elmgrove Ave.*
STREETMAN, FLORA M., A.M. (Rice) 2616 Louisiana St., Houston, Tex.
STROM, Prof. C. W., Ph.D. (Illinois) Luther Coll., Decorah, Iowa.
STRONG, Prof. CORA, A.M. (Michigan) North Carolina Coll., Greensboro, N. C.
SUESMAN, W. P., LL.B. (Lake Forest) 241 Weybosset St., Providence, R. I.
SUFFA, Prof. MARY C., A.M. (Brown) Elmira Coll., Elmira, N. Y.
SULLIVAN, RUSSELL, A.B. (Yale) 1431 N. Meridian St., Indianapolis, Ind.
SWANSON, A. G., A.M. (Nebraska) 314 E. Dartmouth St., Flint, Mich.
SWARTZEL, Prof. K. D., M.S. (Ohio State) Univ. of Pittsburgh, Pittsburgh, Pa. *4360 Center Ave.*
SWEAZEY, Prof. G. B., A.M. (Wabash) Physics and Applied Math., Westminster Coll., Fulton, Mo.
SWEET, E. F. 117 E. Main St., Phelps, N. Y.
SWEET, H. L. Instr., Phillips Exeter Acad., Exeter, N. H.
SWIFT, Prof. ELIJAH, Ph.D. (Göttingen) Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*
SYNGE, Prof. J. L., Sc.D. (Dublin) Univ. of Toronto, Toronto, Ont., Can.

TABER, G. H. 4114 Bigelow Blvd., Pittsburgh (13), Pa.
TALIAFERRO, Prof. CARRIE B., A.M. (Columbia) State Teachers Coll., Farmville, Va.
TALIAFERRO, Prof. T. H., Ph.D. (Johns Hopkins) Dean, Coll. of Arts and Sciences, Univ. of Maryland, College Park, Md.
TAMARKIN, Prof. J. D., Ph.D., M.A.M. (Petrograd) Brown Univ., Providence, R. I.
TAN, Prof. V. A., Ph.D. (Chicago) Univ. of the Philippines, Manila, P. I.
TANGJERD, H. I., A.B. (St. Olaf Coll.) Instr., St. Olaf Coll., Northfield, Minn.
TANZOLA, J. J., A.M. (Columbia) Instr., Cooper Union, New York, N. Y. *2041 Watson Ave. Bronx.*
TAPPAN, Prof. A. HELEN, Ph.D. (Cornell) Dean of Women, Western Coll., Oxford, Ohio.
TARTLER, ALEXANDER, A.B. (Pennsylvania) Instr., Drexel Inst., Philadelphia, Pa. *32nd and Chestnut Sts.*
TATE, JENNIE L., A.M. (Wisconsin) Head of Dept., McMurry Coll., Abilene, Tex.
TAYLOR, Prof. EUGENE, A.M. (DePauw) Univ. of Idaho, Moscow, Ida.
TAYLOR, Prof. E. H., Ph.D. (Harvard) State Teachers Coll., Charleston, Ill.
TAYLOR, Prof. F. J., A.B. (St. Thomas) St. Thomas Coll., St. Paul, Minn.
TAYLOR, Prof. J. H., Ph.D. (Chicago) George Washington Univ., Washington, D. C.
TAYLOR, Prof. J. S., Ph.D. (California) Univ. of Pittsburgh, Pittsburgh, Pa. *103 Alumni Hall.*
TAYLOR, Prof. MILDRED E., Ph.D. (Illinois) Head of Dept., Mary Baldwin Coll., Staunton, Va.
TAYLOR, Prof. W. E., Ph.D. (Syracuse) Syracuse Univ., Syracuse, N. Y. *822 Irving Ave.*
TAYLOR, Prof. W. H., Ph.D. (Iowa) Alabama Coll., Montevallo, Ala.
THECLA, Sister MARY, Ph.D. (Fordham) Teacher, Queen of All Saints High School, Brooklyn, N. Y. *315 Clinton Ave.*
THEOBALD, Prof. JOHN, A.B. (Columbia Coll.), S.T.B. (Catholic Univ.) Columbia Coll., Dubuque, Iowa.
THIESMEYER, MILDRED G., A.M. (Columbia) Teacher, Hunter Coll. High School, New York, N. Y. *81 Anderson Ave., Scarsdale, N. Y.*
THOM, ADELA M., A.M. (Chicago) Elgin, Ill.
THOMAS, Prof. C. F., A.B. (Amherst) Case School of Appl. Sc., Cleveland, Ohio.
THOMAS, Prof. EVAN. Coll. of Eng., Univ. of Vermont, Burlington, Vt. *187 Loomis St.*
THOMAS, Asst. Prof. R. W., M.S. (Washington and Jefferson) Washington and Jefferson Coll., Washington, Pa. *333 Wilson Ave.*
THOMAS, Asso. Prof. T. Y., Ph.D. (Princeton) Princeton Univ., Princeton, N. J. *Fine Hall*
THOME, W. J., C.E. (Mich. State Coll.) 14911 Fairfield Ave., Detroit, Mich.
THOMPSON, Asso. Prof. E. L., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex.
THOMPSON, EVELYN R., A.M. (Columbia) Teacher, Western High School, Washington, D. C.
THOMPSON, Asst. Prof. HELEN, Ph.D. (Columbia) Research Assoc. in Biometry, Clinic of Child Development, New Haven, Conn. *52 Hillhouse Ave.*
THOMPSON, J. E., A.M. (Columbia) Instr., Pratt Inst., Brooklyn, N. Y. *260 Washington Ave.*
THOMPSON, Prof. MIRIAM A., A.M. (North Carolina) Registrar, Limestone Coll., Gaffney, S. C.
THOMSEN, H. IVAH, Ph.D. 1928 Mt. Royal Terrace, Baltimore, Md.

THOMSON, J. F., M.S. (Union Coll.) Instr., Univ. of Michigan, Ann Arbor, Mich. *1014 Cornwell Ave.*
THORNTON, H. B., A.M. (Cincinnati) Head of Dept., Sumner Jr. Coll., Kansas City, Kans. *418 Adams St., Jefferson City, Mo.*
THORNTON, Prof. W. M., LL.D. (Hampden-Sidney) Applied Math., Univ. of Virginia, University, Va.
THORP, ELLA A. M., A.B. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *656 Jefferson St. N. E.*
TIENZO, Asst. Prof. TELESFORO, M.S. (Chicago) Univ. of the Philippines, Manila, P. I. *Dept. of Math.*
TILLEY, Asst. Prof. ARTHUR, M.S. (New York Univ.) Washington Square Coll., New York Univ., New York, N. Y.
TINNEY, Prof. J. C., M.S. (Chicago) Bishop Coll., Marshall, Tex.
TITSWORTH, Mrs. H. L., A.B. (Tulane) Fellow, Newcomb Coll., New Orleans, La. *2320 Nashville Ave.*
TITSWORTH, Prof. W. A., M.S. (Wisconsin) Head of Dept., Alfred Univ., Alfred, N. Y.
TITT, Prof. H. G., A.M. (Michigan) Huron Coll., Huron, S. D. *959 Ohio Ave. S. W.*
TITUS, C. M., A.M. (Stanford) Instr., Univ. Farm School, Davis, Calif.
TOCHER, R. B. Student, Hobart Coll., Geneva, N. Y. *711 S. Main St.*
TOLAR, Prof. M. B., A.M. (Kentucky) Head of Dept., Fenn Coll., Cleveland, Ohio.
DETOLEDO, Prof. L. O. Univ. of Madrid, Madrid, Spain. *Velasques 28-3°, Facultad de Ciencias.*
TOOPS, Prof. H. A., Ph.D. (Columbia) Psych., Ohio State Univ., Columbus, Ohio.
TORRANCE, C. C., A.M. (Cornell) Instr., Cornell Univ., Ithaca, N. Y. *Dept. of Math.*
TORREY, Asst. Prof. MARIAN M., Ph.D. (Cornell) Goucher Coll., Baltimore, Md. *On leave 1931–32, 238 Williams St., Providence, R. I.*
TOUTON, Prof. F. C., Ph.D. (Columbia) Vice-Pres. and Dir. of Educational Program, Univ. of Southern California, Los Angeles, Calif.
TOWNSEND, Prof. E. J., Ph.D. (Göttingen) Emeritus, Univ. of Illinois, Urbana, Ill. *510 John St.*
TRACEY, Asso. Prof. J. I., Ph.D. (Johns Hopkins) Yale Univ., New Haven, Conn. *84 McKinley Ave.*
TRACY, SARAH E., B.L. (Swarthmore) Teacher, John Burroughs School, Clayton, Mo. *7918 Kingsbury Blvd., St. Louis, Mo.*
TREMBLAY, Prof. ALTHÉOD, Higher Math., Laval Univ., Quebec, Can. *472 St. Francois St.*
TREVOR, Prof. J. E., Ph.D. (Leipzig) Thermodynamics, Cornell Univ., Ithaca, N. Y.
TRIPP, Prof. M. O., Ph.D. (Columbia) Wittenberg Coll., Springfield, Ohio.
TUCKER, B. A., A.B. (Mississippi) Southeastern La. Coll., Hammond, La. *Box 726.*
TURNER, Prof. A. B., Ph.D. (Pennsylvania) Coll. of the City of New York, New York, N. Y. *245 N. Mountain Ave., Montclair, N. J.*
TURNER, Prof. BIRD M., Ph.D. (Bryn Mawr) West Virginia Univ., Morgantown, W. Va. *107 High St.*
TURNER, Asso. Prof. J. S., Ph.D. (Chicago) Iowa State Coll., Ames, Iowa. *2514 Knapp St.*
TYLER, A. C., A.B. (Princeton) Headmaster, Providence Country Day School, East Providence, R. I. *Box 106.*
TYLER, H. W., Ph.D. (Erlangen) Consultant, Liby. of Congress; Genl. Secy., Amer. Asso. of Univ. Professors, Washington, D. C. *26 Jackson Pl.*
TYLER, Asst. Prof. JOHN. U. S. Naval Acad., Annapolis, Md. *3 Southgate Ave.*

UHLER, Asso. Prof. H. S., Ph.D. (Johns Hopkins) Physics, Yale Univ., New Haven, Conn. *108 Thornton St., Hamden, Conn.*
ULRICH, F. E., A.M. (Harvard) Instr., Union Coll., Schenectady, N. Y. *37 Ray St.*
UNDERHILL, Asso. Prof. A. L., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn.
UNDERWOOD, P. H. Ball High School, Galveston, Tex. *2527 Ave. I.*
UNDERWOOD, Asso. Prof. R. S., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex.
UNSELD, G. P., A.M. (Colorado) Physics, W. Side High School, Salt Lake City, Utah. *1359 Glenmore St.*
UPTON, Prof. C. B., A.M. (Columbia) Teachers Coll., Columbia Univ., New York, N. Y.
URNER, S. E., Ph.D. (Harvard) Instr., Jr. Coll., Los Angeles, Calif. *2129 Holly Dr.*

VAN ARNAM, R. N., M.S. (Cornell) Asst., Math. and Astr., Lehigh Univ., Bethlehem, Pa. *418 N. New St.*
VAN BUSKIRK, Prof. H. C., Ph.B. (Cornell) California Inst. of Tech., Pasadena, Calif. *390 S. Holliston Ave.*
VAN DEUSEN, DOROTHY H., B.S. in Eng. (Michigan) Teacher, High School, Battle Creek, Mich. *261 Manchester St.*

VANDIVER, Asso. Prof. H. S. Univ. of Texas, Austin, Tex.
VAN FLEET, G. S., A.M. (Michigan) Asst. Actuary, American Natl. Life Ins. Co., Galveston, Tex.
VANHEE, LOUIS, D.D., Jesuit Father. 11 Rue des Récollets, Louvain, Belgium.
VAN HORNE, R. N., Ph.B. (Morningside) Morningside Coll., Sioux City, Iowa. *1501 Sioux Trail.*
VAN ORSTRAND, C. E., M.S. (Michigan) Geophysicist, U. S. Geol. Survey, Washington, D. C. *1667 31st St., N.W.*
VAN VELZER, Prof. C. A., Ph.D. (Hillsdale) Carthage Coll., Carthage, Ill. *903 Buchanan St.*
VAN VLECK, Prof. E. B., Ph.D. (Göttingen) Emeritus, Univ. of Wisconsin, Madison, Wis. *519 N. Pinckney St.*
VASS, J. I., A.M. (Northwestern) Instr., Univ. of Wisconsin, Extension Div., Milwaukee, Wis. *623 W. State St.*
VEATCH, Asst. Prof. R. W., A.M. (Northwestern) Univ. of Tulsa, Tulsa, Okla.
VEBLEN, Prof. OSWALD, Ph.D. (Chicago) Princeton Univ., Princeton, N. J.
VEDDER, Prof. J. N., A.M. Union Coll., Schenectady, N. Y.
VEHSE, Asst. Prof. C. H., M.S. (Brown) West Virginia Univ., Morgantown, W. Va.
VIRTS, R. O., A.M. (Indiana) Teacher, High School, Ft. Wayne, Ind. *4746 Stratford Rd.*
VIVIAN, ROXANA H., Ph.D. (Pennsylvania) 66 Milton Rd., Rye, N. Y.

WAGNER, Asso. Prof. C. C., Ph.D. (Michigan) Pennsylvania State Coll., State College, Pa. *122 S. Atherton St.*
WAGNER, E. H., A.M. (Illinois) 1709 Jackson Ave., Ann Arbor, Mich.
WAGNER, Prof. P. S., Ph.D. (Johns Hopkins) Lebanon Valley Coll., Annville, Pa.
WAHLIN, Prof. G. E., Ph.D. (Yale) Univ. of Missouri, Columbia, Mo. *1401 Anthony St.*
WALDEN, Prof. E. E., A.M. (Colorado) Lambuth Coll., Jackson, Tenn.
WALDER, O. E., A.M. (Nebraska) Instr., South Dakota Coll., Brookings, S. D.
WALKER, Asso. Prof. EVELYN, Ph.D. (Columbia) Hunter Coll., New York, N. Y. *695 Park Ave.*
WALKER, Asst. Prof. HELEN M., Ph.D. (Columbia) Educ. Statistics, Teachers Coll., Columbia Univ., New York, N. Y.
WALKER, L. C., A.M. (Stanford) Ceresco, Nebr.
WALKER, R. J., B.S. (Carnegie Inst. of Tech.) Grad. Student, Princeton Univ., Princeton, N. J.
WALLIS, W. J., A.M. (Columbian Univ.), LL.B. (George Washington) Head of Dept., High Schools, Washington, D. C. *Central High School.*
WALSH, C. M., A.B. (Harvard) Bellport, L. I., N. Y.
WALSH, Asso. Prof. J. L., Ph.D. (Harvard) Harvard Univ., Cambridge, Mass. *547 Widener Liby.*
WALTER, ARTHUR, A.M. (Stanford) City Supt. of Schools, Salinas, Calif.
WALTER, R. M., A.M. (Columbia) Instr., New Jersey Coll. for Women, New Brunswick, N. J.
WALTERS, E. L., B.S. (Washington and Jefferson) Grad. Asst., Syracuse Univ., Syracuse, N. Y. *208 Westminster Ave.*
WALTON, Asst. Prof. T. O., M.S. (Chicago) Kalamazoo Coll., Kalamazoo, Mich.
WALTZ, Asst. Prof. A. K., Ph.D. (Cornell) Clarkson Coll., Potsdam, N. Y. *52 Bay St.*
WAPPLE, Prof. A. R., A.M. (California) Southwestern Univ., Georgetown, Tex.
WARD, ETHEL RUTH, A.M. (North Carolina) Instr., Math. and Chem., State Coll. for Women, Columbus, Miss. *Box 1546, College Sta.*
WARD, Asst. Prof. L. E., Ph.D. (Harvard) Univ. of Iowa, Iowa City, Iowa. *120 Physics Bldg.*
WARD, Asst. Prof. MORGAN, Ph.D. (Calif. Inst. of Tech.) California Inst. of Tech., Pasadena, Calif.
WARNER, I. N., B.S. (Chicago) State Teachers Coll., Platteville, Wis.
WARREN, Asst. Prof. E. C., B.S. (Mass. Inst. of Tech.) Colby Coll., Waterville, Me. *28 Winter St.*
WARREN, Prof. L. A. H., Ph.D. (Chicago) Univ. of Manitoba, Winnipeg, Man., Can.
WASHBURNE, A. C., Grad. (U. S. Milit. Acad.) Actuary, Berkshire Life Ins. Co., Pittsfield, Mass.
WATKEYS, Prof. C. W., A.M. (Harvard) Univ. of Rochester, Rochester, N. Y. *287 Dartmouth St.*
WATKINS, EMILY H., A.M. (Teachers Coll., Columbia) Instr., North Carolina Coll. for Women, Greensboro, N. C.
WATT, Asst. Prof. MARTHA W., A.M. (Columbia) Wheaton Coll., Norton, Mass. *2144 Broad St., Providence, R. I.*

WATTS, C. B., A.B. (Indiana) Astronomer, U. S. Naval Observ., Washington, D. C.
WEAR, Prof. L. E., Ph.D. (Johns Hopkins) California Inst. of Tech., Pasadena, Calif. *68 S. Grand Oaks Ave.*
WEAVER, Prof. J. H., Ph.D. (Pennsylvania) Ohio State Univ., Columbus, Ohio. *Hilliard, Ohio.*
WEAVER, Prof. WARREN, Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *1852 Summit Ave.*
WEBBER, Prof. W. P., Ph.D. (Cincinnati) Louisiana State Univ., Baton Rouge, La.
WEBER, W. W., A.M. (Georgia) Dean and Prof. of Math., Lander Coll., Greenwood, S. C. *Box 111.*
WEBSTER, Asso. Prof. LOUISA M., M.S. (New York Univ.) Hunter Coll., New York, N. Y. *218–41–99 Ave., Queens Village, L. I., N. Y.*
WECHSLER, A. L., A.M. (Columbia) Instr., Univ. Extension, Columbia Univ., New York, N. Y. *Apt. No. 712, 27 W. 72nd St.*
WEDDERBURN, Prof. J. H. M., D.Sc. (Edinburgh) Princeton Univ., Princeton, N. J. *Box 53.*
WEIDA, Asso. Prof. F. M., Ph.D. (Iowa) George Washington Univ., Washington, D. C.
WEIMAR, F. G., M.S. (West Virginia) Grad. Asst., West Virginia Univ., Morgantown, W. Va. *42 Heiskell St., Edgwood, Wheeling, W. Va.*
WEINBERG, Prof. E. F., C.E. (Manhattan) Rollins Coll., Winter Park, Fla. *P. O. Box 952, Winter Park, Fla.*
WEINBERGER, M. S., A.B. (Iowa) Univ. of Michigan, Ann Arbor, Mich. *1024 Church St.*
WEINSTEIN, CLEMENT, A.M. (Pennsylvania) Instr., Univ. of Pennsylvania, Philadelphia, Pa.
WEISNER, Asst. Prof. LOUIS, Ph.D. (Columbia) Hunter Coll., New York, N. Y.
WEISS, Asst. Prof. MARIE J., Ph.D. (Stanford) Newcomb Coll., New Orleans, La.
WELKOWITZ, SAMUEL, B.S. (C.C.N.Y.) Chm. Dept. of Math., F. K. Lane High School, Brooklyn, N. Y. *439 Linden Blvd.*
WELLING, W. C., A.B. (Yale) Dir., Bureau of Vital Stat., State Dept. of Health, Hartford, Conn. *Drawer K. Sta. A.*
WELLS, Asst. Prof. E. D., A.M. (Minnesota) Univ. of Pittsburgh, Erie Center, Erie, Pa. *1008 Weschler Ave.*
WELLS, F. A., B.S. (Virginia) Instr., Univ. of Virginia, Charlottesville, Va. *414 Park St.*
WELLS, Prof. MARY EVELYN, Ph.D. (Chicago) Vassar Coll., Poughkeepsie, N. Y.
WELLS, Prof. R. A., A.M. (Michigan) Park Coll., Parkville, Mo.
WELLS, Asst. Prof. V. H., Ph.D. (Michigan) Williams Coll., Williamstown, Mass. *3 Chapin Ct.*
WELTON, P. L., A.M. (Michigan) 209 N. Ingalls St., Ann Arbor, Mich.
WERNICKE, PAUL, Ph.D. (Göttingen) Patent Examiner, U. S. Patent Office, Washington, D. C. *3600 S. Dakota Ave. N.E.*
WESCOTT, M. E., B.S. (Northwestern) Instr., Northwestern Univ., Evanston, Ill. *126 Callan Ave.*
WEST, Mrs. GRACE R., A.M. (Texas) Teacher, Central High School, Tulsa, Okla. *303 Mary Brockman Apts.*
WEST, Prof. H. A., A.B. (Tennessee) Marion Coll., Marion, Ind. *415 W. 38th St.*
WEST, Asst. Prof. J. M., A.M. (Hamilton) Pennsylvania State Coll., State College, Pa. *10 Liberal Arts.*
WESTER, Prof. C. W. State Teachers Coll., Cedar Falls, Iowa. *823 W. 6th St.*
WESTFALL, Prof. W. D. A., Ph.D. (Göttingen) Univ. of Missouri, Columbia, Mo. *11 S. Glenwood Ave.*
WEXLER, CHARLES, Ph.D. (Harvard) Head of Dept., State Teachers Coll., Tempe, Ariz. *1032 Van Ness Ave.*
WHALEY, MARY RUTH, A.B. (Smith) Dale, N. Y.
WHEELER, A. H., A.M. (Clark) Teacher, North High School, Worcester, Mass. *16 Bellevue St.*
WHEELER, Prof. ANNA PELL, Ph.D. (Chicago) Bryn Mawr Coll., Bryn Mawr, Pa. *109 Broadmead, Princeton, N. J.*
WHEELER, Asso. Prof. CHARLES H. III, B.S. (Washington and Jefferson) Univ. of Richmond, Richmond, Va. *On leave 1931–32, Johns Hopkins Univ., Baltimore, Md.*
WHEELER, Asst. Prof. J. J., A.M. (Kansas) Univ. of Kansas, Lawrence, Kans. *1024 Alabama St.*
WHITE, Prof. A. E., M.S. (Purdue) State Agric. Coll., Manhattan, Kans.
WHITE, Prof. C. E., A.M. (Indiana) Muskingum Coll., New Concord, Ohio.
WHITE, Dean E. W., A.M. (Columbia) Head of Dept., Carson and Newman Coll., Jefferson City, Tenn.
WHITE, Asso. Prof. MARION B., Ph.D. (Chicago) Carleton Coll., Northfield, Minn.

WHITED, WILLIS, M.E. (Ames) Advisory Bridge Engr., State Dept. of Highways, Harrisburg, Pa. *26 S. 3rd St.*
WHITFORD, A. E., A.M. (Wisconsin) Lecturer in Math., Univ. of Wisconsin, Madison, Wis. *101 N. Mills St.*
WHITFORD, Asst. Prof. D. E., A.M. (Brown) Poly. Inst. of Brooklyn, Brooklyn, N. Y. *85–99 Livingston St.*
WHITFORD, Asso. Prof. E. E., Ph.D. (Columbia) Coll. of the City of New York, New York, N. Y. *535 W. 110th St.*
WHITING, MABEL G., A.M. (Oberlin) Registrar and Teacher, Jr. Coll., Santa Ana, Calif. *506 E. Chestnut Ave.*
WHITMAN, Asso. Prof. E. A., A.M. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *521 Locust St., Edgewood.*
WHITNEY, ANNA M., A.M. (Columbia) Teacher, High School, Yakima, Wash. *203 S. 8th Ave.*
WHITNEY, C. S., A.M. (Oklahoma) Head of Dept., Jr. Coll., Miami, Okla.
WHITTED, Prof. J. A., A.M. (Southwestern) Ohio Northern Univ., Ada, Ohio. *219 W. Highland Ave.*
WHITTEMORE, Asso. Prof. J. K., A.M. (Harvard) Yale Univ., New Haven, Conn. *45 Lincoln St.*
WHYBURN, G. T., Ph.D. (Texas) Assoc., Johns Hopkins Univ., Baltimore, Md.
WHYBURN, Asso. Prof. W. M., Ph.D. (Texas) Univ. of California at Los Angeles, Los Angeles, Calif.
WIBLE, Prof. W. M., A.M. (Indiana) Intermountain Union Coll., Helena, Mont. *1735 Winne Ave., Lenox Add.*
WICK, R. F., B.S. in E.E. (A. and M. Coll. of Texas) 10 Sheridan Sq., Apt. 9E, New York, N. Y.
WIDDER, Asst. Prof. D. V., Ph.D. (Harvard) Harvard Univ., Cambridge, Mass. *Widener 572.*
WIELEITNER, HEINRICH, Ph.D. (Munich) Dir., Neues Realgymnasium, Munich, Germany. *Müllerstr. 5, SO 2, Munich.*
WIGGIN, Asso. Prof. EVELYN P., A.M. (Brown) Randolph-Macon Woman's Coll., Lynchburg, Va.
WILBUR, Asst. Prof. W. E., M.S. (Maine) Univ. of New Hampshire, Durham, N. H.
WILCZEWSKI, Rev. JOSEPH, A.M. (St. Louis Univ.) St. Louis Univ., St. Louis, Mo.
WILDER, Asst. Prof. C. E., Ph.D. (Harvard) Dartmouth Coll., Hanover, N. H.
WILDER, G. F. John Adams High School, New York, N. Y. *Rockaway Blvd. and 101 St., Queens.*
WILDER, MARIAN A., A.M. (Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn. *3320 Second Ave. S.*
WILDER, Asso. Prof. R. L., Ph.D. (Texas) Univ. of Michigan, Ann Arbor, Mich. *1209 Packard St.*
WILDERMUTH, Asso. Prof. R. B., A.M. (Ohio State) Capital Univ., Columbus, Ohio.
WILEY, Asso. Prof. ELLEN E., A.B. (St. Lawrence) Middlebury Coll., Middlebury, Vt. *120 Main St.*
WILEY, Prof. F. B., Ph.D. (Chicago) Denison Univ., Granville, Ohio.
WILKINS, Asst. Prof. P. D., M.S. (Case) Bates Coll., Lewiston, Me.
WILLETT, Prof. H. C., A.M. (So. Calif.) Univ. of Southern California, Los Angeles, Calif.
WILLEY, MAUD, A.M. (Mills Coll.) 1810 Alabama St., Lawrence, Kans.
WILLIAMS, Asst. Prof. A. R., Ph.D. (California) Univ. of California, Berkeley, Calif. *455 Wheeler Hall.*
WILLIAMS, Prof. F. B., Ph.D. (Clark) Clark Univ., Worcester, Mass.
WILLIAMS, Prof. F. G., Ph.D. (Cornell) Susquehanna Univ., Selinsgrove, Pa.
WILLIAMS, Asst. Prof. G. A., A.M. (California) Oregon State Coll., Corvallis, Ore. *On leave 1931–32, Univ. of California, Berkeley, Calif.*
WILLIAMS, Prof. J. E., Ph.D. (Virginia) Dean, Virginia Poly. Inst., Blacksburg, Va.
WILLIAMS, Prof. K. P., Ph.D. (Princeton) Indiana Univ., Bloomington, Ind. *523 E. 3rd St.*
WILLIAMS, Prof. S. W., A.B., B.E. (Vanderbilt) Arkansas Coll., Batesville, Ark.
WILLIAMS, Asso. Prof. W. L., A.M. (South Carolina) Univ. of South Carolina, Columbia, S. C.
WILLIAMSON, Prof. C. O., Ph.D. (Chicago) Applied Math., Coll. of Wooster, Wooster, Ohio.
WILLIAMSON, JOHN, Ph.D. (Chicago) Assoc., Johns Hopkins Univ., Baltimore, Md. *Dept. of Math.*
WILLIS, RUBY, A.B. (Wellesley) Head of Dept., Walnut Hill School, Natick, Mass.
WILLSON, Prof. F. N., C.E. (Rensselaer) Emeritus, Princeton Univ., Princeton, N. J.
WILMER, F. L. Odebolt, Iowa.
WILSON, Asso. Prof. A. H., Ph.D. (Chicago) Haverford Coll., Haverford, Pa.

WILSON, C. R., M.S. (Iowa). Address unknown.
WILSON, DOROTHY, A.M. (West Virginia) Head of Dept., Potomac State School, Keyser, W.Va. *305 S. Mineral St.*
WILSON, Prof. E. B., Ph.D. (Yale) Vital Statistics, Harvard School of Public Health, Boston, Mass. *55 Van Dyke St.*
WILSON, ELIZABETH W., A.M. (Radcliffe) Actuary, Cambridge, Mass. *Mather Ct., Apt. 42.*
WILSON, Prof. N. R., Ph.D. (Chicago) Univ. of Manitoba, Winnipeg, Man., Can. *989 Grosvenor Ave.*
WILSON, T. R. C., C.E. (Purdue) Sr. Engr., Forest Products Lab., Madison, Wis. *1911 Kendall Ave.*
WILSON, Prof. W. A., Ph.D. (Yale) Yale Univ., New Haven, Conn. *228 Park St.*
WILSON, Prof. W. H., Ph.D. (Illinois) Acting Dean, Coll. of Arts and Sc., Univ. of Florida, Gainesville, Fla. *1242 N. Franklin St.*
WILTON, Prof. J. R., M.A. (Cantab.) Univ. of Adelaide, Adelaide, South Australia.
WINBIGLER, Prof. ALICE, A.M. (Monmouth) Monmouth Coll., Monmouth, Ill.
WINGER, Prof. R. M., Ph.D. (Johns Hopkins) Univ. of Washington, Seattle, Wash.
WINKELMANN, Rev. G. L., M.S. (Chicago) Head of Dept., St. Johns Univ., Collegeville, Minn.
WINSLOW, Asst. Prof. J. B., A.M. (Michigan) Math. and Astr., Univ. of the City of Toledo, Toledo, Ohio. *Temperance, Mich.*
WINTERS, F. W., A.M. (Harvard) Bell Telephone Labs., New York, N. Y. *150–52 Bayside Ave., Flushing, L. I., N. Y.*
WISHARD, G. W., B.S. (Natl. Normal Univ.) 5134 Carthage Ave., Norwood, Ohio.
WITT, R. E., C.E. (Washington and Lee) Instr., Washington and Lee Univ., Lexington, Va. *Box 674.*
WOLEVER, FRANCES E., A.M. (Illinois) Grad. Asst., Univ. of Illinois, Urbana, Ill. *Tuscola, Ill.*
WOLFE, ALBERTA, M.S. (Iowa State Coll.) Instr., Western Coll., Oxford, Ohio.
WOLFE, Asst. Prof. CLYDE, Ph.D. (California) California Inst. of Tech., Pasadena, Calif.
WOLFE, Asso. Prof. H. E., Ph.D. (Indiana) Indiana Univ., Bloomington, Ind. *316 N. Washington St.*
WONG, Asst. Prof. B. C., Ph.D. (California) Univ. of California, Berkeley, Calif. *1933 Grant St.*
WOO, KAMCHEUNG, A.M. (California) International House, Berkeley, Calif.
WOOD, Prof. FREDRICK, Ph.D. (Wisconsin) Hamline Univ., St. Paul, Minn.
WOOD, Asso. Prof. F. E., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *909 Colfax St.*
WOOD, META A., A.M. (Wisconsin) 106 Morningside Drive, New York, N. Y.
WOOD, ROSE B., A.B. (Barnard) 121 Eighth St. N.E., Atlanta, Ga.
WOOD, Prof. RUTH G., Ph.D. (Yale) Smith Coll., Northampton, Mass.
WOOD, R. R., B.S. (Haverford) Secy., Friends' Peace Committee, Philadelphia, Pa. *272 W. Main St., Moorestown, N. J.*
WOODARD, Prof. D. W., Ph.D. (Pennsylvania) Howard Univ., Washington, D. C.
WOODMANSEE, Prof. W. R., A.M. (Wisconsin) Ripon Coll., Ripon, Wis.
WOODS, Prof. F. S., Ph.D. (Göttingen) Mass. Inst. of Tech., Cambridge, Mass.
WOODS, Asst. Prof. ROSCOE, Ph.D. (Illinois) Univ. of Iowa, Iowa City, Iowa. *221 Physics Bldg.*
WOODSON, G. F., Jr., A.M. (Ohio State) Head of Dept., Johnson C. Smith Univ., Charlotte, N. C.
WOOLARD, Asst. Prof. E. W., Ph.D. (George Washington) George Washington Univ., Washington, D. C.
WORTHINGTON, Asst. Prof. EUPHEMIA R., Ph.D. (Yale) Univ. of California at Los Angeles, Los Angeles, Calif.
WREN, Prof. F. L., Ph.D. (Chicago) Teaching of Math., George Peabody Coll., Nashville, Tenn.
WRESTLER, FERNA E., A.M. (Kansas) Head of Dept., Jr. Coll., El Dorado, Kans. *421 W. Olive St.*
WRIGHT, Asst. Prof. FRANCES M., A.M. (Brown) Elmira Coll., Elmira, N. Y.
WRIGHT, FRANCES W., A.M. (Brown) Asst., Harvard Coll. Observ., Cambridge, Mass. *250 Brattle St.*
WRIGHT, H. N., Ph.D. (California) 514 W. 122nd St., New York, N. Y.
WRIGHT, VERA L., A.M. (Minnesota) 8 Academy St., Bluefield, Va.
WRIGHT, Prof. W. L., A.M. (Princeton) Lincoln Univ., Lincoln University, Pa.
WUNDER, Prof. C. N., Ph.D. (Virginia) Atlantic Univ., Virginia Beach, Va.
WYANT, Prof. E. KATHRYN, Ph.D. (Missouri) Northeastern Teachers Coll., Tahlequah, Okla. *Dept. of Math.*

WYLIE, Asso. Prof. C. C., Ph.D. (Illinois) Astr., Univ. of Iowa, Iowa City, Iowa. *Univ. Observatory.*

YANNEY, Prof. B. F., Ph.D. (Chicago) Coll. of Wooster, Wooster, Ohio. *666 N. Bever St.*
YANOSIK, G. A., C.E. (New York Univ.) Instr., New York Univ., New York, N. Y. *52 Greenvale Ave., Yonkers, N. Y.*
YARBROUGH, H. M., Ph.D. (Indiana) Head of Dept., State Teachers Coll., Bowling Green, Ky. *1224 Laurel Ave.*
YATES, Asst. Prof. R. C., Ph.D. (Johns Hopkins) Univ. of Maryland, College Park, Md. *810 Mt. Vernon Ave., Alexandria, Va.*
YEATON, Prof. C. H., Ph.D. (Chicago) Oberlin Coll., Oberlin, Ohio. *189 Forest St.*
YOTHERS, J. L., A.M. (Otterbein) Head of Dept., Coe Coll., Cedar Rapids, Iowa.
YOUNG, A. L., M.S. (Iowa State Coll.) Farm Mechanics, Univ. of Illinois, Urbana, Ill. *709 Indiana Ave.*
YOUNG, F. G., M.A. (Alberta) Instr., Provincial Inst. of Tech. and Art, Calgary, Alta., Can. *1301 9th St. N.W.*
YOUNG, Asst. Prof. JESSICA M., Ph.D. (California) Math. and Astr., Washington Univ., St. Louis, Mo. *4511 N. 20th St.*
YOUNG, Prof. J. W., Ph.D. (Cornell) Dartmouth Coll., Hanover, N. H.
YOUNG, Asso. Prof. J. W. A., Ph.D. (Clark) Univ. of Chicago, Chicago, Ill. *5422 Blackstone Ave.*
YOUNG, Prof. MABEL M., Ph.D. (Johns Hopkins) Wellesley Coll., Wellesley, Mass. *6 Norfolk Terrace.*
YOUNG, MARGARET M., A.M. (Columbia) Instr., Brooklyn Coll., Brooklyn, N. Y. *1001 Anderson Ave., New York, N. Y.*
YOUTZ, PATRICK, M.S. (Chicago) Instr., Bucknell Univ., Lewisburg, Pa.
YOWELL, E. I., C.E., Ph.D. (Cincinnati) Astronomer, Univ. of Cincinnati, Cincinnati, Ohio. *3127 Griest Ave.*

ZARISKI, OSCAR, Ph.D. (Univ. of Rome) Associate, Johns Hopkins Univ., Baltimore, Md. *Dept. of Math.*
ZELDIN, Asst. Prof. S. D., Ph.D. (Clark) Mass. Inst. of Tech., Cambridge, Mass.
ZIMMERMAN, B. C., A.M. (St. Louis Univ.) Grad. student, St. Louis Univ., St. Louis, Mo.
ZIMMERMAN, Prof. JOHN, B.S. (Princeton) Head of Dept. and Registrar, Univ. of Dubuque, Dubuque, Iowa. *1089 Glen Oak Ave.*
ZINSZER, Prof. H. A., Ph.D. (Indiana) Physics and Astr., State Teachers Coll., Hays, Kans.
ZOCH, R. T., A.B. (George Washington) Principal Scientific Aid, U. S. Weather Bureau, Washington, D. C.

INSTITUTIONAL MEMBERS

Arkansas

University of Arkansas, Fayetteville, Ark.

California

Pomona College, Claremont, Calif.
Occidental College, Los Angeles, Calif.
University of California at Los Angeles, Los Angeles, Calif.
University of Southern California, Los Angeles, Calif.
Menlo School and Junior College, Menlo Park, Calif.
California Institute of Technology, Pasadena, Calif.
State Teachers College, San Diego, Calif.
St. Ignatius College, San Francisco, Calif.

Canada

University of Montreal, Montreal, Canada.

Colorado

University of Colorado, Boulder, Colo.

Connecticut

Wesleyan University, Middletown, Conn.

Delaware

The University of Delaware, Newark, Del.

District of Columbia

George Washington University, Washington, D.C.

Florida

John B. Stetson University, De Land, Fla.

Georgia

University of Georgia, Athens, Ga.
Georgia School of Technology, Atlanta, Ga.
Wesleyan College, Macon, Ga.

Illinois

Armour Institute of Technology, Chicago, Ill.
De Paul University, Chicago, Ill.
St. Xavier College, Chicago, Ill.
University of Chicago, Chicago, Ill.
Northwestern University, Evanston, Ill.
Knox College, Galesburg, Ill.
University of Illinois, Urbana, Ill.

Indiana

Indiana University, Bloomington, Ind.
Butler University, Indianapolis, Ind.
Purdue University, LaFayette, Ind.
University of Notre Dame, Notre Dame, Ind.
St. Mary-of-the-Woods College, St. Mary-of-the-Woods, Ind.

Iowa

Iowa State College, Ames, Ia.
St. Ambrose College, Davenport, Ia.
Drake University, Des Moines, Ia.
Columbia College, Dubuque, Ia.
State University of Iowa, Iowa City, Ia.

Kansas

University of Kansas, Lawrence, Kans.
State Agricultural College, Manhattan, Kans.
Bethel College, Newton, Kans.
University of Wichita, Wichita, Kans.

INSTITUTIONAL MEMBERS (*continued*)

Kentucky

University of Kentucky, Lexington, Ky.

Louisiana

Louisiana State University, Baton Rouge, La.
Louisiana State Normal College, Natchitoches, La.
Loyola University, New Orleans, La.
Tulane University of Louisiana, New Orleans, La.
Louisiana Polytechnic Institute, Ruston, La.

Maine

University of Maine, Orono, Me.

Maryland

St. John's College, Annapolis, Md.

Massachusetts

Amherst College, Amherst, Mass.
Boston University, Boston, Mass.
Northeastern University, Boston, Mass.
Harvard College, Cambridge, Mass.
Wellesley College, Wellesley, Mass.
Worcester Polytechnic Institute, Worcester, Mass.

Michigan

University of Michigan, Ann Arbor, Mich.
College of the City of Detroit, Detroit, Mich.
Michigan State College, East Lansing, Mich.
Kalamazoo College, Kalamazoo, Mich.
Michigan State Normal College, Ypsilanti, Mich.

Minnesota

Augsburg College, Minneapolis, Minn.
University of Minnesota, Minneapolis, Minn.
Carleton College, Northfield, Minn.
St. Benedict's College, St. Joseph, Minn.
College of St. Teresa, Winona, Minn.

Missouri

University of Missouri, Columbia, Mo.
Northeast Missouri State Teachers College, Kirksville, Mo.
Missouri School of Mines and Metallurgy, Rolla, Mo.
The Principia, St. Louis, Mo.
Washington University, St. Louis, Mo.

Montana

University of Montana, Missoula, Mont.

Nebraska

University of Nebraska, Lincoln, Nebr.
Creighton University, Omaha, Nebr.
State Teachers College, Peru, Nebr.

New Hampshire

University of New Hampshire, Durham, N. H.
Dartmouth College, Hanover, N. H.

New Jersey

New Jersey College for Women, New Brunswick, N. J.
Rutgers University, New Brunswick, N. J.
Princeton University, Princeton, N. J.

New Mexico

New Mexico Normal University, Las Vegas, N. M.

INSTITUTIONAL MEMBERS (*continued*)

New York

New York State College for Teachers, Albany, N. Y.
Wells College, Aurora, N. Y.
University of Buffalo, Buffalo, N. Y.
Hamilton College, Clinton, N. Y.
Elmira College, Elmira, N. Y.
Adelphi College, Garden City, N. Y.
College of Mount St. Vincent, Mount St. Vincent-on-Hudson, N. Y.
College of the City of New York, New York, N. Y.
College of the Sacred Heart, New York, N. Y.
Columbia University, New York, N. Y.
The Cooper Union, New York, N. Y.
Hunter College of the City of New York, New York, N. Y.
New York University, New York, N. Y.
University of Rochester, Rochester, N. Y.
St. Bonaventure's College, St. Bonaventure, N. Y.
Union College, Schenectady, N. Y.

Ohio

University of Akron, Akron, Ohio.
University of Cincinnati, Cincinnati, Ohio.
Case School of Applied Science, Cleveland, Ohio.
Western Reserve University, Cleveland, Ohio.
Ohio Wesleyan University, Delaware, Ohio.
Kenyon College, Gambier, Ohio.
Denison University, Granville, Ohio.
Hiram College, Hiram, Ohio.
Oberlin College, Oberlin, Ohio.
Otterbein College, Westerville, Ohio.
The College of Wooster, Wooster, Ohio.
Antioch College, Yellow Springs, Ohio.

Oklahoma

University of Oklahoma, Norman, Okla.
Oklahoma City University, Oklahoma City, Okla.

Oregon

University of Oregon, Eugene, Ore.

Pennsylvania

Bryn Mawr College, Bryn Mawr, Pa.
Lafayette College, Easton, Pa.
Seton Hill College, Greensburg, Pa.
Allegheny College, Meadville, Pa.
Westminster College, New Wilmington, Pa.
Drexel Institute, Philadelphia, Pa.
Pennsylvania College for Women, Pittsburgh, Pa.
Swarthmore College, Swarthmore, Pa.
Washington and Jefferson College, Washington, Pa.

Rhode Island

Brown University, Providence, R. I.
Providence College, Providence, R. I.

South Carolina

University of South Carolina, Columbia, S. C.

Tennessee

George Peabody College for Teachers, Nashville, Tenn.
Vanderbilt University, Nashville, Tenn.

Texas

West Texas State Teachers College, Canyon, Tex.
Southern Methodist University, Dallas, Tex.
Rice Institute, Houston, Tex.

INSTITUTIONAL MEMBERS (*continued*)

Utah

University of Utah, Salt Lake City, Utah.

Vermont

Middlebury College, Middlebury, Vt.

Virginia

Virginia Polytechnic Institute, Blacksburg, Va.
University of Virginia, University, Va.
College of William and Mary, Williamsburg, Va.

Washington

University of Washington, Seattle, Wash.

West Virginia

Bethany College, Bethany, W. Va.

Wyoming

University of Wyoming, Laramie, Wyo.

GEOGRAPHICAL DISTRIBUTION OF INDIVIDUAL MEMBERS

UNITED STATES AND CANADA

ALABAMA. (20)

ATHENS. Stone.
AUBURN. Crenshaw, Hampton, Harkin, Killebrew, Robinson.
BIRMINGHAM. Eagles, Hess, Moore, Sewell.
FLORENCE. Culmer.
MARION. Allen, Murfee.
MONTEVALLO. Jackson, Taylor.
ROANOKE. Mickle.
UNIVERSITY. Dahlene, Lewis, Ott, Sell.

ARIZONA. (9)

FLAGSTAFF. Lampland, Risselman.
PHOENIX. Hannelly.
TEMPE. Wexler.
TUCSON. Graesser, Keyes, Leonard, Shaw, C. F. Smith.

ARKANSAS. (7)

BATESVILLE. Williams.
FAYETTEVILLE. Adkisson, Harding, Hosford, Hughes, Nichols.
LITTLE ROCK. Bigbee.

CALIFORNIA. (89)

ATASCADERO. Anderson.
BAKERSFIELD. Sagen.
BERKELEY. Bernstein, Haskell, Irwin, Keeler, Lehmer, Levy, McCarty, McDonald, Noble, Powelson, Putnam, Sperry, Williams, Wong, Woo.
CHICO. Iloff.
CLAREMONT. Berry, Jaeger, Russell.
DAVIS. Titus.
FRESNO. Morris.
FULLERTON. Ernsberger, Reynolds.
GLENDALE. Bolton.
HOLLYWOOD. Entz.
LA JOLLA. McEwen.
LONG BEACH. Lodwick.
LOS ANGELES. Allen, Ames, Bell, Campbell, Collier, Daus, Garver, Gaver, Glazier, Hedrick, Hill, Hunt, James, McClellan, Mason, Orange, Sherwood, Showman, Steed, Touton, Urner, Whyburn, Willett, Worthington.
MILLS COLLEGE. Alderton.
MODESTO. Pobanz.
MT. HAMILTON. Jeffers.
PALO ALTO. Hoskins, Stager.
PASADENA. Bateman, Bell, Birchby, Meek, Michal, Millikan, Stoner, Van Buskirk, Ward, Wear, Wolfe.
REDLANDS. Albert, Keith.
SALINAS. Walter.
SAN DIEGO. Gleason, Klauber, Livingston.
SAN FRANCISCO. Ivanoff, Libby.
SANTA ANA. Whiting.
SANTA BARBARA. Fenner.
SANTA MARIA. Funk.
STANFORD UNIVERSITY. Blichfeldt, Eells, Green, Lehmer, Moreno.
STOCKTON. Corbin.
TAFT. Robb.
WHITTIER. Hadley, Skarstedt.

CANADA. (37)

CALGARY. Young.
EDMONTON SOUTH. Campbell, Cook, Sheldon.
HAMILTON. Findlay.
KINGSTON. Gummer, Johnston, Matheson, Miller.
LENNOXVILLE. Home, Richardson.
LONDON. Kingston.
MONTREAL. MacLean, Murray, Pelletier.
NIAGARA FALLS. Scharf.
OTTAWA. Dube, Henroteau.
QUEBEC. Savary, Tremblay.
SACKVILLE. McEwen.
SASKATOON. Dines, Ling.
TORONTO. Beatty, De Lury, Fields, Pounder, Roseburgh, Synge.
VANCOUVER. Buchanan, Jordan.
VICTORIA. Gage.
WINNIPEG. Milne, Riter, Warren, Wilson.
WOLFVILLE. Jeffery.

CANAL ZONE. (1)

BALBOA HEIGHTS. Judd.

COLORADO. (30)

BOULDER. Britton, De Long, Hacker, Hutchinson, Jekel, Kempner, Kendall, Lester, Light, McMaster, Nelson, Rainville.
CANON CITY. McNatt.
COLORADO SPRINGS. Folk, Lovitt, Sisam.
DENVER, Carmichael, Gorrell, Lewis, McGinley, Odell, Recht, Risley, Rôte, Sabin.
FORT COLLINS. Clark, Macdonald.
GOLDEN. Everett, Fitterer.
GREELEY. Finley.

CONNECTICUT. (37)

GREENWICH. Burney.
HARTFORD. Andrews, Dadourian, Davis, Elston, B. D. Flynn, J. D. Flynn, Mitchell, Welling.
MIDDLETOWN. Arnold, Camp, Howland.
MILFORD. Burgess, Rosenbaum.
NEW HAVEN. Barney, E. W. Brown, Church, Fisher, Fox, Kovarik, Longley, Miles, Moore, Ore, Pierpont, P. F. Smith, Thompson, Tracey, Uhler, Whittemore, Wilson.

NEW LONDON. Dimick, Leib, Shover.
STORRS. Cheney.
WATERBURY. Pine.
WEST HAVEN. Shook.

CUBA. (1)

HAVANA. Corral y Alemán.

DELAWARE. (7)

DOVER. Short.
NEWARK. Harding, Harter, McDougle, Rees.
WILMINGTON. Gardner, R. W. Jones.

DISTRICT OF COLUMBIA. (47)

BROOKLAND. Landry.
WASHINGTON. Adams, Arnaud, Ashmun, Avers, Bauer, Blake, Braun, Burley, Claire, Cox, Darling, Duerksen, Edmonston, English, Ewin, Federico, Goldberg, Gosnell, Hamilton, Johnston, Keulegan, Kullback, Lambert, Lennahan, Mangold, Mears, Querfeld, Ramler, Rice, Richmond, Roos, Scheier, Shenton, Sinkov, Stevens, Taylor, Thompson, Tyler, Van Orstrand, Wallis, Watts, Weida, Wernicke, Woodard, Woolard, Zoch.

FLORIDA. (13)

DE LAND. Faulkner.
FORT PIERCE. Bullard.
GAINESVILLE. Dostal, Kokomoor, Messick, Pirenian, Simpson, Wilson.
LAKELAND. Cox.
ORMOND BEACH. Palmié.
TALLAHASSEE. Larson, E. R. Smith.
WINTER PARK. Weinberg.

GEORGIA. (21)

ATHENS. Barrow, Cumming, Hill, Stephens,
ATLANTA. Fulmer, Hefner, Hook, Morton, Patton, Skiles, D. M. Smith, Wood.
DECATUR. Field, Gaylord, Robinson.
DEMOREST. Rogers.
EMORY UNIVERSITY. Messick.
FORSYTH. Plymale.
LA GRANGE. Bailey.
MACON. Holder.
ROME. Hightower.

HAWAII. (1)

HONOLULU. Donaghho.

IDAHO. (2)

CALDWELL. Rankin.
MOSCOW. Taylor.

ILLINOIS. (126)

BASCO. Holloway.
BLOOMINGTON. Hunt.
BLUFFS. Carter.
CARLINVILLE. Renner.
CARTHAGE. Van Velzer.
CHAMPAIGN. Moore, Rogers.
CHARLESTON. Taylor.
CHICAGO. Barnard, Bartky, Bay, Bibb, Bliss, Bower, G. M. Brown, Burrows, Butler, Campbell, Christman, Cobb, Davis, Dickson, Dunford, Ettinger, Everett, Georges, Gerst, Graves, Haggard, Jarrett, Kanies, Kinney, Krathwohl, Kurzin, Lane, Lange, Laves, Logsdon, Lunn, MacMillan, McShane, Moore, Moulton, Nicolet, Nixon, Nyberg, Pettersen, Pyke, Reid, Roberts, Roeser, Schweitzer, Slaught, Spencer, Stafford, Stauffer, Young.
CICERO. Richards.
DECATUR. Kiefer.
DE KALB. Parson.
ELGIN. Thom.
EUREKA. Newson.
EVANSTON. Curtiss, Feltges, Garabedian, Griffiths, Holgate, Moulton, Newell, Simmons, Wescott, Wood.
FREEPORT. Martin, Mensenkamp.
GALESBURG. Heren, Sellew, Stephens.
GODFREY. Bromwell.
GURNEE. Johnston.
JACKSONVILLE. Anderson, Miller.
LAKE FOREST. Curtis.
LA SALLE. Rabe, Carus.
LEBANON. Stowell.
LINCOLN. Balof, Denny.
MACOMB. Ginnings, Schreiber.
MAYWOOD. Hildebrandt.
MONMOUTH. Beveridge, Winbigler.
NORMAL. Atkin, Flagg, Mills.
OAK PARK. Escott.
PEORIA. Comstock, Gault.
RIVER FOREST. Dobbin.
ROCKFORD. McGavock.
ROCK ISLAND. Cederberg.
SANDWICH. Rumney.
TAYLORVILLE. Dappert.
TUSCOLA. Wolever.
URBANA. Armstrong, Bailey, Bower, D. M. Brown, Carmichael, Coble, Crathorne, Emch, Grant, Hazlett, Levy, Lytle, Merrill, Miles, Miller, Peters, Rumsey, P. K. Smith, Steimley, Townsend, Young.
WHEATON. Hillard.

INDIANA. (65)

BLOOMINGTON. Davis, Hanna, Hennel, Moore, Rothrock, Williams, Wolfe.
COLUMBIA CITY. Knisely.
CRAWFORDSVILLE. Carscallen.
CULVER. Adkins.
DANVILLE. Cole.
EARLHAM. Long.
EAST CHICAGO. Darragh, Kellam.
FORT WAYNE. d'Unger, Jenkins, Paxton, Reising, Virts.
FRANKLIN. Heath.
GEORGETOWN. Kelly.
GOSHEN. Lehman.
GREENCASTLE. Arnold, Edington, Greenleaf.
HANOVER. Meyer.
INDIANAPOLIS. Aley, Banes, Bowersox, Johnson, Lutz, Sullivan.
LAFAYETTE. Marshall, Mason, Miller.
MARION. West.
MUNCIE. Edwards, Shiveley.

NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparó, McCue, Maurus.
OAKLAND CITY. Jordan.
SCIRCLEVILLE. Orr.
RICHMOND. Grant.
TERRE HAUTE. Kennedy, Pollock, Shriner, Sousley.
VALPARAISO. Copp.
VEVAY. Haskell.
WEST LAFAYETTE. Black, Bolks, Doan, Graves, Hadley, Happell, Hazard, Hodge, Hughes, Klinger, Little, Long, Robbins, Stone.

IOWA. (57)

AMES. Brandner, Colpitts, Daniells, Fleming, Gouwens, Herr, J. V. McKelvey, M. M. McKelvey, Roberts, Robertson, E. R. Smith, Snedecor, Turner.
BRITT. Day.
CEDAR FALLS. Condit, Kearney, Wester.
CEDAR RAPIDS. Coffin, Yothers.
DAVENPORT. Ries.
DECORAH. Strom.
DES MOINES. Corey, Garnett, Neff.
DUBUQUE. Theobald, Zimmerman.
FAIRFIELD. Roberts.
FAYETTE. Deming.
FOREST CITY. Mundhjeld.
GRINELL. Bailey, McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. F. E. Baker, R. P. Baker, Chittenden, Conkwright, Constantine, Fischer, Reilly, Rietz, Ward, Woods, Wylie.
LE MARS. Blue.
MOUNT VERNON. McGraw, Moots.
ODEBOLT. Wilmer.
OSKALOOSA. Sherman.
PELLA. Evers.
SIOUX CITY. Graber, Gwinn, Olson, Van Horne.
WAVERLY. Brezler.
WELLMAN. Kreth.

KANSAS. (52)

ATCHISON. Pretz.
BALDWIN. Garrett.
COFFEYVILLE. Steininger.
EL DORADO. Wrestler.
EMPORIA. Peterson.
HAYS. Colyer, Zinszer.
HESSTON. Driver.
INDEPENDENCE. Bell.
KANSAS CITY. Dougherty, Thornton.
LAWRENCE. Ashton, Babcock, Black, Jordan, Mitchell, G. W. Smith, Stouffer, Wheeler, Willey.
LINDSBORG. Marm.
MANHATTAN. Andrews, Babcock, Hyde, Janes, Lewis, Lyons, Mossman, Ollivier, Porter, Remick, Stratton, White.
MCPHERSON. Arnett.
NEWTON. Richert.
OTTAWA. Bennett, Loewen.
PITTSBURG. Hill, Shirk.
ST. MARYS. Quinn.
SALINA. Fahnestock, McCain, Ploenges.
STERLING. Bell.
TOPEKA. Harshbarger, Householder, McLatchey.
WICHITA. Hoare, Longenecker, Mendenhall, Reagan.
WINFIELD. Myers.

KENTUCKY (31)

BEREA. Hutcherson, Pugsley.
BOWLING GREEN. Johnson, Yarbrough.
CAMPBELLSVILLE. Lyon.
DANVILLE. Fehn.
GEORGETOWN. Hatfield.
JACKSON. Fremd.
LEXINGTON. Allison, Baxter, Boyd, Cohen, Davis, Downing, Hendricks, Latimer, LeStourgeon, Maney, Mathis, Park, Pence, Rees.
LOUISVILLE. Anselm, Bullitt, Hill, Moore, Paulinus, Simester, Stevenson.
MURRAY. Carman.
RICHMOND. Park.

LOUISIANA. (25)

BATON ROUGE. Nichols, O'Quinn, Sanders, H. L. Smith, Webber.
CLINTON. Petty
COLUMBIA. Redditt.
HAMMOND. Tucker.
NATCHITOCHES. Blair, Killen, Maddox.
NEW ORLEANS. Anderson, Buchanan, Dinwiddie, Duren, Frankenbush, Gentry, Many, Menuet, Monasterio, Spencer, Titsworth, Weiss.
PINEVILLE. C. D. Smith.
SHREVEPORT. Maizlish.

MAINE (12)

BRUNSWICK. Hammond, Holmes, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
ORONO. Bryan, Hart, Jordan, Schultz.
WATERVILLE. Ashcraft, Warren.

MARYLAND (41)

ABERDEEN. Dederick.
ANNAPOLIS. Bingley, Bramble, Capron, Clayton, Clements, Dillingham, Kells, Leiper Lyle, Rawlins, Root, Scarborough, Tyler.
BALTIMORE. Aitchison, Bacon, Bassler, Cohen, Harry, Lewis, Morrill, Murnaghan, Reed, Reynolds, Richeson, Thomsen, Torrey, Whyburn, Williamson, Zariski.
CHELTENHAM. Hartnell.
COLLEGE PARK. Alrich, Gwinner, Spann, Taliaferro, Yates.
CUMBERLAND. Hart.
EMITTSBURG. Burke.
FREDERICK. Brown.
PORT DEPOSIT. Haviland.
ROLAND PARK. Morrow.

MASSACHUSETTS. (98)

AMESBURY. Dame.

AMHERST. Esty, Moore, Porter.
ANDOVER. Newton.
ATTLEBORO. Holt.
BELMONT. Douglass, Rutledge.
BOSTON. Andrew, Bruce, Garabedian, Gould, Laurentine, Leavens, Littauer, Mode, Schroeder, Spear, Wilson.
BROOKLINE. Miller.
CAMBRIDGE. Bailey, Ball, Beatley, Birkhoff, Bradley, Brown, Coolidge, Cooper, Coral, Crum, Franklin, Gaylord, Graustein, Huntington, Kellogg, Kennelly, Littauer, Longley, Moore, Morse, Osgood, Passano, Peterson, Phillips, Price, Rice, Robinson, Seidel, Walsh, Widder, Wilson, Woods, Wright, Zeldin.
DANVERS. Majella.
DORCHESTER. Davis, Heins, Quigley.
EVERETT. Bryant.
GLOUCESTER. MacNutt.
GROTON. Nash.
LOWELL. Aurelius.
LYNN. Evans.
MEDFORD. Miller.
NATICK. Willis.
NORTHAMPTON. Benedict, McCoy, Munroe, Rambo, Wood.
NORTH BROOKFIELD. Kimball.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
SOUTH HADLEY. Anderton, Doak, Martin.
TUFTS COLLEGE. Mergendahl, Ransom.
WELLESLEY. Copeland, Doughty, Hall, Merrill, C. E. Smith, Stark, Young.
WILLIAMSTOWN. Agard, Dorwart, Hardy, Wells.
WOLLASTON. Dennison.
WORCESTER. Brown, Gay, Lepeshkin, Melville, Morley, Rice, Wheeler, Williams.

MICHIGAN. (73)

ALBION. Ingalls, Sleight.
ALMA. Clack.
ANN ARBOR. Aberle, Anning, Ayres, Baten, Bradshaw, Coe, Corliss, Craig, Elder, P. Field, S. E. Field, Ford, Getchell, Hildebrandt, Hopkins, Karpinski, Kazarinoff, Love, Nyswander, Peterson, Raiford, Rainich, Rood, Rouse, Running, Schorling, Thomson, Wagner, Weinberger, Wilder.
BATTLE CREEK. Van Deusen.
BAY CITY. Shellenbarger.
DETROIT. Baldwin, Borgman, Bushyager, Darnell, Folley, Frumveller, Johnson, Johnston, McCarthy, Mullen, Nelson, Paula, Schoonover, Thome.
EAST LANSING. Crowe, Emmons, Grove, Olson, Plant, Powell, Speeker.
FLINT. Stout, Swanson.
HILLSDALE. Herron.
HOUGHTON. Roman.
JACKSON. Field.
KALAMAZOO. Ackley, Blair, Everett, Walton.
MARQUETTE. Spooner.
MOUNT PLEASANT. Richtmeyer.
SPRING ARBOR. M. G. Smith.
YPSILANTI. Barnhill, Erikson, Lindquist, Lyman, Matteson.

MINNESOTA. (43)

COLERAINE. Fisk.
COLLEGEVILLE. Winkelmann.
MANKATO. Robbins.
MINNEAPOLIS. Brink, Brooke, Bussey, Carlson, Dalaker, Erwin, Gibbens, Gunstad, Guttman, Hart, Hartig, Jackson, Jensen, Kirchner, Ness, O'Toole, Priester, Scammon, Schey, Schnell, Shuman, Shumway, Thorp, Underhill, Wilder.
MOORHEAD. Leonard.
NORTHFIELD. Gingrich, Solum, Tangjerd, White.
ST. PAUL. Alice Irene, Chellevold, Kingery, Moench, Morgan, Taylor, Wood.
ST. PETER. Rundstrom.
VIRGINIA. Hancock.
WINONA. Bogard.

MISSISSIPPI. (16)

A. AND M. COLLEGE. Fox, C. D. Smith.
BLUE MOUNTAIN. Hutchins.
CLEVELAND. Hickey.
CLINTON. Price.
COLUMBUS. Ward.
GRENADA. Harris.
HATTIESBURG. Dearman.
HOLLY SPRINGS. McDonald.
JACKSON. Babbitt, McCoy, Mitchell.
STARKVILLE. Edmondson.
UNIVERSITY. Bickerstaff, Rhodes.
WESSON. Felder.

MISSOURI. (48)

CANTON. Ingold.
CAPE GIRARDEAU. Johnson, Knepper.
CARTHAGE. Murto.
CLAYTON. Haertter, Tracy.
COLUMBIA. Callaway, Ingold, Robinson, Wahlin, Westfall.
FAYETTE. Fleet.
FULTON. Christian, Sweazey.
KANSAS CITY. Cutting, Higgins, Luby, Murray, Pierson.
KIRKSVILLE. Cosby, Jamison.
KIRKWOOD. Harris, McFarland.
MAYSVILLE. Saunders.
PARKSVILLE. Wells.
ROLLA. Hinsch.
SEDALIA. Mayer.
SPRINGFIELD. Finkel, H'Doubler.
ST. CHARLES. Karr.
ST. LOUIS. Dunkel, Grummann, Huntington, King, Middlemiss, Muehlman, O'Donnell, Osborn, Rider, Roever, Shannon, Siroky, Stephens, Wilczewski, Young, Zimmerman.
WARRENSBURG. Scarborough.
WEBSTER GROVES. Clarke, Pennell.

MONTANA. (6)

BOZEMAN. Hurst.
HELENA. Canning, Wible.
MISSOULA. Carey, Lennes, Merrill.

NEBRASKA. (23)

BURWELL. Opp.
CERESCO. Walker.
GRAND ISLAND. Anderson.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Basoco, Brenke, Camp, Candy, Collins, Congdon, Gaba, Howie, Pierce, Runge.
OMAHA. Bettinger, Campbell, Earl, Gunn.
PERU. Hill.
STRANG. Lefever.
WAYNE. Schmeiser.
YORK. Feemster.

NEVADA. (2)

RENO. Allen, Searcy.

NEW HAMPSHIRE. (20)

CONCORD. Conwell.
DURHAM. Bauer, Kichline, Kimball, Slobin, Wilbur.
EXETER. Barber, Butterfield, Sweet.
HANOVER. Beetle, Bill, Brown, Forsyth, Mathewson, Morgan, Perkins, Robinson, Silverman, Wilder, Young.

NEW JERSEY. (39)

BELLEPLAIN. Durell.
BLOOMFIELD. Hussey.
EAST ORANGE. Stanwick.
HIGHTSTOWN. Litterick.
JERSEY CITY. J. P. Smith.
LAWRENCEVILLE. Mikesh.
LEONIA. Gafafer.
MONTCLAIR. Davis, Mallory.
MOORESTOWN. Wood.
MORRIS PLAINS. Johnson.
NEWARK. Conkling.
NEW BRUNSWICK. Bunyan, Meder, Morris, Nelson, Walter, Wilson.
PRINCETON. Adams, Alexander, Dorroh, Eisenhart, Flood, Gillespie, Hille, Kennison, Knebelman, Lefschetz, Levine, Thomas, Veblen, Walker, Wedderburn, Wheeler, Willson.
SUMMIT. MacDonald.
TRENTON. Colliton, Shuster.
WORTENDYKE. F. E. Smith.

NEW MEXICO. (11)

ALBUQUERQUE. Barnhart, Bauer, Fuller, Graham, Harp, MacKay, Newsom.
EAST LAS VEGAS. Rodgers.
MOUNTAINAIR. Conlee.
SILVER CITY. Mickelson.
SOCORRO. Reece.

NEW YORK. (252)

ALBANY. Alice Irene, Beaver, Birchenough, Do Bell, Lester.
ALFRED. Polan, Seidlin, Titsworth.
ALLEGANY. McLaughlin.
ANNANDALE-ON-HUDSON. Garabedian, Phalen.
AURORA. Barbour, Carroll, Hollcroft.
BALDWIN. Grove.
BAYSIDE. Dean.
BELLPORT. Walsh.
BINGHAMTON. Patten.
BRONX. Tanzola.
BROOKLYN. Berry, Bowden, Charosh, Cowles, Deutsch, Fleisher, Fuller, Haas, R. A. Johnson, Karnow, Koch, Kryder, Langman, Lepowsky, Lieber, Locke, Markowitz, Moore, Ruderman, Schuyler, Shorr, Thecla, Thompson, Welkowitz, Whitford, Young.
BUFFALO. Archer, Gehman, Harrington, Montague, Pound, Rice.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CORONA. Hanson.
DALE. Whaley.
ELMIRA. Suffa, Wright.
FLUSHING. Lehmann, Pruslin.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs, Starrett, Tocher.
HAMILTON. Aude, A. W. Smith.
HOUGHTON. Brockett.
ITHACA. Boothroyd, Cameron, Carver, Dye, Gillespie, Hadlock, Horsfall, Hurwitz, B. W. Jones, Karapetoff, Lowenstein, Paradiso, Purcell, Randolph, Ranum, Snyder, Torrance, Trevor.
JAMAICA. Barrett.
JORDAN. Howe.
KINDERHOOK. Magee.
NEW YORK. Allen, Allison, Anderson, Archibald, Berger, Bergstresser, Berkeley, Berry, Blair, Bradley, Breckenridge, Breit, Brewster, A. B. Brown, Burdick, Burgess, Bushey, G. A. Campbell, G. C. Campbell, Clark, Cooley, Darkow, Dauenhauer, Doermann, Edmonson, Eisele, Farnum, Feld, Fiske, Fite, Flanders, Foster, Frank, Frankel, Fry, Funkhouser, Gentzler, Gill, Glover, Graham, Griffin, Hall, Harper, Hawkes, Henderson, Hill, Hirsch, Hopper, Hoyt, Hughes, Hurwitz, Hutchinson, Jablonower, Joffe, P. C. Jones, Kunte, Larkin, Linehan, MacGregor, Maiden, Miller, Mirick, Molina, Morehouse, Mullins, Paaswell, Packer, Payne, Pedersen, Penn, Plimpton, Pooler, Post, Pride, Quilty, Raudenbush, Reddick, Rees, Reeve, Ritt, Saurel, Schelkunoff, H. M. Schlauch, W. S. Schlauch, Schub, Seely, Shaw, Shewhart, Siceloff, Simons, Skelding, D. E. Smith, R. F. Smith, R. R. Smith, Spies, Steirnagle, Tilley, Turner, Upton, E. Walker, H. M. Walker, Webster, Wechsler, Weisner, Whitford, Wick, Wilder, Winters, Wood, Wright, Young.
NISKAYUNA. Male.
ONEONTA. Schoonmaker.
PARISH. Church.

PHELPS. Sweet.
POTSDAM. Rowe, Waltz.
POUGHKEEPSIE. Cummings, Wells.
ROCHESTER. Betz, Gale, Harding, Long, Smyth, Watkeys, Welton.
RYE. Vivian.
ST. BONAVENTURE. Nickol.
SCARSDALE. Mac Neish, Thiesmeyer.
SCHENECTADY. Burkett, Lerch, Libman, Morse, Oergel, Poritsky, Snyder, Stokes, Ulrich, Vedder.
STATEN ISLAND. Nordgaard.
SYRACUSE. Campbell, Carroll, Decker, Harwood, Jose, Lindsey, Ryan, Sperry, Taylor, Walters.
TARRYTOWN. Putnam.
TROY. Crockett, McGiffert, Street.
VALLEY STREAM. Henry.
WELLSVILLE. Lish.
WEST POINT. Echols.
YONKERS. Hubert, John, Yanosik.

NORTH CAROLINA. (31)

ASHEVILLE. Peck.
CHAPEL HILL. Browne, Henderson, Hill, Lasley, Linker, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. Douglas, Mebane.
DURHAM. Dale, Dressel, Elliott, Hickson, Rankin, Roberts, Robison.
ELON COLLEGE. Amick.
GREENSBORO. Barton, Pegram, Ragsdale, Streator, Strong, Watkins.
GREENVILLE. Graham, ReBarker.
HOT SPRINGS. Meyer.
LOUISBURG. Harris.
MARS HILL. Robinson.
WINGATE. Hendricks.

NORTH DAKOTA. (6)

FARGO. Householder, I. W. Smith.
GRAND FORKS. Leith, Staley.
UNIVERSITY. Hitchcock.
VALLEY CITY. Meyer.

OHIO. (131)

ADA. Fairchild, Whitted.
AKRON. Bender.
ASHLAND. Black.
ATHENS. Borger, Reed.
BEREA. Baur, Dustheimer.
BOWLING GREEN. Mathias, Overman.
BLUFFTON. Hirschler.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Hancock, Hobensack, Justice, Kersten, Kindle, Lubin, Merriman, Moore, Mullings, Rhodes, Salkover, E. S. Smith, Yowell.
CLEVELAND. Baker, Benander, Boyce, O. E. Brown, Burington, Burwell, Focke, Johnson, Jonah, Justin, Morris, Musselman, Nassau, Oldenburger, Sanford, Sauté, Simon, Thomas, Tolar, Trofimov.
COLUMBUS. Alden, Amos, Bamforth, Beatty, Blumberg, Buell, Harmount, Hildner, Horn, M. E. Jones, Kuhn, LaPaz, MacDuffee, Manson, Morris, Radó, Rasor, Rickard, Singer, Toops, Wildermuth.
DAYTON. Hartwick.
DEFIANCE. Caris, MacCullough.
DELAWARE. Crane, Newlin, Rowland.
FINDLAY. Roots.
GAMBIER. Allen, Bumer.
GRANVILLE. Ladner, Peckham, Wiley.
HILLIARD. Weaver.
HIRAM. Clarke, Jerome.
IVORYDALE. Selheimer.
KENT. Manchester, Schaeffer, Stelson.
MARIETTA. Cope, Rea.
MOUNT ST. JOSEPH. Corona.
NEW CONCORD. White.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Anderson, Erickson, Pollard, Spenceley, Tappan, Wolfe.
PAINESVILLE. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Lemme, Mercedes, Winslow.
WESTERVILLE. Glover, Menke.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson, Yanney.
YELLOW SPRINGS. Dawson, Dwyer.
YOUNGSTOWN. Foard.
ZANESVILLE. Riesbeck.

OKLAHOMA. (23)

ALVA. Hall.
CORDELL. Croom.
MIAMI. Whitney.
MUSKOGEE. Barrick.
NORMAN. Brixey, Court, Duval, Hassler, McFarland, Meacham, Reaves.
SHAWNEE. Short.
STILLWATER. Barnett, Drummond, Flanders, Garretson, Gundersen, H. W. Smith.
TAHLEQUAH. Wyant.
TULSA. Byrd, Howard, Veatch, West.
WEATHERFORD. McCormick.

OREGON. (11)

ALBANY. Ramsey.
CORVALLIS. Beaty, Johnson, Kirkham, Williams.
EUGENE. De Cou, McAlister, Milne.
PORTLAND. Griffin, Merriss, Short.

PANAMA. (1)

PANAMA CITY. Linares.

PENNSYLVANIA. (130)

ANNVILLE. Wagner.
BALA-CYNWYD. Sensenig.
BARNESBORO. Fisanick.
BEAVER FALLS. Cleland, McCormick.
BETHLEHEM. Cairns, Cutler, Ewing, Fort, Latshaw, Rau, Raynor, Reynolds, Shook, Smail, Van Arnam.

BRYN ATHYN. Allen.
BRYN MAWR. Lehr.
BUTLER. Robb.
CALIFORNIA. Foberg.
CARLISLE. Ayres, Landis.
COLLEGEVILLE. Clawson, Manning.
DEVON. Clarke.
EASTON. Hall, Hatch, W. M. Smith.
ERIE. Benedicta, Wells.
GETTYSBURG. Reen.
GROVE CITY. Grimes.
HARRISBURG. Whited.
HAVERFORD. Gummere, Reid, Wilson.
KUTZTOWN. Kunkel.
LANCASTER. Charles, Long.
LARIMER. A. A. Jones.
LATROBE. Seubert.
LEWISBURG. Gold, Lindemann, MacCreadie, Richardson, Youtz.
LINCOLN UNIVERSITY. Wright.
LOCKHAVEN. High, S. J. Smith.
MEADVILLE. Akers, Beisel.
MILLERSVILLE. Seiverling.
NEW WILMINGTON. Freas.
PARADISE. Eshleman.
PHILADELPHIA. Arnold, Bristol, Caris, Chambers, Constable, Crawley, Davis, Evans Kevles, Kline, Knedler, Kusner, Latshaw, Linton, Lufkin, Mitchell, Partridge, Rankin, W. Roberts, Rosengarten, Rothermel, Roulton, Safford, Shohat, Tartler, Weinstein.
PITTSBURGH. Baird, Bohnert, Calkins, Geckeler, Hicks, Hoover, Johnson, Kaltenborn, Mathews, Neelley, Olds, Riggs, Rosenbach, Saibel, Spencer, Swartzel, Taber, Taylor, Whitman.
SALTSBURG. Githens.
SCRANTON. Bertrand.
SELINSGROVE. Williams.
SEWICKLEY. Miller.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Hamilton, F. W. Owens, H. B. Owens, Rupp, Sheffer, Shibli, Wagner, West.
SWARTHMORE. Dresden, Marriott, Miller.
SWISSVALE. Bobertz, Foraker.
UPPER DARBY. McDonough.
WASHINGTON. Atchison, Bert, Rasel, Shaub, Thomas.
WAYNESBURG. Bond.
WILLIAMSPORT. Smink.

PHILIPPINE ISLANDS. (6)

LAGUNA. Salvosa.
LEYTE. Icamen.
MANILA. Jimenez, Mills, Tan, Tienzo.

PORTO RICO. (3)

MAYAGUEZ. Sanchez-Diaz.
RIO PEDRAS. Horne.
SAN GERMAN. Ramos.

RHODE ISLAND. (22)

NEWPORT. Chase.
EAST PROVIDENCE. Tyler.
PROVIDENCE. Adams, Agnew, Archibald, Astrachan, Bennett, Carlen, Chace, Currier, Gilman, Manning, Morton, Moskovitz, Moyle, Oakley, Richardson, Rosskopf, Smiley, Suesman, Tamarkin, Watt.

SOUTH CAROLINA. (12)

CHARLESTON. Bond, Coleman.
COLUMBIA. Coleman, Jackson, Williams.
GAFFNEY. Thompson.
GREENVILLE. Earle.
GREENWOOD. Weber.
HARTSVILLE. Reaves.
ROCK HILL. Grant, Pugh.
SALUDA. Ramage.

SOUTH DAKOTA. (11)

BROOKINGS. MacDougal, Miller, Rasmusen, Walder.
GLENHAM. Stagner.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Bowles, March.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.

TENNESSEE. (17)

CHATTANOOGA. Perry.
HARROGATE. Messick.
JACKSON. Carr, Walden.
JEFFERSON CITY. White.
KNOXVILLE. Bond, Ghormley.
MARYVILLE. Knapp.
MEMPHIS. Scott.
NASHVILLE. Blair, S. I. Jones, N. P. Miser, W. L. Miser, Peterson, Wren.
PULASKI. Meade.
SPRING HILL. Hume.

TEXAS. (73)

ABILENE. Burnam, Tate.
ALPINE. Gilley.
AMARILLO. Barrick.
AUSTIN. Banks, Barnes, Batchelder, Benedict, Cooper, Craig, Decherd, Dodd, Ettlinger, Feenberg, Horton, Lubben, Mitchell, Moore, Muller, Norbert, Quinn, Vandiver.
BOERNE. Hathaway.
BORGER. May.
BROWNSVILLE. De la Garza.
CANYON. Murray.
COLLEGE STATION. Blumberg, Camp, Finlay, Halperin.
DALLAS. Cell, Dice, E. H. Jones, Reinsch.
DENTON. M. C. Brown, Duncan, Hughes, Oldham.
EL PASO. Kennedy.
FORT WORTH. Howard, Sherer.
GALVESTON. Underwood, Van Fleet.
GEORGETOWN. Wapple.
HOUSTON. Blau, Blumenthal, Bray, Dean, Evans, Ford, Lovett, P. K. Rees, W. A. Rees, Streetman.
LUBBOCK. Edmonson, Sparks, Thompson, Underwood.

MARSHALL. Tinner.
NACOGDOCHES. Cross, Ferguson, Oxsheer.
PRAIRIE VIEW. Randall.
REALITOS. Pickett.
SAN ANTONIO. Hurry, McNelly.
STEPHENVILLE. McSweeny, Redden.
TYLER. Holmes, Nelson.
WACO. Harrell.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams, Shirley.

UTAH. (3)

SALT LAKE CITY. Gibson, Pehrson, Unseld.

VERMONT. (11)

BURLINGTON. Bullard, Butterfield, Donahue, Millington, Swift, Thomas.
MIDDLEBURY. Hazeltine, Perkins, Wiley.
NORTHFIELD. Holmes.
WINOOSKI. Alliot.

VIRGINIA. (58)

ASHLAND. Simpson.
BLACKSBURG. Brodie, Hatcher, O'Shaughnessy, Rasche, Williams.
BLUEFIELD. Wright.
BRISTOL. Mize.
CHARLOTTESVILLE. Pinkerton, Stone, Wells.
EMORY. Miller.
ETTRICK. Stokes.
FARMVILLE. Taliaferro.
GREENWAY. Hollis.
HAMPTON. Perkins.
HOLLINS. Dickinson.
LEXINGTON. Byrne, Paxton, Purdie, L. W. Smith, Witt.
LYNCHBURG. Berry, Larew, Pattillo, Wiggin.
MONTEREY. Colaw.
RICHMOND. Gaines, Harris, M. L. Smith, Wheeler.
SALEM. Carpenter.
STAUNTON. Taylor.
SWEET BRIAR. Moody, Morenus.
UNIVERSITY. Bruce, Buchanan, Echols, Linfield, Luck, Oglesby, Sparrow, Thornton.
VIRGINIA BEACH. Huber, Wunder.
WILLIAMSBURG. Russell, Stetson.

WASHINGTON. (17)

PULLMAN. Butler, Isaacs.
SEATTLE. Ballantine, Carlson, Cramlet, Jerbert, McFarlan, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Buxton.
TACOMA. Hanawalt, Martin.
WALLA WALLA. Bratton, Stewart.
YAKIMA. Whitney.

WEST VIRGINIA. (18)

CHARLESTON. Lanham.
HARPERS FERRY. Drew.
HUNTINGTON. Hackney.
INSTITUTE. Lacy.
INWOOD. Mish.
KEYSER. Wilson.
LEWISBURG. Boyd.
MONTGOMERY. W. F. Smith.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Stewart, Turner, Vehse.
PHILIPPI. Morris.
WHEELING. Bagby, Weimer.

WISCONSIN. (40)

BELOIT. Conwell, Huffer.
LA CROSSE. Adkins.
MADISON. Allen, Bennett, Evans, Hart, Hartung, Ingraham, Langer, Lowney, Skinner, E. S. Sokolnikoff, I. S. Sokolnikoff, Stromovsky, Van Vleck, Weaver, Whitford, Wilson.
MILWAUKEE. Battig, Bear, Beckwith, Evans, Knight, Lewandowski, Parkinson, Pettit, Quarles, Roth, Simpson, Vass.
OSHKOSH. Beenken.
PLATTEVILLE. Warner.
RACINE. Mary Joan.
RIPON. Woodmansee.
RIVER FALLS. Eide.
SUPERIOR. C. W. Smith.
WAUKESHA. Hopkins.
WEST DE PERE. De Cleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (4)

LARAMIE. Barr, Bellamy, Neubauer, Rechard.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINE. (2)

BUENOS AIRES, Baidaff, Caruthers.

BELGIUM. (1)

ANTWERP. Vanhee.

BURMA. (1)

RANGOON. Campbell.

CHINA. (3)

CANTON. MacDonald.
NANKING. Chang.
PEKING. Konantz.

FRANCE. (3)

PARIS. Borel, Fréchet, Hadamard.

GERMANY. (4)

BERLIN. Cary.
GÖTTINGEN. Boeder, Bond.
MUNICH. Wieleitner.

GREAT BRITAIN. (5)

CAMBRIDGE. Hardy.
DUBLIN. Rowe.
EDINBURGH. Horsburgh.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville.

HOLLAND. (2)

DELFT. Schouten.
ROTTERDAM. Manning.

INDIA. (3)

ALLAHABAD CITY. Mitra.
CALCUTTA. Prasad.
MADURA. Lockwood.

ITALY. (6)

BOLOGNA. Bortolotti, Enriques, Pincherle.
CAGLIARI. Crudeli.
ROME. Laboccetta.
TURIN. Fubini.

JAPAN. (3)

SENDAI. Hayashi.
TOKYO. Mikami.
PYENGYANG, KOREA. Parker.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SOUTH AFRICA. (3)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.
RONDEBOSCH. Muir.

SOUTH AUSTRALIA. (1)

ADELAIDE. Wilton.

SPAIN. (1)

MADRID. de Toledo.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (1)

CONSTANTINOPLE. Mourad.
ISTANBUL. Harshbarger.

UKRAINE. (1)

KIEFF. Kryloff.

RECAPITULATION OF MEMBERSHIP

Individual members November 7, 1931	2,014	
Institutional members November 7, 1931	132	
Total membership November 7, 1931		2,146
Total membership November 20, 1929		2,097

CHARTER MEMBERSHIP

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	969	
Net gain in institutional members	80	
Total net gain over charter membership		1,049
Total net gain since November 20, 1929		49

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL.

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority fo fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement of such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-President such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings, provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Presidents

E. R. Hedrick....................1916
Florian Cajori....................1917
E. V. Huntington....................1918
H. E. Slaught....................1919
D. E. Smith....................1920
G. A. Miller....................1921
R. C. Archibald....................1922
R. D. Carmichael....................1923
H. L. Rietz....................1924
J. L. Coolidge....................1925
Dunham Jackson....................1926
W. B. Ford....................1927–1928
J. W. Young....................1929–1930
E. T. Bell....................1931–

Vice-Presidents

E. V. Huntington....................1916
G. A. Miller....................1916
D. N. Lehmer....................1917, 1918
Oswald Veblen....................1917
J. W. Young....................1918, 1926
R. G. D. Richardson....................1919
H. L. Rietz....................1919
Helen A. Merrill....................1920
E. J. Wilczynski....................1920
R. C. Archibald....................1921
R. D. Carmichael....................1921, 1922
B. F. Finkel....................1922
A. B. Chace....................1923
L. P. Eisenhart....................1923
J. L. Coolidge....................1924
Dunham Jackson....................1924, 1925
A. A. Bennett....................1925
W. B. Ford....................1926
A. J. Kempner....................1927, 1928
Clara E. Smith....................1927
F. D. Murnaghan....................1928
E. T. Bell....................1929, 1930
W. C. Graustein....................1929, 1930
Arnold Dresden....................1931
C. N. Moore....................1931

Secretary-Treasurer.

(Appointed by the Trustees after 1918)

W. D. Cairns....................1916–

Committee on Official Journal.

(Appointed by the Trustees.)

H. E. Slaught....................1916–
R. D. Carmichael....................1916–1918
W. H. Bussey....................1916–1918
R. C. Archibald....................1919–1921
W. A. Hurwitz....................1919–1921
A. A. Bennett....................1922
H. P. Manning....................1921–1922
W. B. Ford....................1923–1925
J. L. Coolidge....................1923
A. J. Kempner....................1924–
W. H. Bussey....................1926–1931
W. B. Carver....................1932–

Elected Members of the Board

D. N. Lehmer....................1916–1918, 1922–1924, 1930–
R. E. Moritz....................1916–1918
K. D. Swartzel....................1916
Oswald Veblen....................1916, 1920–1922, 1926–1931
R. C. Archibald....................1916–1917, 1923–1930
Florian Cajori....................1916, 1918–1923, 1926–1930
M. B. Porter....................1916–1917
J. W. Young....................1916–1917, 1920–1922
B. F. Finkel....................1916–1921, 1930–
E. H. Moore....................1916–1921, 1923–1928
Alexander Ziwet....................1916–1918
E. R. Hedrick....................1917–1922, 1924–1929
J. N. Van der Vries....................1916–1918
Helen A. Merrill....................1917–1919
D. E. Smith....................1917–1919, 1921–1926
Elizabeth B. Cowley....................1918–1920
G. A. Miller....................1918–1920, 1922–1924
E. J. Wilczynski....................1918–1919, 1922–1926
L. P. Eisenhart....................1919–1922, 1925–1930
E. V. Huntington....................1917, 1919–1927
E. L. Dodd....................1920
R. D. Carmichael....................1920, 1924–1929
A. A. Bennett....................1921, 1930–
H. L. Rietz....................1921–1923, 1925–1930
C. F. Gummer....................1921–1925
Dunham Jackson....................1923–1929
Clara E. Smith....................1923–1925
A. B. Chace....................1924–1925
J. L. Coolidge....................1926–1931
E. T. Bell....................1927–1928
E. P. Lane....................1928–
W. B. Ford....................1929–
E. R. Smith....................1929
W. L. Hart....................1930–
Lao G. Simons....................1930–
L. L. Dines....................1931–
T. C. Fry....................1931–
J. W. Glover....................1931–

The Carus Mathematical Monographs

The Carus Monographs are already fulfilling their mission as intended by the generous donor, Mrs. Mary Hegeler Carus, and her son, Dr. Edward H. Carus.

Somewhat more than one-half the members of the Association have taken advantage of the distribution at cost of the first four Monographs already published. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the Association if a large majority of the members would subscribe for the complete series.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations,* by Professor G. A. Bliss. (First Impression, 1925; Second Impression, 1927.)

No. 2. *Analytic Functions of a Complex Variable,* by Professor D. R. Curtiss. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by Professor H. L. Rietz. (First Impression, 1927; Second Impression, 1929.)

No. 4. *Projective Geometry,* by Professor J. W. Young. (First Impression, 1930.)

No. 5. In preparation.

The Rhind Mathematical Papyrus

CHANCELLOR ARNOLD BUFFUM CHACE, of Brown University, has rendered signal honor to the ASSOCIATION by publishing under its auspices his RHIND MATHEMATICAL PAPYRUS. The entire receipts from the sale of this great work will be used to start an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND.

Volume I, over 200 pages (11¼″ x 8″), contains the free Translation, Commentary, and Bibliography of Egyptian Mathematics.

Volume II, 140 plates (11¼″ x 14″) in two colors with Text and Introductions, contains the Photographic Facsimile, Hieroglyphic Transcription, Transliteration, and Literal Translation.

This exposition of the oldest mathematical document in the world will be of great value, not only to students of mathematics but to any one interested in the work of a civilization of nearly 4,000 years ago.

After January 1, 1932, the special price of $20.00 per set (postage prepaid), will be made for individual and institutional members of the ASSOCIATION on application to the SECRETARY; to all others the price will be $25.00 per set (postage prepaid), obtainable only through the OPEN COURT PUBLISHING COMPANY, 339 East Chicago Avenue, Chicago, Illinois. Previous to January 1932, sets may be purchased at the former prices of $15.00 and $20.00, respectively, plus postage.

INDEX TO VOLUME XXXVIII, 1931

THE AMERICAN MATHEMATICAL MONTHLY

By H. S. EVERETT, The University of Chicago

PAPERS, REPORTS OF MEETINGS

QUESTIONS AND DISCUSSIONS—DISCUSSIONS

RECENT PUBLICATIONS—NEW BOOKS RECEIVED

RECENT PUBLICATIONS—REVIEWS

MATHEMATICAL CLUBS—TOPICS

MATHEMATICAL CLUBS—ACTIVITIES

(Including Chapters of The Pi Mu Epsilon Mathematical Fraternity)

PROBLEMS—AUTHORS

Numbers refer to pages, black-face type indicating a problem solved and solution published, italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS

Numbers in black-face type refer to problems; those in light-face to pages.

NOTES AND NEWS

PERSONAL MENTION

PERSONAL MENTION—NECROLOGY

CORRIGENDA

Volume XXXVIII, 1931:

P. 46, thirteenth line from bottom, for "Evarts" read "J. P. Everett."
P. 52, sixth line from bottom, for "J. D. Deutsch" read "J. G. Deutsch."
P. 189, in D_m the second last element in the last row should read "β_{m-1}" instead of "β_m".
P. 193, the last row of D_m should read "$0 \cdots \quad \cdots \quad \cdots a_{m-1}^{(1)}$".
P. 193, second line from bottom, for "Δ" read "Δ_m".
P. 194, line 2, for "Poincaré's" read "von Koch's."
P. 323, line 17, for "one element" read "one or more elements."
P. 338, line 19, for "G. O. Irwin" read "J. O. Irwin."
See also CORRIGENDA on page 182

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE CO-OPERATION OF

ELIZABETH CARLSON
N. A. COURT
OTTO DUNKEL
H. S. EVERETT
B. F. FINKEL
R. E. GILMAN
R. A. JOHNSON
H. W. KUHN
J. R. MUSSELMAN
H. L. OLSON
D. E. SMITH
F. M. WEIDA

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXVIII, 1931

NUMBER 10, DECEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. John's University, Collegeville, Minnesota on Saturday, May 16, 1931. Sessions were held at 11:00 o'clock and at 2:30 o'clock with luncheon in the dining room of St. John's University. Professor C. H. Gingrich, chairman of the Section, presided at both sessions.

Eighty-five persons attended the meeting including the following twenty members of the Association: Sister Alice Irene, R. W. Brink, W. H. Bussey, Elizabeth Carlson, H. H. Dalaker, Margaret C. Eide, C. H. Gingrich, W. L. Hart, H. E. Hartig, Dunham Jackson, C. M. Jensen, W. H. Kirchner, L. W. Moench, L. J. Schnell, F. J. Taylor, Ella Thorp, A. L. Underhill, Marion B. White, G. L. Winkelmann, Fredrick Wood.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of St. John's University and the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman, Margaret C. Eide, State Teachers College, River Falls, Wisconsin; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee, Marion B. White, Carleton College, Northfield; F. J. Taylor, College of St. Thomas, St. Paul; C. M. Jensen, Macalester College, St. Paul.

The following seven papers were presented:

1. "The solution of the differential equation curl curl $A=0$ in spherical and cylindrical coordinates" by Professor Fredrick Wood, Hamline University.

2. "Properties of a wave equation" by Mr. Edwin N. Oberg, University of Minnesota, by invitation.

3. "Report of the committee on college entrance requirements in geometry" by Professor Dunham Jackson, University of Minnesota.

4. "Mathematics and liberal education" by Professor W. H. Bussey, University of Minnesota.

5. "Correlation between the sex of human siblings in twin and triple births" by Miss Borghild Gunstad, University of Minnesota, by invitation.

6. "The nth derivative of a function of a function" by Mr. Max G. Scherberg, University of Minnesota, by invitation.

7. "Placement tests and high school training in mathematics" by Professor Robert F. Schuck, University of Minnesota, by invitation.

Abstracts of some of these papers follow:

1. In considering the vector potential A for a given field the equation curl $A=0$ arises. Its relation to $\nabla^2 A=0$ suggests that a solution may be found by following the usual procedure in solving $\nabla^2 A=0$. The work is carried out in spherical and cylindrical coordinates and the results are found to be closely related to the solutions of $\nabla^2 A=0$. That the work can be carried out in other systems of coordinates seems quite possible.

2. The wave equation for a one dimensional range is of the form

$$\frac{d^2\psi}{dx^2} + \frac{\pi^2\mu}{h^2}[w - r(x)]\,\psi = 0, \text{ where } \mu, h, \text{ and } w \text{ are constants.}$$

The purpose of this paper is to discuss a form for the solutions of the above equation, when $r(x)$ is sufficiently restricted, from which one can determine the asymptotic form of the solutions as x becomes infinite.

3. This committee report was published in the May issue of this Monthly.

5. Studies have been undertaken in the biometric laboratory at the University of Minnesota for the purpose of determining whether the slight positive relationship previously demonstrated to exist between the sex of children of the same family might be caused by the presence of identical twins in the general population. The correlation between the sex of twins is high—about +.40 (determined from the distribution of sex in 994,760 pairs of twins) as contrasted with the correlation of +.01 calculated from Geissler's data on the sex of 4,794,304 children. The percentage of identical twins range from 20% to 37% for the various nationalities, while the ratio of twin births to total births lies between 1 to 80 and 1 to 90. The theoretical number of identical twins in the general population was calculated on the assumption that 1/80th of all births were twin births and that 40% of these were identical twins.

Then the elimination of these identical twins reduced the original correlation by about one third but the sign remained positive. There seems to be a definite, though very slight, tendency for members of the same sibship to be of the same sex rather than to represent a chance combination of the two sexes.

6. The operator $(a_n+b_{n-1}+b_{n-2}\cdots b_1)(a_{n-1}+b_{n-2}\cdots b_1)\cdots(a_2+b_1)a_1$ when operating on the product $x(t)\,y(x)$ will give the nth derivative of $y(x)$ with respect to t and in terms of the derivatives of $y(x)$ with respect to x and $x(t)$ with respect to t. It operates in the following manner. The form is expanded and the terms interpreted by the following equations:

$$b_i{}^j a_i{}^k = a_i{}^{j+k} \quad (k \neq 0), \quad b_i{}^j a_i{}^0 = 0,$$

$$a_i{}^l a_j{}^k a_n{}^m x(t) y(x) = \frac{d^l x}{dt^l}\cdot\frac{d^k x}{dt^k}\cdot\frac{d^m x}{dt^m}\cdot\frac{d^3 y}{dx^3}\,.$$

The order of the derivative of y with respect to x is equal to the number of distinct subscripts on the a's.

A. L. UNDERHILL, *Secretary*

THE PROJECTIVE APPROACH TO THE CLIFFORD SURFACE

By LAURA E. CHRISTMAN, Chicago, Ill.

1. *Introduction.* Euclidean space differs from the two non-Euclidean spaces (hyperbolic and elliptic) in that the metric relations of the former are developed with respect to a degenerate quadric (which as an envelope of planes is a conic, the space dual of a point cone), while the metric relations of the latter geometries are developed with respect to proper quadrics as the absolute, real in the case of hyperbolic geometry, imaginary in the case of elliptic geometry.

Since both non-Euclidean and Euclidean figures may be translated into, or developed from, projective figures, "the projective approach to non-Euclidean geometry seems at once the most natural and the most elegant[1]". Using this approach and taking figures of circles and spheres, concerning which much is known in Euclidean geometry, I have developed several interesting facts concerning the Clifford surface, facts which connect it with a particular Euclidean ellipsoid rather than with the Euclidean cylinder.

I shall begin by developing one theorem of non-Euclidean plane geometry from its Euclidean equivalent. Consider the Euclidean theorem:

The locus of intersection of two tangents to a circle, which meet at a constant angle, is a concentric circle.

It will be remembered that Euclidean plane geometry presupposes, as an absolute, a degenerate conic which as a point locus is a line used twice, and as a line locus is two conjugate imaginary points; and that all circles pass through these two points called the I, J points. The homogeneous coordinates for these two points are $(1, \pm i, 0)$. The following quotations from the Euclidean-projective dictionary which mathematicians have developed will apply to our theorem.

Euclidean	*Projective*
The measure of an angle.[2]	$1/2i \log r$, where r is the cross ratio of the pencil formed by the sides of the angle and the rays to I and J.
Circle.	Conic passing through I and J.
Concentric circles.	Conics having double contact at the points I and J.

We then have the theorem in projective form as follows:

The locus of the centre of a pencil of constant double ratio, formed by two variable tangents to a conic and lines to the extremities of a fixed chord is a second conic having double contact with the first at the extremities of the fixed chord.[3]

Again we must refer to a dictionary, this time to translate the projective form into the non-Euclidean form.

[1] R. M. Winger, *Projective Geometry*, p. 419.

[2] Quoted by Winger (p. 97) from E. Laguerre.

[3] Winger, p. 150.

Projective	*Non-Euclidean*
Lines intersecting on the absolute conic.	Parallel lines.
Lines tangent to the absolute conic.	Isotropic lines.
A conic having double contact with the absolute.[1]	A circle.
The chord of contact.	The axis of the circle.
The absolute pole of the chord of contact (i.e., the pole with respect to the absolute).	The centre of the circle.

Allow the first conic to become the non-Euclidean absolute and the theorem assumes this form:

The locus of the centre of a pencil of constant double ratio, formed by two isotropic lines and two lines parallel to a given line, is a circle having the given line as axis.

This can be restated thus:

The locus of a point from which the two lines parallel to a given line have a constant angle of parallelism is a circle having the given line as axis.

Allow the second conic to become the absolute and the theorem becomes:

Every pair of parallel tangents to a circle and the two lines which are simultaneously parallel to the axis and both tangents form a parallel pencil of constant cross ratio.

The same machinery will develop non-Euclidean theorems of three dimensions. For example, consider the Euclidean theorem:

If two concentric spheres S_1 and S_2 meet a quadric Q, and one of them, S_1, is tangent to the quadric along a circle, then the other will intersect the quadric along two plane sections or circles.

The following equivalents from the Euclidean-projective dictionary will apply.

Euclidean	*Projective*
The infinitely distant plane.	A plane possessed of only ordinary projective properties, but designated by the symbol P_∞.
The absolute conic on the infinitely distant plane.	An ordinary conic designated by A on the plane P_∞.
All spheres.	All quadrics which pass through A.
Concentric spheres.	Quadrics tangent to one another along the plane section A.

The projective form of the theorem is then:

If two quadrics Q and S_2 are each tangent along a conic to a third quadric S_1, they will intersect in two conics.[2]

---

[1] Proved in Winger's book, pp. 405–409 and p. 262.

[2] Proved in Salmon's *Geometry of Three Dimensions*, Edition of 1912, p. 103.

To this form of the theorem apply the single fact from the projective non-Euclidean dictionary, that a quadric tangent to the absolute along a plane curve is a sphere, and allow S_1 to be used as the absolute; then we have this interesting non-Euclidean form for the theorem: *Two spheres intersect in two circles.*

2. *Development of a Clifford Surface in Elliptic Geometry from a Certain Euclidean Figure.* For the Clifford surface problem we will turn to figures 1 and 2 which illustrate a certain figure in Euclidean and projective space.

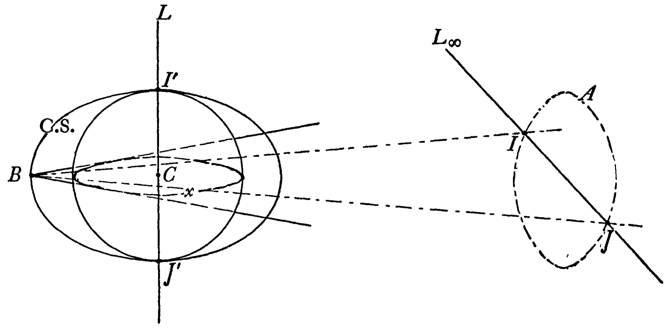


FIG. 1

In Figure 1 there is given sphere S, centre C, and diameter L, and the absolute conic A. x is any one plane section of the sphere, perpendicular to L. In the plane of this section, assume tangents to x meeting at a constant angle which is equal to B, and also the infinitely distant line L_∞ on which are the circular points I and J. Then the locus of B is a circle concentric to x.

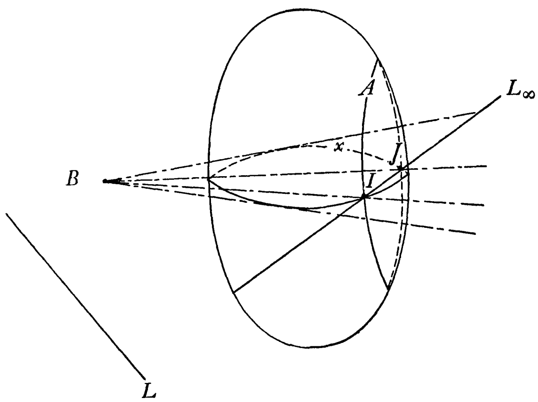


FIG. 2

Now (Fig. 2) project the sphere S into the quadric S through the fixed conic A, and let the same lettering denote throughout projectively corresponding parts. Then the pencil of lines at B used in measuring the size of angle B in the Euclidean figure will project into a pencil of lines at B, in the projective figure, of constant cross ratio, two lines tangent to x and two lines to I and J respectively, and the projective locus of B in the plane of x will become a conic of

double contact with x at the points I and J. (In fact the whole figure in the plane of x is precisely the figure dealt with in the introduction in the circle theorem.) The diameter L of the sphere S is still (after the projection) a line conjugate to the line of I and J (L_∞) with respect to S.

In the projective figure, let S be isolated as the non-Euclidean absolute (elliptic case). Let IJ be a real line intersecting this imaginary elliptic absolute in conjugate imaginary points. Then the two lines from B, a real point, tangent to x(a section of the absolute) become isotropic lines from B. The two lines from B to I and J become two parallels to IJ or L_∞, which are imaginary, having a constant angle at B. (Since the cross ratio of the sides and the isotropic lines is constant.) The locus of B becomes a real circle with axis L_∞ and centre the absolute pole of L_∞. So the centre of the original circular section (x) of the Euclidean sphere (pole of IJ) goes into the centre of the new non-Euclidean circle locus of B.

Revert again to the Euclidean figure and let the plane of angle B and circle x move continuously parallel to itself; i.e., revolve around the chord IJ or L_∞. As long as it makes a cross section of S it will project into a non-Euclidean circle just as in the plane of section x first considered. This circle will always have IJ as its axis and the absolute pole of IJ in its own plane (which, as the plane moves, traces the line L) as its centre. Therefore this surface, which in Euclidean geometry is an oblate spheroid, projects into a non-Euclidean surface, with the points (around each section through IJ) equidistant from IJ, and with the absolute pole of IJ in that plane, as centre.

Now the question is: Are the distances of point B, from axis and the centre of the circle in *one* cross section respectively equal to the corresponding distances of point B in another cross section? In Figure 3(a) let lines BC and $B'C'$ be conjugates of L with respect to the absolute, since L is perpendicular both to BC and $B'C'$ in the Euclidean figure. So our problem reduces to the question in cross ratios: Is $(Bc\,|ba) = (B'c'\,|b'a')$? In Figure 3, (b) and (c), suppose we have the two Euclidean circles of which BC and $B'C'$ respectively are the radii. Since angle $\theta =$ angle θ', the two figures are similar, and therefore not only cross ratios but ratios are equal.

Therefore every non-Euclidean circle with axis IJ, on which point B may lie, as the plane of x revolves around IJ, has the same radius. Therefore this surface, generated by x as the plane of x revolves around IJ, is the locus of points in space equidistant from IJ and equidistant from L. But since there are paratactic lines to IJ (or lines in space equidistant from IJ) they must lie on this surface. Therefore they are its generators and this surface is a *Clifford surface*. Perhaps the construction of this surface, projectively, may be most concisely restated in the form of two theorems or statements, each expressed in three equivalent forms:

I. *Euclidean*. If through a sphere is passed a plane, the locus of the intersection of two lines in the plane tangent to the section which meet at a constant angle is a circle concentric to the plane section.

Projective. If through a quadric is passed a plane, the locus of the centre of a pencil of constant cross ratio, formed by the tangents to the section, and two lines to two fixed points on the section, is a second conic tangent to the section at the two fixed points.

Non-Euclidean. The locus of the real intersection of two conjugate imaginary lines in a plane parallel, in opposite senses, to a real line of the plane, which meet at a constant angle of parallelism, is a circle having the given line as axis and its absolute pole, in that plane, as centre.

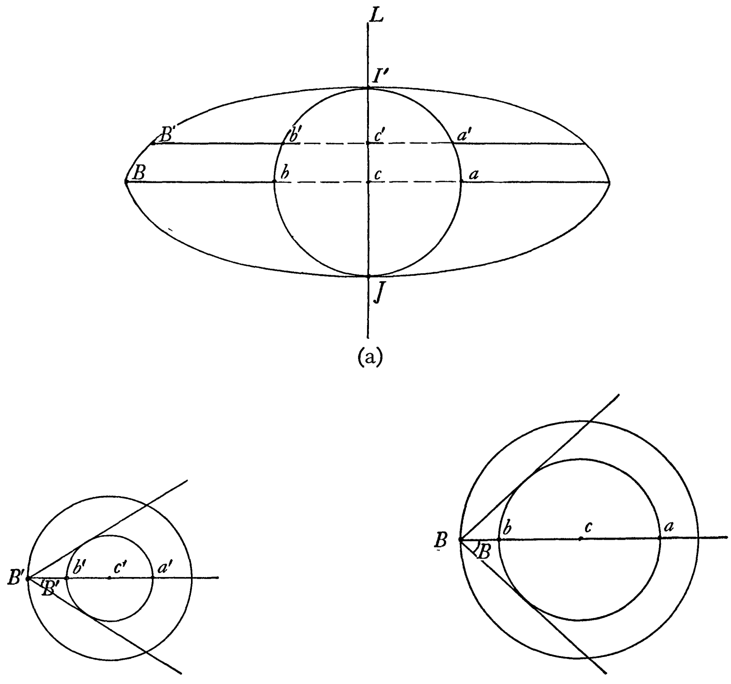


FIG. 3

II. *Euclidean.* If the plane is moved parallel to itself, the locus of the intersection of tangents which meet at a constant angle (described in I above) is the plane's intersection with an oblate spheroid.

Projective. If the plane is moved in the various positions of a pencil of planes, whose axis is the given line, the locus is the locus generated by the conic of the above theorem, a surface tangent to the "absolute" at the points I and J and at the extremities of L.

Non-Euclidean. The locus in space of the vertex of two conjugate imaginary lines parallel in opposite senses to a given real line, which meet at a constant angle of parallelism, is a surface equidistant from the given line; and also from its absolute polar, a surface whose cross section through either axis is a circle or equidistant curve.

3. *Facts Concerning the Clifford Surface Deducible from the Euclidean Figure.* The last theorem may be restated to furnish our first fact:

The Clifford surface is not only the locus of real right and left paratactic lines at a given distance from either axis, but also the locus of real meeting points of conjugate imaginary lines parallel right and left to the axis and coplanar with it.

I am here defining paratactic lines to be lines at a constant distance one from the other and skew to one another, parallel lines to be lines meeting one another on the absolute (and therefore in the same plane). Of course the space is elliptic.

Again from the Euclidean figure we can say:

The Clifford surface meets each of its axes in conjugate imaginary points of the absolute.

Reverting to the Euclidean figure, we can write these equations for it.

Let the equation of the sphere be $x^2+y^2+z^2=r^2$.

Then we find the equation of the Euclidean equivalent of the Clifford surface which I shall call C.S.

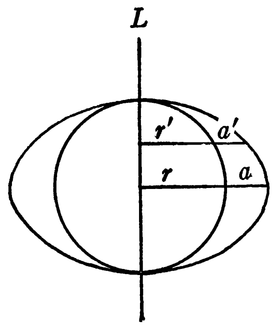


FIG. 4

Consider the diametric section of the sphere perpendicular to L. From the equation of the sphere the length of the radius of the co-planar section of the C.S. is r+some quantity which we may call a. See Fig. 4 for a cross section of all these sections of the sphere and C.S.

Since we have proved the ratio of r and $r+a$ constant whether the section is the original diametric section perpendicular to L or an ordinary plane section parallel to the diametric plane, if we call the radius of any other parallel section r' and the additional length necessary to give us a radius of C.S., a' we have:

$$r + a:r = r' + a':r'; \text{ or } a'/r' = a/r.$$

But in the sphere

$$r'^2 = r^2 - z^2.$$

Therefore in the plane $z=z'$ the equation of the section of C.S. is

$$x^2 + y^2 = (r' + a')^2 = r'^2[1 + a'r'^{-1}]^2 = (r^2 - z'^2)\{1 + ar^{-1}\}^2.$$

Hence the general equation of C.S. is

$$x^2 + y^2 = (r^2 - z^2)\{1 + ar^{-1}\}^2.$$

Let $1+ar^{-1}=R$ and the equation of C.S. becomes

$$x^2r^{-2}R^{-2} + y^2r^{-2}R^{-2} = 1 - z^2r^{-2},$$

while the equation of the sphere is

$$x^2r^{-2} + y^2z^{-2} = 1 - z^2r^{-2}.$$

Therefore the imaginary generators of the C.S. are represented by

(A) $xr^{-1}R^{-1} - izr^{-1} = \xi(1 - yr^{-1}R^{-1})$; $1 + yr^{-1}R^{-1} = \xi(xr^{-1}R^{-1} + izr^{-1})$

and

(B) $xr^{-1}R^{-1} - izr^{-1} = \eta(1 + yr^{-1}R^{-1})$; $1 - yr^{-1}R^{-1} = \eta(xr^{-1}R^{-1} + izr^{-1})$;

and the imaginary generators of the sphere S by

(C) $xr^{-1} - izr^{-1} = \xi'(1 - yr^{-1})$; $1 + yr^{-1} = \xi'(xr^{-1} + izr^{-1})$;

and

(D) $xr^{-1} - izr^{-1} = \eta'(1 + yr^{-1})$; $1 - yr^{-1} = \eta'(xr^{-1} + izr^{-1})$.

Now we ask: Is it possible to choose ξ and ξ' so that the lines A and C are identical? If so, then a λ can be found so that for those values of ξ and ξ'

$$xr^{-1}R^{-1} - izr^{-1} + \xi yr^{-1}R^{-1} - \xi + \lambda(\xi xr^{-1}R^{-1} + \xi izr^{-1} - yr^{-1}R^{-1} - 1) = 0$$

is the same equation as

$$xr^{-1} - izr^{-1} + \xi' yr^{-1} - \xi = 0.$$

By equating ratios of corresponding coefficients in the two equations, we find that

$$\xi = \xi' = \pm i, \text{ and } \lambda = (R-1)i^{-1}(R+1)^{-1}.$$

Therefore with these values the first plane of C passes through the line A. Using $\xi=\xi'=\pm i$ we find that for $\lambda=(1+R)i^{-1}(R+1)^{-1}$ the second plane of C also passes through the line A. Hence the lines A and C are identical when $\xi=\xi'=\pm i$.

In the same way it can be developed that the lines B and D are identical when $\eta=\eta'=\pm i$.

Therefore the sphere and C.S. have two generators of each system in common.[1]

Klein[2] states that *the Clifford surface has in common with the absolute a tangent tetrahedron.* This fact is also evident in the Euclidean equivalent. Refer again to Figure 1. Since the ellipsoid C.S. has circular sections if planes are passed parallel to plane x, and since these sections are concentric with the coplanar sections of S, each section of C.S. is tangent to the coplanar section of S at points I and J. Therefore C.S. is tangent to S at points I and J. From the left-

[1] Sommerville, *Non-Euclidean Geometry*, p. 113, proves the fact synthetically.

[2] Klein, *Vorlesungen über Nicht Euklidische Geometrie* (1928), p. 241.

hand portion of the figure, C.S. is evidently tangent to S at points I' and J'. Therefore if these four points are taken as the vertices of a tetrahedron, C.S. has in common with S a tangent tetrahedron composed of planes through points $I'IJJ'$. For consider planes $I'IJ$ and $J'IJ$. They are parallel to section x through I' and J'. Therefore they are tangent both to C.S. aud S, one at I' and one at J'. Consider planes $I'JJ'$ and $I'IJ'$. They are tangent to C.S. and S at points I and J. Therefore we have in the Euclidean equivalent of the Clifford surface a tangent tetrahedron which is also a tangent tetrahedron to the Euclidean equivalent of the absolute.

ON THE EXPANSION OF A CERTAIN TYPE OF DETERMINANT

By DONALD L. McDONOUGH

1. In a well known textbook in the theory of invariants[1] the following theorem appears:

"*If* a_{ix}, *where* $1 \leqq i \leqq r$, *are* r *distinct (binary) linear forms, and* A_i *are binary forms of orders* α_i, *where* $\sum_{i=1}^{r} \alpha_i = n-r+1$, *then any binary form* f *of order* n *can be expressed in the form*

$$f = \sum_{i=1}^{r} a_{ix}^{n-a_i} A_i, \tag{1}$$

and the expression is unique."

It is understood that the linear forms a_{ix} are given and the binary forms A_i are to be determined.

Upon equating coefficients of like products of powers of the variables in the expanded form of (1), $n+1$ linear non-homogeneous equations arise in the coefficients of the binary forms A_i which are to be regarded as unknowns. The determinant of the coefficients of these unknowns has a peculiar form. It has been thought that this determinant could be expanded as a numerical multiple of a product of powers of the differences of the roots of the linear forms. It seems, however, that this has never been proved.

It is the desire of the writer to prove:

Theorem I. *Any determinant that arises in this way can be expressed as a numerical multiple of a product of powers of the resultants of the linear forms taken by pairs.*

Theorem II. *Any such determinant is different from zero, and the numerical multiplier can be expressed in terms of binomial coefficients.*

The truth of both of these theorems is immediately evident in the case where $r=2$. In this case the determinant in question is merely the resultant of the two

[1] O. E. Glenn's *The Theory of Invariants*, page 120. For a discussion of a somewhat similar problem for $r=2$ see a paper by the same author, *The Symbolical Theory of Finite Expansions*, Transactions of the American Mathematical Society, vol. 15 (1914), especially pp. 80–82.

forms $(a_{i1}x_1+a_{i2}x_2)^{n-\alpha_i}$, where $\alpha_1+\alpha_2=n+1$. Consequently this determinant is equal to $(a_{11}a_{22}-a_{21}a_{12})^{(n-\alpha_1)(n-\alpha_2)}$. Since the linear forms are distinct, this cannot be zero.

2. For the sake of convenience in notation let the r linear forms be represented in the form $x_1+a_ix_2$, and set $n-\alpha_i=\eta_i$, and $\alpha_i+1=n_i$. Let

$$A_i = \sum_{k=1}^{n_i} p_{ik}x_k.$$

(1) will then take the form

$$\sum_{i=1}^{r}\sum_{j=0}^{\eta_i}\sum_{k=1}^{n_i}\binom{\eta_i}{j} a_i^j p_{ik} x_1^{n-j-k+1}\, x_2^{j+k-1} = f. \tag{2}$$

The general form of the determinant that arises when we interchange rows and columns is now

$$\left|\begin{array}{cccccccccc}
1 & \eta_1a_1 & \binom{\eta_1}{2}a_1^2 & \cdots & \cdots & a_1^{\eta_1} & 0 & \cdots & \cdots & 0\\
0 & 1 & \eta_1a_1 & \cdots & \cdots & \eta_1a_1^{\eta_1-1} & a_1^{\eta_1} & 0 & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdots & 1 & \eta_1a_1 & \binom{\eta_1}{2}a_1^2 & \cdots & \cdots & \cdots & a_1^{\eta_1}\\
1 & \eta_2a_2 & \binom{\eta_2}{2}a_2^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 1 & \eta_2a_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdots & \cdots & 1 & \eta_2a_2 & \binom{\eta_2}{2}a_2^2 & \cdot & a_2^{\eta_2}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
1 & \eta_ra_r & \binom{\eta_r}{2}a_r^2 & \cdots & a_r^{\eta_r} & 0 & \cdots & \cdots & \cdots & 0\\
0 & 1 & \eta_ra_r & \binom{\eta_r}{2}a_r^2 & \cdot\ \eta_ra_r^{\eta_r-1} & a_r^{\eta_r} & 0 & \cdots & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & \cdots & \cdots & 1 & \eta_ra_r & \binom{\eta_r}{2}a_r^2 & \cdot & a_r^{\eta_r}
\end{array}\right| \tag{3}$$

It will be noted that in each row the numerical coefficients are the binomial

coefficients that occur in the expansion of $(x_1+a_ix_2)^{\eta_i}$. Consider the rows involving any particular a_i. It will be noted that in the jth of these rows the first non-zero element appears in the jth column, where $1\leqq j\leqq n_i$, and n_i is the number of rows in a_i. The order of the determinant is

$$n+1=\sum_{i=1}^{r} n_i.$$

It can be considered as the eliminant of the $n+1$ forms

$$x_1^{\lambda}x_2^{\mu}(x_1+a_ix_2)^{\eta_i}, \tag{4}$$

where $1\leqq i\leqq r$, λ takes on all values from n_i-1 to 0 inclusive for each value of i, $\lambda+\mu=n_i-1$, and $n_i+\eta_i-1=n$.

3. We shall now prove:

Lemma 1. *The determinant* (3) *is a homogeneous polynomial in the quantities* $a_i(1\leqq i\leqq r)$.

In considering the rows involving any particular a_i it will be observed that the element in the jth of these rows and the $r_j^{(i)}$ th column of the determinant, where $r_j^{(i)}\geqq j$, involves a_i to the $(r_j^{(i)}-j)$th power. The degree of any term of the expansion of (3) in a_i is therefore

$$\sum_{j=1}^{n_i}(r_j^{(i)}-j)=$$

$$\sum_{j=1}^{n_i} r_j^{(i)}-\tfrac{1}{2}n_i(n_i+1) \tag{5}$$

and the total degree of the term is

$$\sum_{i=1}^{r}\ \sum_{j=1}^{n_i}(r_j^{(i)}-j)=\tfrac{1}{2}(n+1)(n+2)-\sum_{i=1}^{r}\tfrac{1}{2}n_i(n_i+1). \tag{6}$$

Since this expression depends only upon the order of the determinant and the number of rows in each quantity a_i, the lemma is proved.

Lemma 2. *Any minor of* (3) *of order of* n_i *involving only one of the quantities* a_i, *is homogeneous in that quantity.*

As may be seen from (5), the degree of any term of such a minor is dependent only upon the number of columns of (3) represented in the minor and on the value of n_i. The truth of the lemma is therefore obvious.

4. Instead of considering the forms (4), we shall consider the equivalent set of $\sum_{i=1}^{i} n_i$ forms that result when for each particular value of i we multiply the j_ith and following forms involving a_i respectively by the terms of $(1+a_i)^{n_i-j_i}$ and add, where j_i ranges from 1 to n_i. The resulting forms are

$$x_2^{k_i}(x_1+a_ix_2)^{n-k_i}, \quad \text{where } 0\leqq k_i\leqq n_i-1. \tag{7}$$

The eliminant of these forms is

$$
\begin{vmatrix}
1 & \binom{n}{1}a_1 & \binom{n}{2}a_1^2 & \cdots & \cdots & \cdots & \cdots & a_1^n \\
0 & 1 & \binom{n-1}{1}a_1 & \cdots & \cdots & \cdots & \cdots & a_1^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots 0 & 1 & \binom{n-k_1}{1} & a_1 \cdots & a_1^{n-k_1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots 0 & 1 & \binom{\eta_1}{1} & a_1 \cdots & a_1^{\eta_1} \\
1 & \binom{n}{1}a_2 & \binom{n}{2}a_2^2 & \cdots & \cdots & \cdots & \cdots & a_2^n \\
0 & 1 & \binom{n-1}{1}a_2 & \cdots & \cdots & \cdots & \cdots & a_2^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots 0 & 1 & \binom{\eta_2}{1} & a_2 \cdots & a_2^{\eta_2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \binom{n}{1}a_r & \binom{n}{2}a_r^2 & \cdots & \cdots & \cdots & \cdots & a_r^n \\
0 & 1 & \binom{n-1}{1}a_r & \cdots & \cdots & \cdots & \cdots & a_r^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots 0 & 1 & \binom{\eta_r}{1} & a_r \cdots & a_r^{\eta_r}
\end{vmatrix} . \tag{8}
$$

This determinant is equal to (3), since it is merely the result of adding to each row of (3) certain multiples of the rows following it that contain the same a_i.

Replacing $x_1^\lambda x_2^\mu$ by y_μ where $0 \leqq \mu \leqq n$ and $\lambda+\mu=n$, in the expanded form of (7), we have a system of $n+1=\sum_{i=1}^{r} n_i$ linear forms in as many variables:

$$
\sum_{i=0}^{n-k_i} \binom{n-k_i}{j} a_i^j y_{j+k_i}, \tag{9}
$$

where $0 \leqq i \leqq r$, and $0 \leqq k_i \leqq n_i - 1$ for each value of i.

If the forms (7) are subjected to the transformation

$$(10) \qquad x_1 = x_1' - a_1 x_2', \quad x_2 = x_2',$$

the resulting forms will be

$$x_2'^{\,k_i}[x_1' + (a_i - a_1)x_2']^{n-k_i},$$

where $0 \leqq k_i \leqq n_i - 1$. Hence

$$y_\mu = (x_1' - a_1 x_2')^\lambda x_2'^{\,\mu} = \sum_{k=0}^{\lambda}(-1)^k \binom{\lambda}{k} a_1^k x_1'^{\,\lambda-k} x_2'^{\,k+\mu}.$$

Let $x_1'^{\lambda-k} x_2'^{k+\mu} = y'_{k+\mu}$. The transformation induced on the y_μ''s by (10) is

$$(11) \qquad y_\mu = \sum_{k=0}^{\lambda}(-1)^k \binom{\lambda}{k} a_1^k y'_{k+\mu},$$

where $0 \leqq \mu \leqq n$, and $\lambda + \mu = n$. This is a transformation of determinant unity.

The transformation (11) reduces the forms (9) to the forms y_j' for $1 \leqq j \leqq n_1$, and

$$\sum_{j=k_i}^{n} \binom{n - k_i}{j - k_i} (a_1 - a_i)^{j-k_i} y_j',$$

where $1 < i \leqq r$ and $0 \leqq k_i \leqq n_i - 1$ for the remaining forms. The eliminant of these forms is equal to the eliminant of the forms (9) as a consequence of the following well known theorem:

"*If a system of n linear forms in n variables with matrix a is subjected to a linear transformation with matrix c, the resulting system has as its matrix ac.*"

This eliminant is obviously equal to its $\sum_{i=2}^{r} n_i$-rowed minor that lies in the lower right hand corner. If this minor be expanded according to its n_i rows involving $(a_i - a_1)$, it is evident from (5) that each of the terms in this expansion will be divisible by $a_i - a_1$ raised to the power

$$\sum_{j=1}^{n_i}(n_1 + j) - \tfrac{1}{2}n_i(n_i + 1) = n_1 n_i.$$

If we replace a_1 in the transformation (11) by the other a_i's in turn, it is evident that the resulting eliminant, and hence the eliminant of the forms (9), is divisible by $(a_i - a_j)^{n_i n_j}$, and hence by

$$\prod_{i=1}^{r-1} \prod_{j=i+1}^{r} (a_i - a_j)^{n_i n_j}.$$

The degree of any term of this product is

$$\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_i n \, .$$

But from (6) the degree of any term of the expansion of the determinant (3) is

$$\tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1) = \tfrac{1}{2}\sum_{i=1}^{r} n_i\left(\sum_{i=1}^{r} n_i + 1\right)$$

$$- \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1) = \tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} n_i n_j + \tfrac{1}{2}\sum_{i=1}^{r} n_i - \tfrac{1}{2}\sum_{i=1}^{r} n_i(n_i+1)$$

$$= \tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} n_i n_j - \tfrac{1}{2}\sum_{i=1}^{r} n_i^2 = \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} n_i n_j,$$

and is thus the same as that of the product referred to above. Hence (3) is equal to

$$k\prod_{i=1}^{r-1}\prod_{j=i+1}^{r}(a_i - a_j)^{n_i n_j}, \tag{12}$$

where k is a numerical multiplier to be determined in the next section. This completes the proof of Theorem I.

5. It remains to prove Theorem II, i.e. to determine k in (12). This can be done by comparing a particular term of (12) with the corresponding term of the determinant (8), which has been shown to be equal to (3). The proofs of Lemmas 1 and 2 hold true mutatis mutandis for (8).

We shall designate the n_i-rowed minors in a_i that lie along the leading diagonal of (8) by M_i, where $1 \leqq i \leqq r$; the $\sum_{j=1}^{i} n_j$-rowed minors in the upper left hand corner of (8) by D_i. D_r is evidently the determinant itself. All the terms of the expansion of M_i are of the same degree in a_i by Lemma 2. M_i is a unique minor of D_i of maximum degree in a_i by (5). Now all the terms of (8) of maximum degree in a_r arise as the product of M_r by D_{r-1}. Of these terms, those that involve a_{r-1} to a maximum degree arise as the product $M_r M_{r-1} D_{r-2}$. Of these terms, those that involve a_{r-2} to a maximum degree arise as the product $M_r M_{r-1} M_{r-2}$ D_{r-3}. Continuing this process we see that $\prod_{i=1}^{r} M_i$ is a unique term in the expansion of (8) of which the coefficient is k.

6. In order to find k it is merely necessary to find the values of the determinants that result when in the minors M_i we omit the quantities a_i. For convenience we shall now assume that, if necessary, the rows in (8) have been so permuted that each $n_i \geqq n_{i+1}$. These determinants are:

$$
(13)\qquad \begin{vmatrix}
\binom{n}{\sum_{j=1}^{i-1} n_j} & \binom{n}{\sum_{j=1}^{i-1} n_j + 1} & \cdots & \binom{n}{\sum_{j=1}^{i-1} n_j + n_i - 1} \\
\binom{n-1}{\sum_{j=1}^{i-1} n_j - 1} & \binom{n-1}{\sum_{j=1}^{i-1} n_j} & \cdots & \binom{n-1}{\sum_{j=1}^{i-1} n_j + n_i - 2} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j - n_i + 1} & \binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j - n_i + 2} & \cdots & \binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j}
\end{vmatrix}
$$

where $1 < i \leqq r$.

If the last row of (13) is brought into the first place, the next to the last row into the second place, etc., and then the last column is brought into the first place, the next to the last into the second place, etc., (13) is unchanged since there are as many interchanges of rows as of columns. If we now bear in mind that

$$\binom{n}{r} = \binom{n}{n-r},$$

we find that

$$\binom{n-n_i+1}{\sum_{j=1}^{i-1} n_j} = \binom{n-n_i+1}{n - \sum_{j=1}^{i} n_j + 1}.$$

If similar changes are made in all of the elements, the determinant becomes

$$
(14)\qquad \begin{vmatrix}
\binom{n-n_i+1}{n-\sum_{j=1}^{i} n_j + 1} & \binom{n-n_i+1}{n-\sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n-n_i+1}{n-\sum_{j=1}^{i-1} n_j} \\
\binom{n-n_i+2}{n-\sum_{j=1}^{i} n_j + 1} & \binom{n-n_i+2}{n-\sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n-n_i+2}{n-\sum_{j=1}^{i-1} n_j} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\binom{n}{n-\sum_{j=1}^{i} n_j + 1} & \binom{n}{n-\sum_{j=1}^{i} n_j + 2} & \cdots & \binom{n}{n-\sum_{j=1}^{i-1} n_j}
\end{vmatrix}
$$

The values of these determinants that result from M_1 and M_r aree vidently 1. In general, (14) is a determinant of a type already discussed.[1] It has been shown that

$$\begin{vmatrix} \binom{m}{p} & \binom{m}{p+1} \cdots \binom{m}{p+r} \\ \binom{m+1}{p} & \binom{m+1}{p+1} \cdots \binom{m+1}{p+r} \\ \cdots & \cdots \\ \binom{m+r}{p} & \binom{m+r}{p+1} \cdots \binom{m+r}{p+r} \end{vmatrix} = \frac{\binom{m+r}{r+1}}{\binom{p+r}{r+1}} \cdot \frac{\binom{m+r-1}{r+1}}{\binom{p+r-1}{r+1}} \cdots \frac{\binom{m+r-p+1}{r+1}}{\binom{r+1}{r+1}}.$$

Hence (14) is equal to

$$\frac{\binom{n}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j}{n_i}} \cdot \frac{\binom{n-1}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j - 1}{n_i}} \cdots \frac{\binom{\sum_{j=1}^{i} n_j}{n_i}}{\binom{n_i}{n_i}}.$$

and the numerical coefficient, k, referred to at the end of the preceding paragraph is

$$(15) \qquad \prod_{i=2}^{r-1} \frac{\binom{n}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j}{n_i}} \cdot \frac{\binom{n-1}{n_i}}{\binom{n-\sum_{j=1}^{i-1} n_j - 1}{n_i}} \cdot \cdots \cdot \frac{\binom{\sum_{j=1}^{i} n_j}{n_i}}{\binom{n_i}{n_i}}.$$

And this is the value of k in (12).

7. We shall now determine the invariant factors of the matrix (4), and as a corollary find the rank of the system of equations (1) for the case where two of the a_i's are equal.

We shall refer to the eliminant of the forms that result when the transformation (11) is applied to the forms (9) as (8′). It has been pointed out that (8)

[1] E. Pascal in *Die Determinanten*, pages 133, 134.

is obtained from (4) by elementary transformations, and that the determinant of the transformation (11) is independent of the a_i's. Hence the greatest common divisor of all minors of a given order in (4) is the same as in (8′). It is desired to choose a minor of given order in (8′) that contains the lowest power of a_2-a_1. (There is no real loss in generality in dealing with a_2-a_1 instead of a_i-a_j. It will again be convenient so to arrange the quantities that n_1 shall be greater than or equal to n_2.) It is evident from Laplace's development in terms of minors from the first rows that any minor of (8′) that is not equal to zero that contains certain of the first n_1 rows must contain the columns with the same numbers. If any minor does not contain any rows involving powers of a_2-a_1 it cannot be divisible by a_2-a_1, since none of the other rows involves a_2. From Laplace's development of a minor of (8′) in terms of minors from the rows involving a_2-a_1 it follows that the minors will be of minimum degree in a_2-a_1 when the number of rows in a_2-a_1 is as small as possible.

Consider a determinant of order $n-l+1$ of the matrix which results when in (8′) l of the rows in a_2-a_1 are omitted ($l \leqq n_2$). Each of these determinants is equal to its $(n-n_1-l+1)$-rowed minor in the lower right hand corner as follows from Laplace's development. For the n_1-rowed minor in the upper left hand corner has the value 1, and all the other n_1-rowed minors from the first n_1 rows are zero. Now expand this minor in terms of minors of order n_2-l from the n_2-l rows involving a_2-a_1. Any minor chosen entirely from rows containing powers of a_2-a_1 will, when expanded be a homogeneous function of a_2-a_1. This can be seen from arguments essentially the same as those used in the proof of Lemma 2. The degree of any element in this minor will be N_i-M_j, where N_i is the number of the column of (8′) from which the element is taken and M_j is the number of the row in a_2-a_1. $\Sigma N_i-\Sigma M_j$ is thus the degree of any term in the expansion of this minor. To make the minor of minimum degree we must take the N's as small as possible and the M's as large as possible.

Theorem III. *Every minor of order $n-l+1$ is divisible by $(a_2-a_1)^{(n_1-l)(n_2-l)}$, and there is no higher power of a_2-a_1 that divides all such minors.*

We now wish to show that there is a minor of minimum degree in a_2-a_1 and of order $n-l+1$ that is different from zero (if the a_i's are all distinct) and that is divisible by exactly the power specified in the theorem. To obtain this minor we omit rows numbered $n_1+1, n_1+2, \cdots n_1+l$, and the columns numbered $n_1+n_2-l+1, n_1+n_2-l+2, \cdots, n_1+n_2$. As was pointed out above, this minor is equal to its $(n-n_1-l+1)$-rowed minor in the lower right hand corner. In the latter, which we shall refer to as (8″), it will be noted that the (n_2-l)-rowed minor in the upper left hand corner is the minor of minimum degree in a_2-a_1 described above, and its complementary minor of order $(n-n_1-n_2+1)$ in the lower right hand corner is the corresponding minor of (8′). Expand (8″) according to Laplace's Development in terms of the (n_2-l)-rowed minors from the first (n_2-l) rows. Each of these minors is homogeneous in a_2-a_1. The term of this sum that involves the leading (n_2-l)-rowed minor involves a_2-a_1 to a lower

power than any other term provided it is not identically zero. Since this term involves a_2-a_1 to the power

$$\sum_{i=1}^{n_2-l}(n_1+i)-\sum_{j=0}^{n_2-l-1}(n_2-j)=\frac{n_2-l}{2}(2n_1+n_2-l+1)$$
$$-\frac{n_2-l}{2}(n_2+l+1)=(n_2-l)(n_1-l),$$

the expansion is divisible by $(a_2-a_1)^{(n_1-l)(n_2-l)}$. The argument given in paragraph 5 shows that the complementary minor of this leading minor is not identically zero. The leading minor itself is not zero, for the determinant of the numerical coefficients is a determinant of the type discussed in paragraph 6.

If we now replace a_2-a_1 by a_i-a_j and allow i and j to take on all possible values from 1 to r inclusive, it is evident that if the product is taken over pairs of values $i, j, (j>i)$ for which $n_i>1$, $n_j>1$, the highest common factor of all determinants of order $n-l+1$ of the matrix (3) is

$$k\prod_i\prod_j(a_i-a_j)^{(n_i-l)(n_j-l)},$$

where k is a numerical constant. From this result the invariant factors and the elementary divisors can readily be determined.

Corollary: *The rank of the system of equations (1) for the case where two of the a_i's are equal is $n+1-min(n_i, n_j)$.*

For suppose that two of the a_i's are alike, in particular, that $a_1=a_2$. It is evident from the preceding results that any minor of (3) will contain (a_2-a_1) as a factor, and hence vanish if $l<n_2\leqq n_1$. If $l=n_2$ there is a minor of order $n-l+1$, selected in the manner described in the proof of the preceding theorem, that involves a_2-a_1 to the zeroth power, and does not vanish. Hence, in general if two of the a_i's are alike, and all the other a_i's are different from each other and from these two, the rank of (3) is reduced by the lower of the two corresponding n_i's.

It is evident from (8) that, if two of the a_i's are equal, its rank is reduced by at least this much, since there are then exactly that number of pairs of rows that are identical. An extension may obviously be made to the more general case in which there are a larger number of equalities connecting the a_i's. Here, after striking out from (8) all but one row in each set of identical rows, the rank of the resulting matrix may be shown to be equal to the number of rows that it contains by a method similar to that employed above.

EARLY LITERARY EVIDENCE OF THE USE OF THE ZERO IN INDIA[1]

By BIBHUTIBHUSAN DATTA, University of Calcutta

In a previous article in this Monthly,[2] the writer collected from the earlier literature of the Hindus certain passages proving the existence of the zero as a distinct symbol and its use as a numeral.[3] Those evidence were mostly from the *Pañcasiddhāntikā* (505 A.D.) of Varāhamihira and works anterior to it. In this second article we propose to collect similar evidence from the posterior Hindu works.

To the earlier period also belongs the Bakhshālī Manuscript.[4] It contains epigraphical as well as literary evidence of the use of the modern decimal place value numerals. For example,

"... become $\left|\begin{array}{c|c|c|c|c} 120 & 90 & 80 & 75 & 72 \\ 60 & 60 & 60 & 60 & 60 \end{array}\right|$. On leaving out (the denominators) of them, result $120 \mid \mathit{90} \mid \mathit{80} \mid \mathit{75} \mid \mathit{72} \mid$. The sum of these being taken, becomes 437 . . ."[5]

"$\left|\begin{array}{c|c} 880 & 964 \\ 84 & 168 \end{array}\right|$ multiplied become $\left|\begin{array}{c} 848320 \\ 14112 \end{array}\right|$. The square of forty different places is $\left|\underline{1600}\right|$. On subtracting this from the number above, the remainder is $\left|\begin{array}{c} 846720 \\ 14112 \end{array}\right|$. On removal of the common factor, becomes $\left|\underline{60}\right|$."[6]

There are innumerable passages of this kind in the work. It will be noticed that in each of these instances, without the figures, the sentences would be grammatically incomplete and the whole composition incoherent. So the figures must have been put there at the time of the original composition of the text, and can not be suspected of being later interpolations For an explicit reference to the zero and an operation with it, we take the following instance:

"... $\left|\begin{array}{c|c|c|c|c} \cdot & 2 & 3 & 4 & \text{visible } 200 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right|$. *Adding unity to the zero* $1 \mid 2 \mid 3$ . . . ."[7]

This will dispel the doubts of even those critics who, while admitting the figures to be authentic, would still doubt whether they were originally given in this form. The Bakhshālī treatise on mathematics was written in the early centuries of the Christian era.[8] Hence the decimal place value numerals with the zero must have been well known to the Hindu scholars of the time.

One of the earliest known writers of the second period to bear witness to the use of zero, is a contemporary of Varāhamihira, Jinabhadra Gaṇi by name. He lived

---

[1] This is the writer's second article with this title. The first was published in this Monthly, vol. 33 (1926), pp. 449–454.

[2] Loc. cit.

[3] The writer is gratified to see that his paper inspired the distinguished German savant, Professor Julius Ruska, to collect similar evidence from the Arabic literature (*Zahl und Null bei Jabir ibn Haiyan*, *Archiv f. Ges. Math. Natur. Tech.*, Bd. ii, pp. 256–264). Jabir wrote about 770 A.D.

[4] *The Bakshālī Manuscript—Parts I & II*, ed. G. R. Kaye, Calcutta, 1927.

[5] *Ibid.*, folio 1, verso. Portions in italics in this passage have been restored.

[6] *Ibid.*, folio 56, verso.

[7] *Ibid.*, folio 22, verso.

[8] Bibhutibhusan Datta, *The Bakhshālī Mathematics*, *Bull. Cal. Math. Soc.*, vol. 21 (1929), pp. 1–60; R. Hoernle, *The Bakhshālī Manuscript*, *Ind. Ant.*, vol. 17, pp. 33ff, 275ff. [But c.f. the Kaye ed. Editor.]

in 529–589 A.D. While mentioning large numbers with several zeros, Jinabhadra Gaṇi often enumerates, obviously for abridgement, the number of zeros contained. For instance 224400000000 is mentioned as "twenty-two, forty-four, eight zeros;[1] 3200400000000 as "thirty-two, two zeros, four, eight zeros."[2] There are several instances of this kind in his work.[3] In certain calculations relating to the mensuration of the northern Bhāratavarsa, which is of the form of a segment of a circle bounded by two parallel chords, it is necessary to extract the square root of 58545048750. Now

$$\sqrt{58545048750} = 241960\frac{407150}{483920} = 241960\frac{40715}{48392}\cdot$$

This result has been described by Jinabhadra Gaṇi thus: "Two hundred thousand forty-one thousand nine hundred and sixty, *removing the zero*, the numerator is four-zero-seven-one-five, and the denominator four-eight-three-nine-two."[4] It should be noted that the term *apavartana* ("removal") means in Hindu mathematics what is called in modern mathematics, the reduction of a fraction to lowest terms by removing the common factors from the numerator and the denominator. Hence the zero of Jinabhadra Gaṇi is certainly not a mere concept of nothingness but is a specific symbol.

Another contemporary mathematician, the elder Bhāskara (not the one living in the 12th century), refers like Varāhamihira, to the subtraction of zero.[5] This writer should not be confused with his celebrated namesake, the author of the *Siddhānta-śiromani* (1150). The former, on his own admission, is a direct disciple of Âryabhaṭa (born 476). We also learn from Pṛthudakasvāmī[6] (860) that he was anterior to Brahmagupta (*b.* 598). So the elder Bhāskara must belong to the sixth century of the Christian era.[7]

Here are two typical extracts from the commentary of Siddhasena Gaṇi on the gloss of Umāsvāti on his *Tattvārthādhigama-sūtra*[8]

"The number of *yojana* in the circumference multiplied by twenty-five thousand becomes this 7905675000. Three *gavyuti* (in the circumference) multiplied by twenty-five thousand becomes this 75000. The number of *gavyuti* are

---

[1] *Bṛhat Kṣetra-samāsa* of Jinabhadra Gani, edited with the commentary of Malayagiri, Bombay, i. 69.

[2] *Ibid.*, i. 71.

[3] *Ibid.*, i. 90, 97, 102, 108, 113, 119, etc.

[4] *Ibid.*, i. 83.

[5] *Mahābhāskarīya*, i. 34. This work is still unpublished. Mss. of it and of the other work of the same author *Laghubhāskarīya* are found in the Government Oriental Mss. Library, Madras. The present writer has transcripts of these two works.

[6] *Brāhma-sphuṭa-siddhānta*, xi. 26 (*com*).

[7] See the writer's article, "The Two Bhâskaras," in the *Indian Historical Quarterly*, vol. 6 (1930), pp. 727–736.

[8] *Tattvārthādhigama-sūtra* of Umāsvāti, with his own gloss, elucidated by Siddhasena Gani, edited by H. R. Kapadia, Bombay, 1926, iii. 11 (*com*).

to be changed to *yojana*, hence on dividing by four, the quotient is 18750. These are the *yojanas*. The number of *dhanu* being also multiplied by twenty-five thousand, become this 3200000, . . ."

"Now the square of the chord is this 75600000000; the square of the diameter multiplied by three hundred and sixty-one is 3610000000000. On subtracting the square of the chord from that, the remainder is this 3534400000000. The square root of this is extracted; *half of the eight zeros are four zeros*; the root of the remaining portion is one-eight-eight; hence the resulting root is this 1880000."

It will be clear from these extracts that Siddhasena Gaṇi employed in his calculation the decimal place-value numerals. In the second extract there is an enumeration of the number of zeros. Elsewhere the writer says, "On removing the four zeros, the quotient obtained after that is this 100000." According to Professor Hermann Jacobi, Siddhasena lived at the middle or end of the sixth century.[1]

The evidence of another distinguished writer, who flourished about the close of the century, is equally conclusive. It is still more noteworthy inasmuch as it comes not from a mathematician but from a romance writer, Subandhu by name. In the *Vāsavadattā* of Subhandhu,[2] we meet with the following metaphor:

"And at the time of the rising of the moon with its blackness of night, bowing low, as it were, with folded hands under the guise of closing blue lotuses, immediately the stars shone forth, . . . like ciphers [*śūnya-bindu*] because of the nullity of metempsychosis, scattered in the sky as if on the ink-blue skin rug of the Creator who reckoneth the sum total with a bit of the moon for chalk."

In the beginning of the seventh century, it is found that a section in the chapter on algebra of the astronomical treatise, *Brāhma-sphuṭa-siddhānta* (628) of Brahmagupta is devoted to the treatment of all the fundamental operations with zero, including involution and evolution. Similar sections are indeed found in the mathematical treatises of almost all the posterior Hindu writers, such as *Triśatikā* of Śrīdhara (*C.* 750), *Gaṇita-sāra-saṁgraha* of Mahāvīra (850), *Mahāsiddhānta* of Âryabhaṭa II (950), etc.[3] These arithmetical operations with the zero certainly presuppose the existence of the zero as a numeral. Brahmagupta's term *taccheda* for the quotient of a quantity divided by zero is especially noteworthy in this connexion. That term literally means "having

---

[1] *ZDMG*, Vol. 60, 1906, p. 289.

[2] *Vāsavadattā* of Subandhu, edited by F. Hall (Calcutta, 1859, p. 182) and translated into English by Louis H. Gray (New York, 1913, pp. 99f). It may be noted that the romantic love of Queen Vāsavadattā and King Udayana had been the theme of several Sanskrit works. The earliest of them seems to be a drama, called *Vāsavadattā*, by Subhandhu, the minister of King Bindusāra (280 B.C.) of Pātalīputra. The famous Patañjali (150 B.C.) has referred to an epic poem of the same name. The available prose romance *Vāsavadattā* was written after the middle of the sixth century. (See *Proc. 2nd Orient. Conf., Calcutta, 1922*, pp. 197–7, 203–213.

[3] Bibhutibhusan Datta, "Early history of the arithmetic of zero and infinity in India," *Bull Cal. Math. Soc.*, vol. 18 (1927), pp. 165–176.

that for denominator"; having in this instance cipher (*kha*) in the denominator, it is equivalent to and has been obviously meant for *kha-cheda*. Now to describe a certain quantity as a fraction having zero for the denominator, requires a knowledge of the characteristics of the zero. Like Jinabhadra Gaṇi, Brahmagupta is sometimes found to have enumerated the number of zeros contained in a particular large number. For example; 4320000 is spoken of as "four zeros, *rada*, *veda*"[1] (*rada* = 32, *veda* = 4); 4320000000 as "seven zeros *rada*, *veda*;"[2] 57753300000 as "five zeros, *guṇa*, *guṇa*, five, *muni*, *svara*, *śara*"[3] (*guṇa* = 3, *muni* = 7, *svara* = 7, *śara* = 5); etc.[4]

In the arithmetical treatise of Mahāvīra, the use of zero occurs also in other connexions besides in connexion with its arithmetic. For instance in explaining a method of finding the sum of a series in geometrical progression, he says:

"The number of terms in the series is caused to be marked (in a separate column) by *zero* and by *one* (respectively) corresponding to the even (value) which is halved and to the uneven (value from which *one* is subtracted till by continuing these processes *zero* is ultimately reached);"[5]

This passage reappears verbatim on two other occasions in the *Gaṇita-sāra-saṁgraha*.[6] It should be observed that the last of these rules is the same as that of Piṅgala (before 200 B.C.).[7]

In enumerating the various "technical terms" (*saṁjñā*) signifying the "numbers" (*saṁkhyā*), for the purpose of a glossary on word-numerals, Mahāvīra first mentions the terms for 1, 2, · · · , 9 and then gives the terms denoting the zero.[8] Hence it is clear that, according to him, the zero is as much a number as any of 1, 2, · · · , 9 is. So Professor Tropfke is not correct in thinking that the zero was not regarded as a number before the seventeenth century A.D.[9]

In describing certain large numbers, Mahāvīra writes:

"The (figures) 7, 0, 2, 2, 5, and 1 are *put down* (in order from the unit's place upwards); and then this (number) which is to be multiplied by 73 . . ."[10]

"In this (problem) *put down* (from the unit's place upwards) 1, 1, 0, 1, 1, 0, 1, and 1, which (figures so placed) give the measure of a (particular) number; . . ."[11]

". . . write down the figures 4, 0, 6, 0, 5, and 9 in order (from right to left) and work out the cube of the number . . ."[12]

Such large numbers have been described as " numbers occupying two or more notational places."[13]

---

[1] *Brāhma-sphuṭa-siddhānta*, i. 7.

[2] *Ibid.*, i. 15.

[3] *Ibid.*, i. 16.

[4] *Ibid.*, i. 22, 51–5, etc.

[5] *Ganita-sāra-samgraha*, ii. 94.

[6] vi. 311½, 333½.

[7] See the first article.

[8] *Ganita-sāra-samgraha*, i. 53–62.

[9] J. Tropfke, *Geschichte d. Elementar-Mathematik*, Bd. II (1921), p. 56. For the Hindu definition of the zero, the reader is referred to the article, *Early History of the arithmetic of zero etc.*, *loc. cit.*

[10] *Ibid.*, ii. 15.

[11] *Ibid.*, ii. 17.

[12] *Ibid.*, ii. 52.

[13] *Ganita-sāra-samgraha*, ii. 30.

In the writings of a contemporary scholar, we find innumerable references to the zero and the decimal place-value numerals. He is no other than the eminent mathematician and commentator Pṛthudakasvāmī. Pṛthudakasvāmī wrote commentaries of the two works of Brahmagupta, viz., *Brāhma-sphuṭa-siddhānta* and *Khaṇḍakhādyaka*. Copies of the commentary on the latter, written in 864 A.D., are available in the Imperial Library, Berlin, and the Deccan College Library, Poona. One copy, collected by Bhau Daji, is now in the library of the Calcutta University, but the earlier parts of it are worn out. We select the following extracts from this commentary:[1]

"That being divided by *dvi-nava-rasa*[2] (namely) by this 692, the quotient in days etc. is 0 |34 |56. That being added to the remainder of the intercalary months, (namely) to this 433 |29 |13, becomes this 434 |4 |9. That being multiplied by thirty becomes this 13022 |4 |30. That on division by *rtu-kha-dik*,[3] (that is) by this 1006, the quotient in days etc, called 'the corrected second,' is this 12 |56 |39."[4]

"The *ahargaṇa* is unity 1; multiplied by *kha-kha-vasu*[5] becomes |800 |. Dividing this by *muni-nakha-dvi-nanda-yama*,[6] (we get) as revolution 0. The remainder multiplied by twelve becomes 9600. (Dividing) by that divisor, (we get) as signs 0. The remainder, this 9600, multiplied by thirty becomes 288000. On division of this by the same divisor, degrees are 0. The remainder 288000 being multiplied by sixty 60 and with the same divisor, minutes are 59. The remainder being multiplied by sixty and divided by the same divisor, seconds are 8. Thus the daily motion of the sun in signs etc., is 0 |0 |59 |8."[7]

"Thus the daily motion of the sun is fifty-nine minutes and eight seconds; that of the moon is seven hundred and ninety minutes and thirty-four seconds; that of the moon's perigee is six minutes and forty seconds; and that of moon's apogee is three minutes and eleven seconds. These are again written in figures (*aṅkenāpi likkhyate*):

Sun	Moon	Moon's Perigee	Moon's Apogee
59	790	6	3
8	34	40	11

These minutes etc., are the daily motions of the sun, the moon, the moon's perigee and moon's apogee, respectively."[8]

---

[1] A photostat copy of the Berlin Mss. of *Khanḍakhādyaka* commentary has recently been secured by the Calcutta University. It is being edited by Professor P. C. Sengupta. Our references here are to the folios of that copy. The *Khanḍakhāyaka* with the commentary of Âmarāja has been edited previously by Pandit Babua Misra for the Calcutta University. The numbering of the verses of the original text in this copy is slightly different from the Berlin copy.

[2] *Dvi* = 2, *nava* = 9, *rasa* = 6; *dvi-nava-rasa* = 692.

[3] *Rtu* = 6, *kha* = 0, *dik* = 10; *rtu-kha-dik* = 1006.

[4] Folios 9 verso-10 recto; i. 11–12 (C.U. ed. i. 8–9).

[5] *Kha* = 0, *vasu* = 8.

[6] *Muni* = 7, *nakha* = 20, *dvi* = 2, *nanda* = 9, *yama* = 2.

[7] Folio 12 recto and verso; i. 14 (C.U. ed. ii. 1).

[8] Folio 13, recto & verso; i. 14 (C.U. ed. ii. 1).

"These are again written in figures:

Sun	Moon	Mars	Mer.	Jup.	Ven.	Sat.	Moon's Peri.	Moon's Apo.
59	790	31	245	4	96	2	6	3
8	34	26	32	59	7	0	40	11

These being multiplied by the difference in longitudes, (namely) by this 120, become respectively (beginning with the moon) 94870 |800 |380 |3772 |29464 | 598 |11534(?6) |240."[1]
There are numerous other instances of this kind in the work.[2]

Finally coming to the tenth century of the Christian era, we find a considerable volume of evidence of the use of zero and the decimal place-value numerals. The commentators such as Praśastidhara[3] (964) and Bhaṭṭotpala[4] (966) have freely used them in calculation. For illustration we take the passage, "adding this to the constant quantity *dvi-utkrti-kha*, (namely) to this 2260 . . ."[5] from the former and "by the number *kha-kha-aṣṭa-dik*, (namely) by this 10800 . . ."[6] from the latter. Nemicandra has several times referred to the addition and subtraction of the zero; e.g.,

"One hundred plus seventeen, eleven, *zero*, and four. . . ."[7]

"Deduct *zero*, one, four and five (in each of the four rows, respectively). . . ."[8]
The zero also enters into certain combinations of things described by him:

"One, two, three and five; *zero*, five, ten and fifteen; *zero*, twenty, forty and sixty; putting down (*saṃsthāpya*) these in three rows for three classes of carelessness, find out the elements or number of analysis and synthesis, i.e., combinations."[9]

"One, two, three and four; *zero*, four, eight and twelve; *zero*, sixteen, thirty-two, forty-eight and sixty-four; putting down these in three rows for three classes of carelessness, find out the number of analysis and synthesis."[10]
We find the use of the decimal place value numerals with the zero in the works of Mādhavacandra Traividyadeva, a contemporary disciple of Nemicandra. It may be noted that the latter has defined a very large number, "one-nine-seven-

[1] Folio 15, verso.

[2] Compare folios 16(verso), 18(recto), etc.

[3] Praśastidhara of Kashmir wrote in 964 A.D. a commentary on the *Laghumānasa* (932) of Mañjula. Ms. of this work is found in the Calcutta University Library, which has been "copied from Ms. No. 583 and compared with other Mss. in the Oriental Library, Mysore."

[4] *Brhat Samhitā* of Varāhamihira, edited, with the commentary of Bhattotpala, by Sudhakara Dvivedi, in two parts, Benares, 1895, chap. ii; *Brhajjātaka* of Varāhamihira, edited, with the commentary of Bhattotpala, by Sitarama Jha, Benares, 1923, vii. 1–2, viii. 4, ix. 1–7, etc.

[5] *Laghumānasa*, ii. 5 (com.); *dvi* = 2, *utkrti* = 26, *kha* = 0.

[6] *Brhajjātaka*, vii. 2 (com.); *kha* = 0, *aṣṭa* = 8, *dik* = 10.

[7] *Gommaṭa-sāra* of Nemicandra, edited with English translation and notes in two parts, by J. L. Jaini, Lucknow, 1927; *Karmakānda*, Gāthā 276, 282.

[8] *Ibid.*, *Karma-kāṇḍa*, Gāthā 383; cf. also Gāthā 94, 322, 376.

[9] *Ibid.*, *Jīva-kānda*. Gāthā 43.

[10] *Ibid.*, *Jīva-kāṇḍa*, Gāthā 44.

nine-twelve-zero-nine-two-nine-nine-nine-six-eight- and thirty-one zeros"[1] (19791209299968000000000000000000000000000) as "4294967296 multiplied by the square of sixteen and then multiplied by twice nine, together with thirty-one zeros." Mādhavacandra has expressed the number in figures.

Another very conclusive proof is supplied by the "Kaṭapayādi" variety of the Hindu alphabetic notations. In this system, as is well known, groups of nine letters of the Devanāgarī alphabets beginning with *ka* and *ṭa*, five letters beginning with *pa* and eight letters from *ya*, denote successively the numerals beginning with unity. The letters *ña* and *na*, sometimes also the vowels standing alone, denote the zero.[2] Thus we have on the whole symbols for nine numeral characters and the zero. These are employed, of course, on the principle of place value. Instances of the application of this system of numeral notation are found as early as the beginning of the sixth century of the Christian era in the Laghu-Bhāskarīya[3] (522) of Bhāskara I. According to the Commentator Sūryadeva Yajvā, the system was in use still earlier, before the time of Ârya-bhaṭa I (*c.* 476).

To recapitulate, in these two articles, we have collected from the earlier literature of the Hindus a volume of substantial evidence proving the existence, as also the use as a numeral, of the zero. They range over a long period beginning before 200 B.C. and ending with the tenth century A.D. It is needless to produce evidence of later writers. On closer analysis, it will be found that the volume of evidence of scholars coming after the fifth century A.D. is more than that of the earlier scholars. Again the evidence of a few, such as Varāhamihira (505), can possibly be explained away as referring to the mere concept of nothingness, but those of many others like Piṅgala (before 200 B.C.), the Bakhshalī Ms., Bhāskara I (*c.* 525), Jinabhadra Gaṇi and others, cannot be so done. With the latter writers the zero is undoubtedly a distinct numeral. There is nothing to prevent us from supposing that the former also used it in the same sense. On the contrary, one will easily be led to this conclusion by the fact that the contemporary writers employed the zero as a numeral.

---

[1] *Triloka-sāra* of Nemicandra, edited, with the commentary of Mādhavacandra, by Manoharalal Sastri, Bombay, 1919, Gāthā 20–1.

[2] For further particulars on this point, the reader is referred to the writer's article, *The alphebetic numeral system* (in Bengali) in the *Sāhitya Parisat Patrikā*, Calcutta, vol. 36, pp. 22–50. The article gives a comprehensive history of the subject with special notice of the Hindu systems. Compare also J. F. Fleet, *The Katapayadi system of expressing numbers*, *JRAS*, 1911, pp. 788ff.

[3] i. 18.

THE DESCRIPTION OF A SURFACE OF CONSTANT CURVATURE

By ROBERT C. YATES, The University of Maryland

1. *Introduction*

The normal planes to a surface cut out curves of radii of curvature ρ. Of such sections, there will be a maximum and a minimum radius given by a certain quadratic relation. If these be called ρ_1, and ρ_2, then $\rho_1^{-1}+\rho_2^{-1}$ is the mean curvature and $(\rho_1\rho_2)^{-1}$ the well known Gauss or total curvature. This last is designated simply the curvature.

2. *Discussion*

We shall be interested here in surfaces of revolution for which the curvature is a positive constant, say

$$(\rho_1\rho_2)^{-1} = a^{-2}. \tag{1}$$

If we reduce both principal radii by an equal amount "a,"

$$(\rho_1 - a)^{-1}(\rho_2 - a)^{-1} = a^{-2} \tag{2}$$

or

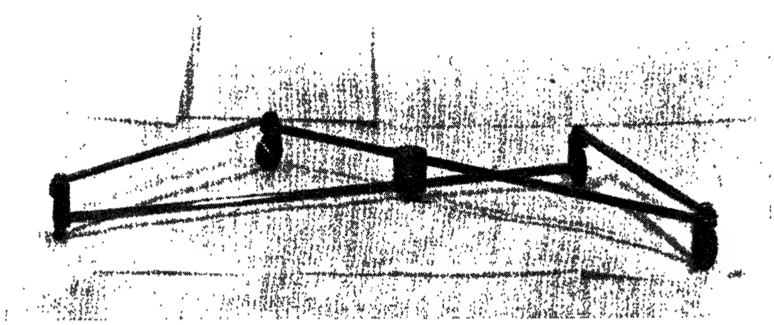
$$\rho_1^{-1} + \rho_2^{-1} = a^{-1},$$

we thus find an associated surface defined with mean curvature constant. Furthermore, this second surface is parallel to the first.[1]

A meridian section of a surface of this sort may be generated by rolling an ellipse along a straight line and taking the curve traced out by a focus.[2] This will give, when revolved about the line as an axis, a surface for which the mean curvature is constant.

3. *Mechanical Description*

Upon the suggestion of Professor Frank Morley, a mechanical device for



constructing both meridian sections was made that will exhibit a fairly large portion of each curve.

[1] A theorem due to Bonnet: Nouvelles annales de mathématiques, Ser. 1, vol. 12 (1853), p. 433.

[2] See, e.g., Greenhill: *Elliptic Functions*, Macmillan and Co., New York (1892), Chapter III.

Two pairs of rods are joined together, the rods in one pair being made equal in length to the major axis, $2a$, and in the other to the distance between the foci, $2c$, of an ellipse. Toothed wheels are placed at the extremities (or at any convenient point) of the rods representing the axis of the ellipse in order that each rod may move at right angles to itself. These wheels cut out two of the four degrees of freedom. The intersection of the rods may be made to move along a straight line, for instance, with the help of a Peaucellier cell.

The wheels themselves, being foci of the ellipses, will trace out the curves for which $\rho_1^{-1}+\rho_2^{-1}=\pm a^{-1}$. The center points of the longer rods will give the sections of the surface of constant curvature. The surface of revolution of the former curves is the unduloid.

It is well to note in closing that these surfaces of revolution of constant mean curvature are those that occur in the theory of soap films.[1]

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

College Algebra. By Lloyd L. Smail. McGraw-Hill Book Co , New York, 1931. xvi+450 pages. Price $2.50.

Here is something really new: a textbook which does not present the subject matter of college algebra as a sequence of disconnected topics, but instead introduces these topics as various aspects of the study of polynomials and other functions. In this way a unified presentation is achieved, the arrangement of material being based upon the systematic study of various types of functions and equations in various numbers of unknowns.

All the traditional topics, with a few exceptions to be noted later, have been included, altho this might not be apparent to one who reads merely the list of chapter titles. Topics which do not seem to fit into the author's outline have been relegated to a final chapter on miscellaneous topics. Perhaps later writers can improve on the present author's arrangement, and can find a more logical place to insert some of these miscellaneous topics. But at the present time, Professor Smail deserves much praise as an innovator, as a pioneer who has had the courage to deviate from the traditional topical arrangement. He has succeeded in so rearranging the topics of College Algebra that the student may feel that he is really going somewhere and is not merely skipping from one independent topic to another.

---

[1] See Maxwell: *Capillary Attraction*, Encyclopaedia Britannica (1924).

The book is divided into thirteen chapters. In the case of each chapter, topics will be noted which are included therein but which would not be naturally suggested by the title of the chapter. The first chapter, an "Introduction," is chiefly concerned with clarifying concepts with which the student has already had some experience. The latter half of this chapter is devoted to complex numbers and their graphical representation, concluding with a section on the geometrical applications of complex numbers based on an article by Professor Smail in this MONTHLY, vol. 36 (1929), pp. 504–511.

The second chapter, "Variables and Functions," contains a more precise discussion of variables and constants than is given in the usual elementary text. Poles, as well as zeros, of a function of one variable are defined. Beginning with section 40, the discussion is concerned with polynomials only and not with functions in general, a fact which the reviewer believes has not been sufficiently emphasized. In fact, the word "polynomial" is used on page 59 ff., but is not defined until page 65. Synthetic division and graphical methods for finding roots of an equation are two topics introduced in this chapter.

Chapters 3 and 4 are entitled "Linear Functions of One Variable" and "Quadratic Functions of One Variable." Under the first of these the topic of arithmetic progressions is introduced; the latter contains the usual development of the quadratic formula.

The next two chapters, 5 and 6, are entitled "Linear Functions of Several Variables" and "Quadratic Functions of Several Variables." In chapter 5, determinants are discussed and are used to solve systems of linear equations. Chapter 6, despite the title, considers functions of two variables only. The treatment of functions is almost entirely graphical and leads to a brief study of the conic sections. In the solution of systems of equations, both algebraic and geometric methods of solution are given.

The next five chapters are concerned with functions of one variable, the titles being: Chapter 7, "Polynomial Functions;" Chapter 8, "Fractional Functions;" Chapter 9, "Irrational Functions;" Chapter 10, "Power Functions and Exponential Functions;" Chapter 11, "Logarithmic Functions." In Chapter 7 is given the binomial theorem for positive integral exponents, Descartes' rule, Horner's method, relations between roots and coefficients, and the other minor topics of the theory of equations. Mathematical induction is casually introduced on page 178, no reference being given there to the more complete discussion of induction given on page 307. The next chapter on "Fractional Functions" shows how to determine the zeros and poles of a rational fractional function, and studies the nature of the graph in the neighborhood of a pole. Another topic considered is that of partial fractions. Chapter 9 contains the method of finding all the nth roots of a number. Topics contained in Chapter 10 are: variation, geometric progressions, compound interest, and annuities. The chapter on "Logarithmic Functions" contains the usual discussion of logarithms and logarithmic computation.

The twelfth chapter is entitled: "Miscellaneous Topics," and discusses these

four topics: permutations and combinations, probability, mathematical induction, binomial theorem for any exponent.

Chapter 13 consists of seventy pages of "Supplementary Exercises," without answers. At the end of each set of exercises in the first twelve chapters is given a reference to the proper page in Chapter 13 on which additional exercises of the same type may be found.

The book concludes with two appendices: (1) "Outline of a systematic logical theory of the number-system of Algebra," and (2) "Summary of Formulas;" a set of tables, answers to most of the odd numbered problems in Chapters 1–12, and an index.

Concerning the details of this book, there is much to be said in praise and much in dispraise, and there are other matters concerning which there are differences of opinion. The reviewer believes that the author is to be praised for making a distinction between the identity and the conditional equality. The symbol introduced on page 27 for "is approximately equal to" deserves to come into more general use. Besides its obvious usefulness in numerical computations, it might also serve a useful purpose in calculus. The author is also to be praised for giving definite references to more advanced books for matters which are "beyond the scope of the present work." These references should serve to show the more ambitious pupils that there is still something to learn after this course has been completed. The abundant use of graphical methods thruout the book is another matter deserving of praise.

But on the other hand, the book has a number of defects. It is marred by numerous misprints. The student may be confused, to say the least, by being told on page 30 that $i^3=-1$, on page 127 that $(-2)(-6)=-12$, and on page 200 that the product of the roots of a quartic is equal to *minus* the constant term. There are still other errors which are not due to misprints. On pages 6 and 7, Law XV (Division is always possible and unique) is hardly consistent with Law XVIII (Division by zero is not defined and is not permissible). The statement on page 95 that the area $OALN$ represents the sum of the first n terms is *false*. Under the topic of probability, the solution of Example 2 on page 299 is entirely wrong. It has been noted before that *polynomial* has been used before being defined. Similarly, at the bottom of page 183 certain information about $P(-x)$ is used which is not given until page 186. The discussion of geometric progressions on page 244 is a little awkward, because a is necessarily positive in the exponential function ka^x, and it may not be obvious therefore to the student that the ratio in a geometric progression may be negative. It is also confusing to have a used in two different senses on pages 244–5, where ka^x is replaced by ar^{n-1}.

As has been noted before, arithmetic progressions and geometric progressions are not taken up together, as is the usual custom. Whether this is a good or a bad point seems to be largely a matter of opinion. Nothing is said about harmonic progressions. Other topics omitted are: infinite series (except for the topic of infinite geometric progressions in Chapter 10), limits, scales of notation, and

continued fractions. Nothing is mentioned concerning the history of algebra. Here again, it is largely a matter of opinion as to whether or not the author is to be blamed for omitting any of these topics.

From the above discussion, it can be seen that the underlying principles upon which the framework of the book has been constructed are sound ones, but that some of the details of construction are faulty. The book is an unusual one, and it is to be hoped that it will prove to be popular and successful.

HARRY MERRILL GEHMAN

The Mathematical Part of Elementary Statistics. By Burton Howard Camp. D. C. Heath & Co., Boston, 1931. 409 pages. $3.60.

As the author tells us in his preface, this book consists of two half-year courses in statistics: the first strictly elementary and the second more advanced. Students are presumed to have a knowledge of analytic geometry, but not necessarily of calculus. It is the practice at Wesleyan for many students to take only the first half, and it is apparently the author's belief that only those who are fairly proficient in mathematics should take the second half. Opinions will of course differ as to the sufficiency of such a preparation and the advantages of such a division.

Before a detailed discussion of the two parts is begun, some general statements are in order. The critic's reported remark regarding one text that, "if this is mathematical, so is the *Book of Numbers*," is not applicable here. This is decidedly a text-book in mathematics; it is organized in the usual way, with plenty of model examples and with problems that can be freely assigned as in a course in calculus. Teachers of mathematics who have wrestled with undergraduate courses in statistics will realize that this is probably the most important thing that I shall say. Features not so good are the lack of an index and of consecutive numbering of paragraphs through the book, resulting not infrequently in a sort of prolonged "treasure hunt" when tracking down a reference. There ought also to the provided a set of answers in a separate pamphlet for the benefit of both teachers and students.

Part I comprises about half the book. It is divided into ten chapters, of which four are given to moments, averages, measures of dispersion, etc.; two to the normal law and its simpler applications; one to time series, especially the fitting of trend lines; and the remaining three to correlation between two variables. On the whole the emphasis is on methods of calculation and the meaning of the many terms. In the first four chapters most of the formulas used are proved, a strong feature being considerable training in the use of the summation sign. In dealing with the normal curve the lack of calculus is partly compensated for by introducing the definite integral as a symbol for the area between two ordinates, a device which provides a convenient notation and permits the checking of some of the properties of the curve with the aid of the tables. Other properties are checked approximately with the aid of a histogram constructed from the tables. It seems to the reviewer that it would have been well here as well as later to in-

sert the demonstrations by calculus as supplementary sections. The remaining chapters seem to call for no special comment.

A salient feature of the treatment is that the author lives up to his title; we have here a well planned and well presented introductory course, unmarred by long discussion of problems from other fields of learning of which the student may know nothing. The student who has completed this course knows the rudiments and is ready to take up the special problems of economic or educational statistics, if he is interested in them, with the departments concerned. The only objection which the reviewer has to offer is his personal belief that a course in statistics which does not include probability and sampling is woefully incomplete.

Part II consists of seven chapters: I, "Probability"; II, "Approximations to the point binomial"; III, "Frequency curves"; IV, "Sampling"; V, "Further topics in correlation"; VI, "Multiple correlation"; VII, "Finite differences." This is followed by a set of tables, which are also published separately. The titles of Chapters I, II, IV, and VI sufficiently describe their contents. The Gram-Charlier expansion is the basis of Chapter III; the further topics in correlation include errors of estimate, the correlation ratio, and polychoric correlation; and Chapter VII is mainly interpolation methods.

On the whole the commendatory remarks on Part I apply also to this part, but with certain reservations. The first and less important is a doubt that students whose training does not include calculus have sufficient mathematical maturity to comprehend the theoretical work involved. If this doubt is well founded, the reviewer believes that it would be better to use calculus freely, instead of the ingenious substitutes for it.

A more important cause for concern is the treatment of fundamentals in the theory of probability. The author avoids the vicious circle in the classical definition by first defining equi-probability of two events by a limiting process and then introducing the classical definition. As he says on page 186, this is equivalent to the limit definition of probability, used by many writers on statistics. The objections which the reviewer wishes to raise are then really directed against current practices in statistics, rather than against this particular book.

The limit definition assumes that the probability p of an event is the limit of u/n, as n approaches infinity, where u is the number of successes in the first n trials of a random sequence. For example, we may imagine an immortal tossing a "good" penny, in which case we believe that the limit will be $\frac{1}{2}$. If it is legitimate to speak of the limit of such a sequence, can we not say that for any positive ϵ, say $\epsilon = .01$, there will be an integer n' (unknown, of course) such that $|\frac{1}{2} - u/n| < \epsilon$ for every $n \geqq n'$? Fixing n, we have $\frac{1}{2}n - n\epsilon < u < \frac{1}{2}n + n\epsilon$. Suppose now that we watch the first $2m$ trials and that these yield k successes, where $k = m + s$, $s \geqq 1$. We then know that the next $n - 2m$ trials will yield r successes, where

$$\tfrac{1}{2}n - m - n\epsilon - s < r < \tfrac{1}{2}n - m + n\epsilon - s.$$

As far as logic or our knowledge go, the results of the next $n - 2m$ trials may

come out in 2^{n-2m} ways. Are these not equally probable? But only those are *possible* for which r satisfies the above inequality. Waiving this difficulty, we readily find by algebra that among the possible results of the next $n-2m$ trials, those in which the first yields a failure are more numerous. Is the tyro right in believing that after a run of heads, a tail is more likely than a head?

The phrase "infinite set of balls in the ratio p white to q black" comes as a shock to a disciple of Cantor, but is of course defensible as a convenient abbreviation in connection with the surrounding context (pp. 241, 293). It does, however, lead one to forget that the author's work does not embrace probability applied to infinite sets and the teacher may find it necessary to clear up the following sort of difficulty. Suppose that the balls in question have the integers 1, 2, 3, $\cdots$, ad infinitum printed on them. What is the probability that, if two are drawn *at random* in succession, the second will have a higher number than the first? Is it obviously $\frac{1}{2}$ and, if so, why? Whatever the integer n on the first ball drawn, there is an infinity of balls having a higher number left and only $n-1$ having a lower one. What is the explanation?

Presumably an expert in statistics will find it easy to dispose of the questions raised in the last two paragraphs. However, many, if not most, teachers of elementary statistics are not experts and are forced to rely more on the text used than in other branches of mathematics. For their sakes the reviewer has raised these objections, chiefly with the hope that they will be speedily demolished and with no desire to detract from the merits of one of the most important textbooks of recent years.

WALLACE A. WILSON

Differential and Integral Calculus. By N. J. Lennes. Harper and Brothers. 1931. xiii+471 pages. Price \$3.00.

Once again Professors Lennes and Slaught have produced an excellent textbook in mathematics. This time Professor Slaught functioned as the editor of the series published by Harper and Brothers.

This book is beautifully set up and printed and paged. There are but very few typographical errors and only one or two figures that might have been added or improved. On page 118 $du^u = uu^{u-1}\, du$ should read $du^n = nu^{n-1}du$. On pages 47 and 49 a good drawing or two to illustrate one or more of the examples on tangents and normals to curves would have been an improvement. On page 120 the drawing of the cycloid looks distorted.

It is easy to see that the author of this text is a good teacher of long experience and is also a good mathematician who delights in rigor of proofs. The text shows this plainly by a combination of great teachableness with the greatest possible logical rigor for an elementary text.

Experienced teachers of the calculus will notice the following good points of this book. The figures are placed on the same page with the accompanying text or on the directly opposite page. The notation D_xy is used for the derivative of y with respect to x, and dy/dx is not introduced until differentials are discussed.

Several drawings are given on pages 69 and 81 showing the first derived curves below the original curves. Inverse functions are given a full and careful treatment. A chapter on the handling of series is given, but the proofs of the theorems used therein are postponed to a later chapter. The derivatives and integrals of infinite series are well discussed. There is a good explanation of the use of implicit functions in the solution of problems in maxima and minima. A good treatment is given of the total differential.

A sound "elementary core" of material characterizes the book, more advanced material being separated and put into later chapters or under the heading of Supplementary Topics. The applications to physics and other subjects are copious but are segregated. This makes the book adapted for use with both Liberal Arts and engineering students. The chapters of this text are so arranged that the integral calculus can be taught after the differential calculus or derivative and antiderivative can be taught together. This should please two distinct groups of teachers and increase the clientele who will use the book.

The constant of integration is emphasized and used freely before the definite integral is introduced. A splendid discussion of the different elements of integration is given on page 224, based upon infinitesimals. A chapter on solid analytics is introduced for the sake of students who have missed or are weak in this subject. A collection of formulas from algebra and trigonometry is inserted. No answers are given, but hints on the working of certain problems are given. Hyperbolic functions and their relation to a rectangular hyperbola are well treated. Sight work is given; huge sets of problems labelled Sets A, B, and C are an excellent feature. A cumulative review (designed to help the student relearn the subject matter) is included in the book, also a very good historical sketch.

Enough problems are worked through in the text to help the student. The book is very honest as to what is proved and what is assumed. There are long and careful explanations and common-sense approaches to the different topics treated in the text. There is continual harping on the fundamental problem of the differential calculus as being a study of change and rate of change. Such difficulties as that connected with the concept of an irrational exponent are enlarged upon in the book. The author does not hesitate to repeat discussions in different settings and with different wording for the sake of emphasis.

The excellence of such definitions as those of function, continuity, tangent should be mentioned. It is pleasing to find in an elementary book such concepts as those of neighborhood, interval, pole of a function, and the like. The chapter entitled Supplementary Topics contains the methods of approximate integration, subnormals, the second mean value theorem, Pappus' theorem on areas, and many other excellent topics. Following this chapter there is a brief discussion of differential equations.

Let us now consider some few criticisms of the text. The idea of multiple integration is led up to by a problem on moments of inertia. It would have been better to use also a problem on volumes. On page 87 an approximation to e is to

be found, with the unsupported statement that the result is certainly correct to eleven places of decimals. Under differentials we find the "proof" that $\Delta x = dx$. It would have been better to say "We may take $dx = \Delta x$." Under the discussion of

$$\int du/(u^2 - a^2) = (1/2a) \log [(u - a)/(u + a)]$$

no mention is made of the cases where $u - a < 0$ or $u + a < 0$. On page 206 the integral $\int dx/\sqrt{(x^2 + a^2)}$ is obtained by a method that belongs more suitably with differential equations. On page 218 no justification is given for the change of variable in integration, and the discussion here is too brief. The discussion of multiple integrals seems too short. The students wants to know how to select the limts of integration, why certain elements of integration are used, and what the first integration gives. In one place the term "least upper bound" is given without any definition of its meaning. The index is incomplete. In the discussion of limits the independent variable is not allowed to reach its limit. This makes for a neater discussion, but it puzzles the student when he comes to other fields of mathematics where the independent variable is allowed to reach its limit and where such an expression as $1/0$ does have a meaning.

Returning to praise of this text-book, we note that it has a wealth of material all well-arranged and a wealth of problems all well-graded. A student should be able to master the subject using this text, with very little or no help from an instructor. It is easy to believe the story of the growth of this book (as recited in the author's preface), namely that the text was taught in mimeographed form for several years and that the students were asked to criticize the various ways of presenting any topic and the different sets of problems. This coöperation has resulted in a very teachable book and one as rigorous as is possible for an elementary text-book on the calculus.

ALAN D. CAMPBELL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities should be sent to F. M. Weida, The George Washington University' Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

LOCAL MATHEMATICS CLUBS

1930–1931

The Mathematics Club of Hunter College

The Mathematics Club of Hunter College has for its object the stimulation of interest in mathematics through the presentation of papers on topics allied with, but not treated in, the regular college courses, and through social contacts of all interested persons. Program meetings are

held every other Tuesday and are open to the students at the Main Building, most of whom are juniors and seniors.

The meetings and programs were as follows:

September 30, 1930: "Algebraic differentiation" by Anna Golden.
October 14, 1930: "Geometric applications of complex numbers" by Sylvia Radlich.
October 28, 1930: "An example of the analytic theory of numbers" by Roselyn Solomon.
November 11, 1930: "The catenary" by Miriam Piesman; "Suspension bridge problem" by Muriel Levine.
November 25, 1930: "Factorization of determinants" by Bertha Sperber.
December 9, 1930: "Affine transformations" by Harriet Cohen.
January 5, 1931: "Summation of series" by Ida Berenson.
February 10, 1931: "Farey series" by Annette Vassel.
February 24, 1931: "Nets of triangles" by Caroline Ullman.
March 10, 1931: "Partial fractions" by Florence Eisinger.
March 24, 1931: "Orbits of Heavenly bodies" by Dorothy Steinberg.
April 14, 1931: "Summation of p-series" by Esther Sass.
April 28, 1931: "Transcendental numbers" by Professor Jewell C. Hughes.
May 12, 1931: "Approximate integration" by Professor J. Hobart Bushey.

The officers for 1930–1931 were: Violet Moskowitz, President; Esther Sass, Vice President; Bertha Sperber, Secretary; Irene Larson, Treasurer; Miriam Peisman, Publicity Manager; Professor Louis Weisner, Faculty Adviser.

The officers are elected in the Spring semester at a special business meeting held after the program meeting.

Among the social affairs of the club was a fall "get-together" party to which the Mathematics Clubs at the Annexes and new lower juniors in the Mathematics Club were especially invited. The entertainment was furnished by the Annex Clubs. Twenty-ninth Street Annex presented a play "Analytics College," while Thirty-second Street Annex delivered a fantastic dramatic poem dealing with the various numbers seen by a student in a nightmare. There was also a little afternoon tea held in the council room, to which the officers of all the clubs were invited. The purpose of this tea was to discuss the advisability of having a series of educational talkies under the auspices of the Mathematics Club. The talkies were given on April 24th..The series comprised: "Woodwind Choir"; "Diagnosis of difficulties in arithmetic"; "The play of imagination in geometry"; "Finding the right vocation"; "Acoustic principles"; "Finding his voice."

The president, in the name of the club, presented a corsage of roses to Professor Simons, the head of the mathematics department and sponsor of the talkies. She also thanked the members of the committee, which consisted of the officers with Margorie Miller and Marion Leary. In addition to the interest in the talkies the affair was a great social success, as was testified by the number attending the little social after the pictures. The strong department spirit was in evidence in this event outside the usual routine as it is in the every day demands on the mathematics majors.

The activities of the clubs at the several annexes are along the same lines as those in the Main Building. It is possible to keep the topics well within the understanding of the groups of students who are located in each building. Some of the outside speakers at these clubs were: Dr. Payne, who spoke on "Relativity simply explained," and Professor Gill, who spoke on "Continued fractions." Both were from New York University.

BERTHA SPERBER, *Secretary*

The Albion College Mathematics Club

The activities of the Albion College Mathematics Club for the year 1930–1931 were as follows:

The officers were: Virginia Bray, President; Pearl Yates, Vice President; Dalon Ely, Secretary-Treasurer.

The meetings and programs were:

October 1930: "The application of mathematics to every day life" by Professor Gilbert A. Bliss of the University of Chicago.
November 1930: Roll Call—Algebraic formulas. Two reports: "The Pythagorean theorem and its proofs"; "The history of the magic square."
January 1931: Roll Call—Geometric formulas. A talk, "Graphs and charts" by Professor E. E. Ingalls.
February 1931: Roll Call—A statement by some prominent mathematician. A contest with trick mathematical problems.
March 1931: A report, "The history of Pi"; Initiation of new members.
April 1931: Roll Call—Geometric theorems. A report, "The simple uses of the slide rule."

DALON ELY, *Secretary*

Sigma Delta Rho, Honorary Mathematics Fraternity of Southern Methodist University.

The club has been organized only since March so our activities for this year are few.

The officers, elected March 2 by a majority vote are: Alice Gillespie, President; Claude Albritton, Vice President; Frances Francis, Recording Secretary; Grace Decker, Corresponding Secretary; Artemus Roberts, Treasurer.

The primary aim of the club is to promote interest in the study of mathematics.

Members must be in the upper third of their class and must have an average of 2.5 in mathematics and 2 in all other courses for their entire university record. They must have completed at least six hours of advanced mathematics and be taking another advanced course. Only one-third of graduate students are eligible. There are twenty-one active members on the roll.

The meetings and programs were as follows:
April 27, 1931: "Partial fractions" by Dr. Edwin Mouzon.
May 18, 1931: "Why mathematics" by Fitz-Hugh Marshall.
June 7, 1931: Banquet.

We had an open house on March 24, 1931.

GRACE DECKER, *Corresponding Secretary.*

The Mathematics Club of the College of the City of Detroit.

The officers for 1930–1931 were: William W. Beeman, President; Rosina Mohaupt, Secretary; Joe Kursman, Treasurer.

The meetings and programs were as follows:
December 9, 1930: "Transcendental numbers" by Dr. K. W. Folley; election of officers.
January 13, 1931: "Scales of notation" by Mr. Chazkel Falik.
March 10, 1931: "The domain of the imaginary" by Mr. Hyman Wagman.
May 5, 1931: "A mathematical theory of monopoly" by Miss Rosina Mohaupt.
June 2, 1931: "Some elementary transformations of conics" by Mr. H. H. Pixley, Instructor in mathematics.

ROSINA MOHAUPT, *Secretary.*

Delta Nabla Fraternity of Westminster College.

In order to be eligible for membership in Delta Nabla Fraternity of Westminster College, one must intend to major in mathematics and must have completed four semesters or three semesters with an average of "B" in the subject. The active membership is of such of the college students of good character and scholarship as have been unanimously approved by the active members of the fraternity, and have been regularly initiated in conformity with the fraternal ritual. At present there are eighteen active members. Meetings are held the first and third Tuesdays of each month.

The purpose of Delta Nabla is to foster mathematics, striving to bring about a deeper interest in and a more extensive study of the subject on the campus.

The officers for 1930–1931 were Pearl Hoagland, '31, President; Edmund Barnes, '31, Vice

President; Hazel Bergland, '31, Secretary-Treasurer. The officers are regularly elected the second Tuesday in April, by secret ballot, and by a majority vote.

The meetings and programs were as follows:

November 18, 1930: "Mathematical wrinkles" by Donald Cleland.

November 30, 1930: Formal initiation and dinner.

December 9, 1930: "Discussion on conics" by Pauline Robinson; Alice Bell told of her experiences while doing her practice teaching; "Mathematical puzzles" by Edmund Barnes.

March 21, 1931: "Mathematics in music" by Mr. Donald Cameron, an assistant in the Conservatory of Music.

April 14, 1931: Election of officers for 1931–1932.

May 1, 1931: Election of new members.

May 13, 1931: Formal initiation.

HAZEL BERGLAND, *Secretary.*

The Mathematics Club of Creighton University.

The mathematics club of Creighton University was organized in the fall of 1929 by Professor Alvin K. Bettinger, head of the mathematics department. It adopted a two-fold purpose, to benefit the mathematics department by bringing all those interested in mathematics together for a discussion and expression on problems and views of the science. In addition, it endeavored to create in each student member a deeper interest and to give a broader outlook on the science in general.

The club operates under a constitution and a board of officers who are elected in May for the following year. The present active membership is eighteen.

Membership in the club shall be open to any student in the Creighton University who is interested in mathematics. Candidates for membership shall file application with the membership committee. The membership committee shall have the power to admit or reject the application of any candidate. If any member shall disagree with the decision of the membership committee the decision shall be placed before the house and decided by a majority vote of the members present.

The meetings and programs were as follows:

October 15, 1930: "The history of geometry" by Mr. Frank E. Marrin; "The definite integral" by Mr. Wendell A. Dwyer.

November 18, 1930: "The Rhind Papyrus" by Mr. Wendell A. Dwyer; "Numbers" by Professor Alvin K. Bettinger.

December 9, 1930: "The life of Steinmetz" by Joseph C. McCarthy; "The life of Descartes" by Joseph A. Flynn.

January 15, 1931: "The Roger Bacon manuscript" by Torrence D. Kay; "The parabola and some of its applications" by Gordon Hannon; "A unique application of the parabola" by Mr. Frank E. Marrin.

February 12, 1931: "The squaring of the circle" by Samuel Steinberg; "The solar system" by Hugh M. Schwaab; "The mechanical investigations of Leonardo da Vinci" by Mr. Wendell A. Dwyer.

March 26, 1931: "The Newton-Leibniz controversy" by James P. O'Brien; "The slide-rule" by Robert J. Myers; "Different methods of rapid calculation" by Henry Sterling.

April 28, 1931: "Some applications of mathematics to physical chemistry" by James P. O'Brien; "Cryptograms and code messages" by Edward J. Solomonow; "Report on the Mathematics-Chemistry symposium held in Indianapolis by the American Chemical Society on March 31, 1931" by Robert J. Myers.

The officers for 1930–1931 were: Robert J. Myers, President; Henry Sterling, Vice President; Torrence D. Kay, Secretary and Treasurer.

TORRENCE D. KAY, *Secretary.*

Mathematics Club of the Eastern Illinois State Teachers College.

The officers for 1930–1931 were: Ralph Evans, President; William Peters, Vice President;

Pauline Schmidt, Secretary-Treasurer; Mr. E. H. Taylor, Head of the mathematics department, Faculty Advisor.

The meetings and programs were as follows.

September 24, 1930: "The mathematics club" by Professor E. H. Taylor; Social hour.

October 8, 1930: "The mathematical cross word puzzle" by Forest Montgomery; "The nine point circle" by Agnes Gray.

October 22, 1930: "History of the Calendar" by Thelma Quicksall; "Life and work of Archimedes" by Odin Stogsdill; Group picture taken from the school annual.

November 5, 1930: "Unsolved problems" by Martha Spangler; "Mathematical fallacies" by Gertrude Hendrix.

December 3, 1930: "Life and work of Galileo" by Ralph Evans; "Theory of Probability" by Professor S. B. Townes.

January 7, 1931: "The sextant" by John Black; "The transit" by Milton Baker; "The application of mathematics to Fine Arts" by Alfrida Schuetz.

January 21, 1931: "The story of arithmetic" by Glenna Albers; "The land of the fourth dimension" by Odin Stogsdill; "Mathematical wrinkles" by Katheryn Sebright.

February 4, 1931: The annual ciphering contest between the various high school and college classes. This contest was won by the seniors of the college.

March 4, 1931: "Purpose and work of mathematical associations" by Professor E. H. Taylor; "Why students fail in mathematics" by William Peters.

March 18, 1931: "The slide-rule" by Professor O. L. Railsback of the Physics department; "School Science and Mathematics" by Virgin King; "The Mathematics Teacher" by Katheryn Sebright.

April 8, 1931: "The Planetarium at Berlin" by Helen Richey; "The great pyramid" by Paul Henry.

May 6, 1931: "Fanciful hypothesis of the origin of numerical forms" by Anna Balmore; "History of Pi" by Robert Newman; "History of the minus sign" by Eva Schaehrer.

May 20, 1931: Election of officers for 1931–1932.

PAULINE SCHMIDT, *Secretary*.

Mathematics Club of Denison University.

The officers for 1930–1931, elected in the spring of the previous year to serve one full year, were: John Rowerton, President; Harrison Korner, Vice President; Robert Bridge, Treasurer; Mary Jane Lamson, Secretary.

The club was founded in 1911. At present we have 37 active members. The purpose of this organization is to bring together those of the faculty and students of Denison University who have a common interest in mathematics to consider mathematical topics which are not treated in scheduled courses in this university.

Any student in Denison University is eligible to membership. The names of candidates for membership are presented at a regular meeting of the club and voted upon at the next succeeding regular meeting.

The meetings and programs were as follows:

October 7, 1930 "Imaginary numbers" by Dr. F. B. Wiley, Head of Department of Mathematics.

October 27, 1930: "Large numbers" by Dr. Wiley.

December 2, 1930: "Probability" by Russel Geil and Harrison Korner.

December 16, 1930: "Oriental mathematics" by Chosaburo Kato, Instructor of mathematics.

January 20, 1931: "Napier's bones" by Mr. Donald Fitch, Registrar.

February 10, 1931: "Mathematical philosophy and its contribution to life" by Dr. Wiley.

April 21, 1931: "Projections" by Anna B. Peckham, Instructor of mathematics.

March 3, 1931: "Exposition vs. expose in mathematics" by Professor Henry Blumberg, Ohio State University.

March 17, 1931: "Units" by Professor Leon Smith, Physics department, Denison University.

At Christmas time, one meeting is given over to puzzles and games, and at the end of the

school year a club banquet is given. In 1931, Professor Rado of Ohio State University gave the address.

MARY JANE LAMSON, *Secretary.*

The Napierian Club of De Pauw University.

The officers for 1930–1931 were: Howard F. Fetters, '31, President; Lenore Ruark, '31, Vice President; Mary McCord, '31, Secretary; Horace Barnett, '31, Treasurer.

The meetings and programs were as follows:

October 2, 1930: Election of new members.

October 9, 1930: History of Napierian club and life of John Napier" by Lenore Ruark; "Napier's logs" by Robert Heritage.

November 13, 1931: "Some problems of mathematical recreations" by Horace Barnett; "The squaring of the circle" by Mary McCord.

December 11, 1930: "Greek in relation to mathematics" by Professor Stephenson of the Greek department; "Hyperbolic functions" by Robert Heritage.

January 15, 1931: "Determinants" by Willard Smith; "Women in the history of mathematics" by Ruth Brookshire.

February 19, 1931: "History of the teaching of mathematics in the Indiana High School" by Hubert Trisler; "Non-Euclidean geometry" by Richard Humphreys.

March 19, 1931: "Some mathematical fallacies" by Dorothy Wurst; "Projectivity" by Professor W. C. Arnold.

April 23, 1931: Social meeting.

May 14, 1931: Joint meeting with the Science Club.

MARY MCCORD, *Secretary.*

The Carleton College Mathematics Club and the School of Crotona.

The Carleton Mathematics Club is an open organization directed by a group of advanced mathematics students, called the School of Crotona. This group is self perpetuating as all nominations for membership in it are made by the members themselves with the ratification and approval of the teachers of the Carleton department of mathematics. Before admission to the School of Crotona, the candidate must pass a rigorous oral examination given by the staff of the mathematics department—the scope of which examination covers all his college work in mathematics. To be eligible for nomination to the School of Crotona, the student must have completed the usual college courses through integral calculus.

The School of Crotona sponsors every month or six weeks a meeting of the Carleton Mathematics Club. The purpose of these meetings is to arouse in the beginning mathematics student an appreciation for higher mathematics and analytical reasoning. This purpose is furthered by limiting the talks by the professors and the papers read by undergraduates to subjects which are not too technical. The usual attendance at a club meeting is about forty.

The officers of the mathematics club are elected by the School of Crotona every June for the next school year. The officers for the college year 1930–1931 were: Lester S. Sinness, President; Harry J. Burton, Vice President; Luther Ford, Secretary-Treasurer.

There were four meetings of the Carleton Mathematics Club during 1930–1931 as follows:

October 10, 1930: Lester Ford led a discussion on the subject "Geometric fallacies."

December 10, 1930: Dr. C. H. Gingrich, Professor of mathematics, talked on the "Square roots of unity with extensions."

March 5, 1931: An open discussion meeting was held on the subject "The history of numbers and notation." H. J. Burton read a survey paper briefly sketching the field of mathematical notation. Miss Eleanor Walker gave in more detail the history of rational numbers. William Lee discussed "Irrationals" and S. Kjontvedt told how imaginary numbers were discovered. Miss Marion B. White, Assistant Professor of mathematics, concluded the discussion with a more detailed talk on imaginary numbers and their significance.

April 24, 1931: Dr. Oliver Dimon Kellog, Professor of mathematics at Harvard University and

Harvard exchange Professor at Carleton, spoke on the "Four color problem of map making."

W. C. CONLEY, *School of Crotona.*

The Irrational Club of the University of Wyoming.

The Irrational Club of the University of Wyoming meets twice each month to discuss various phases of mathematics for which there is no time in regular classes. The only requirement for eligibility is interest in mathematics. Three social meetings are held during the year, one each quarter.

At the first social meeting, which was the first meeting of the year, held early in the fall, officers for the year were elected as follows: Earl Johnson, Positive Square Root; Lillian Carlson, Negative Square Root; Ada Burke, Keeper of the Logs and Bones; Mr. C. F. Barr, John Ferrero and Evelyn Moore, Custodians of the Indices.

This year the club numbered nearly sixty active members, the largest membership the club has known.

Among the interesting talks and reports were included the following topics: "The trisection of an angle," "The history of the decimal point," "Logarithms—the history of their tables," "Mathematics of finance," "The history of Pi," "The degree of accuracy obtainable with engineering tools," "Fourth dimension—for this topic we had a film illustrating Einstein's theory," "The nine point circle," "Euler line," "The use of mathematics in business," "The slide-rule," "Astronomy."

The traditional irrational club beefsteak fry was held late in the spring at Telephone canyon which brought to a close a most successful year for the club.

ADA BURKE, *Keeper of the Logs and Bones.*

The Mathematics Club of the North Carolina College for Women.

The mathematics club of the North Carolina College for Women had the following meetings and programs during the year 1930–1931:

October 1930: "Review of the 'Pastures of Wonder'" (Keyser) by Pickett Henderson and Edna Reams; "The mathematics problems of Edison's test" by Kathleen Nowell.

November 1930: "Review of 'Number,' the language of science" (T. Danzig), by Edna Livingstone and Blanche Fisher; "A daylight meteor" by Virginia Allen.

December 1930: "Review of 'The watchers of the sky'" (Noyes) by Annie Fawcette; "The Adler Planetarium" by Mrs. George Fullerton.

February 1931: Initiation of new members. Play—"The mock trial of B versus A."

March 1931: "Review of 'Mathematics and Poetry'" (Buchanan) by Annie Lee Thompson; "Number Lore" by Kathleen Cox.

April 1931: "The aesthetic measure of a polygon" (from the sixth year book of the National Council) by Mary Ellen Boss and Elizabeth Morgan. The reports were illustrated by an interesting set of figures in the construction and mounting of which the class in the Teaching of Geometry cooperated.

May 1931: Election of officers. Social meeting was in charge of the new freshmen members.

The officers for 1930–1931 were: Annie Fawcette, President; Virginia Barker, Vice President; Martha Pickett Henderson, Secretary-Treasurer; Cornelia Strong, Faculty Counsellor for the club.

CORNELIA STRONG, *Faculty Counsellor.*

Euclidean Circle, Indiana University.

The officers for 1930–1931 were: Mildred Corrie, '31, President; Oliver Dixon, '31, Vice President and Treasurer; Ida Mae Lloyd, '31, Secretary.

The meetings and programs were as follows:

October 1930: Business meeting. "The history of Euclidean Circle" by Professor Hennel.

November 1930: Fall initiation.

January 1931: "The new planet 'Pluto'" by Professor W. A. Cogshall of the Astronomy department.
February 1931: Members of the club presented the historical play—"The evolution of numbers." Initiation for second semester students.
March 1931: "Mathematics and the laws of nature" by Professor H. T. Davis.
April 1931: "The development of the subject 'Mathematics'" by Professor S. C. Davisson.
May 1931: "Mathematics and ornament" by Professor T. W. Moore. Election of officers for 1931–1932 was held.
June 1931: The annual club picnic was held at the Cascades Park.

MILDRED CORRIE, *President.*

Tuftconic, Tufts College.

The officers for 1930–1931 were: Eleonora L. Czerniewska, '31, President; Gordon Clark, '32, Vice President; Pearl A. Russell, '31, Secretary; Raymond Smith, '33, Treasurer.

The meetings and programs were as follows:
October 9, 1930: "Is X a whole number?" by Professor H. L. C. Leighton.
November 6, 1930: "Numbers of Bernoulli" by Raymond Yeaton, '31.
December 16, 1930: "How those Frenchmen did it" by Helen Sakin, '31.
January 13, 1931: "Indeterminatism" by Richard Tousey, '28.
February 26, 1931: Freshman meeting. "Factoring the sums of squares" by Doris Pender, '34; "Trilinear coordinates" by Israel Lawsine, '34.
March 26, 1931: "Time" by Robert Nichols, Instructor in geology.
April 23, 1931: "Our-curves" by Professor Carl H. Holmberg.
May 18, 1931: Picnic at Lake Walden, Concord, Mass.

The purpose of Tuftconic is to promote closer contact between professors and students interested in higher mathematics. Meetings are held monthly at which students or professors present topics of interest. This is followed by a social hour. The club boasts of a song adapted by Professor William R. Ransom.

PEARL A. RUSSELL, *Secretary.*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3518. *Proposed by H. A. Simmons, Northwestern University.*

In Böcher's *Higher Algebra*, p. 11, Exercise 2, the student is expected to make a statement which may be expressed as follows: If f is a polynomial in m variables $x_1, x_2, \cdots, x_m$ which is known not to be of higher degree than n_i in

x_i, $(i=1, \cdots, m)$, and if f vanishes at the $(n_1+1)(n_2+1) \cdots (n_m+1)$ distinct points

$$(1) \qquad [x_1^{(i_1)}, x_2^{(i_2)}, \cdots, x_m^{(i_m)}], \ i_j = 1, 2, \cdots, n_j + 1; j = 1, \cdots, m,$$

then f vanishes identically

We ask the question: What is the greatest number of the points (1) at which a polynomial f of the type hypothesized can vanish and yet not be identically zero?

3519. *Proposed by Norman Anning, University of Michigan.*

AB is a fixed diameter and CD is a moving diameter of a given circle $ACBD$. E is the mid-point of arc BC. DE and AC intersect in P. Prove that DP is normal to the locus of P.

3520. *Proposed by Elijah Swift, University of Vermont.*

To construct a triangle, given the base, AB, the opposite angle, C, and the length of the bisector of the angle C. Show the construction is possible with ruler and compass and give a simple construction.

SOLUTIONS

238 [1916, 19; 1931, 289]. *Proposed by Clifford N. Mills.*

Determine the rational value of x that will render x^3+px^2+qx+r a perfect cube. Apply the result to $x^3-8x^2+12x-6$.

Solution by J. D. Hill, University of California at Los Angeles

By placing $f(x)=x^3+px^2+qx+r$, we will have, as is well known, $f(y-p/3) = y^3+sy+t=F(y)$, where $s=q-p^2/3$, and $t=r-pq/3+2p^3/27$. Since $F(-t/s)$ is a perfect cube, namely, $(-t/s)^3$, we have as a solution of $f(x)=z^3$, the expressions

$$x = - t/s - p/3 = (p^3 - 27r)/(27q - 9p^2),$$

and

$$z = (9pq - 2p^3 - 27r)/(27q - 9p^2).$$

For the special case, $f(x)=x^3-8x^2+12x-6$, we find that $x=25/18$, and that $z=-23/18$.

The method employed here is essentially that given by Fermat. Euler reproduced Fermat's procedure and showed also how a second solution may be obtained from a first one. Dickson's History of the Theory of Numbers[1] gives an outline of these methods with references, while Carmichael[2] has presented a rather complete discussion of this equation with a treatment of the more general case in which the coefficient of x^3 is not necessarily unity.

A Note by the Editors: A solution of this problem similar to the above was

[1] Volume 1, pp. 566–7.

[2] *Diophantine Analysis*, pp. 55–62.

printed [1928, 378]; and this solution was overlooked when the problem was reprinted as unsolved in this Monthly. This solution does not profess to be complete, and probably does not give *all* the rational values of x for which $f(x)$ is a perfect cube.

This same problem appeared also as 243 [1916, 120] and this offering of the problem was solved [1917, 133]. In the note appended by the editors to this solution there is additional information and references.

Problem 299 [1914, 267] was also reprinted through oversight of a solution which was printed [1916, 23].

Also solved by W. E. Buker, E. B. Escott, Edward Fleisher, A. L. McCarty, A. Pelletier, and F. Underwood.

308 [1915, 268; 1931, 229]. *Proposed by H. S. Uhler.*

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on the principal section, the ray inside the prism generates a cone of elliptical right section. It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

Solution by the Proposer

To save repetition reference will be made to the article entitled *Oblique Deviation and Refraction Produced by Prisms*, this Monthly, vol. 28 (1921), pp. 1–10.

From equations (7) and (8),

$$\sin h_1 \pm \sin h_4 = n'[\sin (x + \tfrac{1}{2}\beta) \pm \sin (x - \tfrac{1}{2}\beta)].$$

Hence

$$\sin \tfrac{1}{2}(h_1 + h_4) \cos \tfrac{1}{2}(h_1 - h_4) = n' \sin x \cos \tfrac{1}{2}\beta,$$

$$\cos \tfrac{1}{2}(h_1 + h_4) \sin \tfrac{1}{2}(h_1 - h_4) = n' \cos x \sin \tfrac{1}{2}\beta.$$

Substituting $h_1 - h_4 = D' + \beta$, which is relation (6), we get

$$\sin \tfrac{1}{2}(h_1 + h_4) = n' \sin x \cos \tfrac{1}{2}\beta / \cos \tfrac{1}{2}(D' + \beta),$$

$$\cos \tfrac{1}{2}(h_1 + h_4) = n' \cos x \sin \tfrac{1}{2}\beta / \sin \tfrac{1}{2}(D' + \beta).$$

Consequently

$$(a) \qquad \frac{n'^2 \sin^2 x \cos^2 \frac{1}{2}\beta}{\cos^2 \frac{1}{2}(D' + \beta)} + \frac{n'^2 \cos^2 x \sin^2 \frac{1}{2}\beta}{\sin^2 \frac{1}{2}(D' + \beta)} = 1.$$

It is appropriate now to introduce rectangular coördinates. This may be done conveniently in the following manner. In figure 2 (*loc. cit.*) let O denote the center of the sphere of reference. Let a plane be drawn tangent to this sphere at the point L. Also let L be the origin of coördinates, the axis of x' being taken as the intersection of the principal plane with the tangent plane, and the axis of y' as the line common to the plane of minima, LOH, and the tangent plane.

Now $\beta+D'=h_1-h_4$ and $|h_1|\leqq\pi/2$, $|h_4|\leqq\pi/2$, $h_1-h_4\leqq\pi$, $\beta+D'\leqq\pi$; hence *a fortiori*, $\beta+\frac{1}{2}D'<\pi$ and $D'<\pi$. Consequently the expression (j) is the sum of two positive terms so that it is essentially positive.

It may be shown that the bracketed factor of the coefficient of a^2 is positive in the following way. Let ϵ_0 symbolize the absolute minimum deviation common to D_0 and D_0'. We have then the classical formula

$$n = \sin\tfrac{1}{2}(\epsilon_0+\beta)/\sin\tfrac{1}{2}\beta$$

which gives

$$\sin^2\tfrac{1}{2}\beta = \sin^2 c\sin^2\tfrac{1}{2}(\epsilon_0+\beta).$$

Therefore

$$(k)\quad \begin{aligned} &\sin^2 c\sin^2\tfrac{1}{2}(D'+\beta) - \sin^2\tfrac{1}{2}\beta \\ &\equiv \sin^2 c\sin^2\tfrac{1}{2}(D'+\beta) - \sin^2 c\sin^2\tfrac{1}{2}(\epsilon_0+\beta) \\ &\equiv \sin^2 c[\sin\tfrac{1}{2}(D'+\beta)+\sin\tfrac{1}{2}(\epsilon_0+\beta)]\cdot[\sin\tfrac{1}{2}(D'+\beta)-\sin\tfrac{1}{2}(\epsilon_0+\beta)] \\ &\equiv \sin^2 c\sin\tfrac{1}{2}(D'+\epsilon_0+2\beta)\sin\tfrac{1}{2}(D'-\epsilon_0). \end{aligned}$$

Now $D'>\epsilon_0$ hence $(D'+\epsilon_0+2\beta)/2<(D'+D'+2\beta)/2$, also $D'+\beta\leqq\pi$ so that $(D'+\epsilon_0+2\beta)/2<\pi$. Hence expression (k) is positive, and so it has been shown that each of the coefficients of a^2, x'^2, y'^2 in equation (i) is positive, which is tantamount to proving that the curve under investigation is an ellipse. The coefficient of x'^2 exceeds that of y'^2 by $\sin\frac{1}{2}D'\sin(\beta+\frac{1}{2}D')\sin^2\frac{1}{2}(\beta+D')$, which shows that the major axis of the ellipse lies along the axis of y'.

432 [1917, 287]. *Proposed by R. P. Baker.*

The expressions

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right)$$

and

$$x^{-(i+1)}\left(x^3\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right)$$

are formally equivalent for every integral value of i.

Solution by Neal H. McCoy, Princeton University

For convenience let d/dx be denoted by D and interpret each equation involving D and x as operating on an arbitrary function of x. We shall show that

$$(1)\qquad x^{\alpha(i+1)/2}(x^{1-\alpha}D)^i = x^{-\alpha(i+1)/2}(x^{\alpha+1}D)^i x^{-\alpha(i-1)},$$

where i is any positive integer and α is any constant, not necessarily an integer. As a special case if $\alpha=2$ and we proceed to operate on the function $(c_1e^{ax}+c_2{}^{-ax})/x$, we have the result required by the problem.

A fundamental relation is the following;

$$(2)\qquad Df(x)-f(x)D=f'(x),$$

where $f(x)$ is any function of x and $f'(x)$ indicates the derivative of $f(x)$ with respect to x. This is simply a statement of the formula for the derivative of the product of two functions.

Let us multiply relation (1) on the left by $x^{-\alpha(i+1)/2}$, thus getting the equivalent form,

$$(x^{1-\alpha}D)^i = x^{-\alpha(i+1)}(x^{\alpha+1}D)^i x^{-\alpha(i-1)}. \tag{3}$$

Let P_i denote the expression on the right side of (3). Then

$$P_{i+1} = x^{-\alpha(i+2)}(x^{\alpha+1}D)^i x^{\alpha+1}Dx^{-\alpha i}.$$

But from the relation (2) we have

$$Dx^{-\alpha i} = x^{-\alpha i}D - i\alpha\, x^{-(\alpha i+1)}.$$

Hence

$$P_{i+1} = x^{-\alpha(i+2)}(x^{\alpha+1}D)^i x^{-\alpha(i-1)}xD - \alpha i x^{-\alpha(i+2)}(x^{\alpha+1}D)^i x^{-\alpha(i-1)}$$

We thus find the recursion formula,

$$P_{i+1} = x^{-\alpha}P_i(xD - \alpha i). \tag{4}$$

The relation (3) is easily verified for $i=1$. We accordingly assume it for $i\leqq n$ and show that it holds for $i=n+1$. We have

$$\begin{aligned}(x^{1-\alpha}D)^{n+1} &= (x^{1-\alpha}D)^n x^{1-\alpha}D \\ &= x^{-\alpha}P_{n-1}[xD - (n-1)\alpha]x^{1-\alpha}D,\end{aligned} \tag{5}$$

by relation (4). Now

$$\begin{aligned}xDx^{1-\alpha}D &= x[x^{1-\alpha}D + (1-\alpha)x^{-\alpha}]D \\ &= x^{1-\alpha}(Dx - 1)D + (1-\alpha)x^{1-\alpha}D \\ &= x^{1-\alpha}D(xD - \alpha).\end{aligned}$$

Substituting this in (5), we get

$$(x^{1-\alpha}D)^{n+1} = x^{-\alpha}P_{n-1}x^{1-\alpha}D(xD - n\alpha).$$

But by the hypothesis of the induction we have,

$$P_{n-1}x^{1-\alpha}D = P_n,$$

hence

$$(x^{1-\alpha}D)^{n+1} = x^{-\alpha}P_n(xD - n\alpha) = P_{n+1},$$

by relation (4). This completes the proof of (3).

It is interesting to note that this proof remains valid for $\alpha=-1$ which gives the formula,

$$(x^2D)^i = x^{i+1}D^i x^{i-1}.$$

Also solved by J. P. Dalton, Eugene Stephens, and Morgan Ward.

A Note by Otto Dunkel: In many problems of this nature certain general considerations provide a uniform method of proof which is often used. Here we have to prove the formal identity of two differential operations, or what is the same thing to prove that their difference gives an operator which is zero. An equation such as (1) in the solution above has the form, after expansion and collection of the terms on the left,

$$\sum_{j=0}^{i} R_j(x)D^i \equiv 0, \tag{1}$$

where $R_j\ (x)$ is an algebraic function of x; and we have to prove that the operator on the left for given forms of $R_j(x)$ reduces formally to zero any function which admits the required derivatives.

Consider the result of the operation on $e^{\beta x}$. We find, after discarding the factor $e^{\beta x}$,

$$\sum_{j=0}^{i} R_j(x)\beta^i; \tag{2}$$

and, if (1) is true, (2) is identically zero. Conversely, if (2) is zero for all values of β, it follows easily that (1) is true. Since $e^{\beta x}$ may be expanded into a power series in x, we have merely to examine the result of the given operation on a single term of $e^{\beta x}$ say $A_k x^k$.

Consider the given example of the above solution

$$x^{\alpha(i+1)/2}(x^{1-\alpha}D)^i - x^{-\alpha(i+1)/2}(x^{1+\alpha}D)^i x^{-\alpha(i-1)}. \tag{3}$$

It is easily found that

$$(x^{1-\alpha}D)^i x^k = k[k-\alpha]\cdots[k-(i-1)\alpha]x^{k-i\alpha} \tag{4}$$

Replacing in (4) first α by $-\alpha$, and then k by $k-(i-1)\alpha$, we have also

$$(x^{1+\alpha}D)^i x^{-(i-1)\alpha}x^k = [k-(i-1)\alpha][k-(i-2)\alpha]\cdots kx^{k+\alpha} \tag{5}$$

After multiplying (4) by $x^{\alpha(i+1)/2}$, and (5) by $x^{-\alpha(i+1)/2}$ we obtain $x^{(2k+\alpha-i\alpha)/2}$ with the same coefficient in each case. This concludes the proof of (1).

In the solution [1931, 115] of problem 3425 [1930, 260] a proof was given of the identity

$$x^iD^i \equiv \prod_{j=0}^{i-1}(xD-j).$$

Each side applied to x^k gives at once

$$k(k-1)\cdots(k-i+1)x^k,$$

and hence the formula is true.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Annual Meeting of the Association, New Orleans, Louisiana, Dec. 30-31, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May 8.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN